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THE MIXED PROBLEM FOR A PIECEWISE-HOMOGENEOUS ORTHOTROPIC PLANE WITH A CUT, INTERSECTING PERPENDICULARLY THE LINE OF INTERFACE

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Abstract. In the present paper we solve the mixed problem of the theory of elasticity for a piecewise-homogeneous orthotropic plane having a cut of finite length which intersects the interface boundary at right angles. Symmetric normal displacements are prescribed on the cut ends lying in the left half-plane, and symmetric normal stresses are prescribed on those lying in the right half-plane. Tangential stress along the whole cut is equal to zero.

Using the theory of analytic functions and the Fourier transformations, the problem is reduced to the solution of the problem of linear conjugation. An effective solution of the problem is obtained.

რეზიუმე. ნაშრომში ამოხსნილია დრეკადობის თეორიის შერეული ამოცანა უბნობრივ ერთგვაროვანი ორთოტროპული სიბრტყისათვის, რომელსაც აქვს გამყოფი წრფის მართი კუთხით გადამკვეთი სასრულო სიგრძის ჭრილი. მარცხენა ნახევარსიბრტყეში მოთავსებული ჭრილის ნაპირებზე მოცემულია სიმეტრიული ნორმალური გადაადგილება, ხოლო მარჯვენა ნახევარსიბრტყეში მოთავსებული ჭრილის ნაპირებზე მოცემულია სიმეტრიული ნორმალური ძაბვა. მხები ძაბვა ნულის ტოლია ჭრილის ნაპირებზე.

ანალიზურ ფუნქციათა თეორიის და ინტეგრალური გარდაქმნების გამოყენებით ამოცანის ამოხსნა მიყვანილია წრფივი შეუღლების სასაზღვრო ამოცანის ამოხსნაზე. ამოცანის ამონახსნი მიღებულია ეფექტური სახით.

Let the domain S occupied by a piecewise-homogeneous orthotropic elastic body represent a whole plane of a complex variable $z = x + iy$, cut along the segments $[-1, 1]$ on the x-axis. We assume that the left ($\operatorname{Re} z < 0$) and the right ($\operatorname{Re} z > 0$) half-planes are homogeneous, and principal directions of the elasticity axes coincide with the coordinate axes.

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By S_1 and S_2 we denote, respectively, the right and the left part of the domain S. The stress and displacement components, as well as elastic constants and another variables and functions referring to S_k, will be denoted by the index k $(k = 1, 2)$.

Let to the cut ends $0 \le x < 1$ be applied symmetric normal stresses

$$\big(\sigma_y^{(1)}\big)^+ = \big(\sigma_y^{(1)}\big)^- = p_1(x), \quad \big(\tau_{xy}^{(1)}\big)^+ = \big(\tau_{xy}^{(1)}\big)^- = 0$$

and on the cut ends $-1 \le x < 0$ be given symmetric normal displacements

$$v_1^+ = -v_2^- = g_2(x), \quad \big(\tau_{xy}^{(1)}\big)^+ = \tau_{xy}^{(1)} = 0.$$

$p_1(t)$ and $g_2'(t)$ are the given on ℓ_1 and ℓ_2 functions of the class H (Hölder class), where ℓ_1 and ℓ_2 are the segments $0 \le x \le 1$ and $-1 \le x \le 0$. The signs $(+)$ and $(-)$ denote boundary values of functions on the upper and lower cut ends,

$$v_1(0) = v_2(0) = h, \quad v_1(1) = v_2(-1) = 0. \tag{1}$$

According to [1], stresses and displacements are represented in terms of two holomorphic functions $\Phi_k(z_k)$ and $\Psi_k(\zeta_k)$ as follows:

$$\left.\begin{aligned} \sigma_x^{(k)} &= -2\operatorname{Re}\big[\beta_k^2\,\Phi_k(z_k) + \gamma_k^2\,\Psi_k(\zeta_k)\big]; \\ \sigma_y^{(k)} &= 2\operatorname{Re}\big[\Phi_k(z_k) + \Psi_k(\zeta_k)\big]; \\ \tau_{xy}^{(k)} &= 2\operatorname{Im}\big[\beta_k\,\Phi_k(z_k) + \gamma_k\,\Psi_k(\zeta_k)\big]; \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} u_k &= 2\operatorname{Re}\big[p_k\,\varphi_k(z_k) + r_k\,\psi_k(\zeta_k)\big]; \\ v_k &= -2\operatorname{Im}\big[\beta_k\,r_k\,\varphi(z_k) + \gamma_k\,p_k\,\psi_k(\zeta_k)\big]; \end{aligned}\right\} \tag{3}$$

$$\Phi_k(z_k) = \frac{d\varphi_k(z_k)}{dz_k}, \quad \Psi_k(\zeta_k) = \frac{d\psi_k(\zeta_k)}{d\zeta_k} \tag{4}$$

$z_k = x + \beta_k y$, $\zeta_k = x + i\gamma_k y$, $(x, y) \in S_k$, $\pm\beta_k$ and $\pm\gamma_k$ are the roots of the equation

$$\mu^4 + \big(E_k/G_k - 2\nu_k\big)\mu^2 + E_k/E_k^* = 0, \quad \beta_k > \gamma_k,$$
$$r_k = \frac{(\gamma_k^2 + \nu_k)}{E_k} \quad p_k = -\frac{(\beta_k^2 + \nu_k)}{E_k},$$

E_k, E_k^* are the Young module respectively for tension and pressure in the x and y direction, G_k is shear modulus, and ν_k is the Poisson coefficient.

Owing to the condition of symmetry with respect to the x-axis, the boundary conditions of the problem can be written in the form

$$\begin{gathered}\big(\sigma_y^{(1)}\big)^+ + \big(\sigma_y^{(1)}\big)^- = 2p_1(x), \quad \big(\tau_{xy}^{(k)}\big)^{\pm} = 0, \quad k=1,2,\\ \frac{\partial v_k^+(x,0)}{\partial x} - \frac{\partial v_k^-(x,0)}{\partial x} = 2\frac{\partial v_k^+(x,0)}{\partial x}, \quad k=1,2,\end{gathered} \tag{5}$$

$$\big(\sigma_y^{(k)}\big)^+ - \big(\sigma_y^{(k)}\big)^- = 0, \quad \frac{\partial u_k^+(x,0)}{\partial x} - \frac{\partial u_k^-(x,0)}{\partial x} = 0. \tag{6}$$

Using the conditions (5) and (6) and the equalities (2), (3) and (4), we obtain the following relations:

$$\Phi_k^+(x) - \Phi_k^-(x) = i\frac{f_k(x)}{\beta_k(p_k-r_k)}, \quad x\in\ell_k, \quad k=1,2, \tag{7}$$

$$\Psi_k^+(x) - \Psi_k^-(x) = i\frac{f_k(x)}{\gamma_k(r_k-p_k)}, \quad x\in\ell_k. \quad k=1,2, \tag{8}$$

$$\operatorname{Re}\big[\Phi_1^+(x)+\Phi_1^-(x)+\Psi_1^+(x)+\Psi_1^-(x)\big] = p_1(x), \quad x\in(0,1), \tag{9}$$

where $f_k(x) = \frac{\partial v_k^+(x,0)}{\partial x}$, $k=1,2$; $f_1(x)$ are the unknown and $f_2(x) = g_2'(x)$ are the given functions.

A general solution of the problem is given in the form

$$\begin{aligned}\Phi_k(z_k) &= W_k(z_k) + \frac{W_k^0(z_k)}{\beta_k},\\ \Psi_k(\zeta_k) &= \Omega_k(\zeta_k) - \frac{\Omega_k^0(\zeta_k)}{\gamma_k},\end{aligned} \qquad k=1,2, \tag{10}$$

where $W_1(z_1), \Omega_1(\zeta_1), W_2(z_2), \Omega_2(\zeta_2)$ are the functions, analytic respectively in the half-planes $\operatorname{Re} z_1 > 0$, $\operatorname{Re} z_2 < 0$, $\operatorname{Re}\zeta_1 > 0$, $\operatorname{Re}\zeta_2 < 0$,

$$\begin{aligned}W_k^0(z_k) &= \frac{1}{2\pi(p_k-r_k)}\int_{\ell_k}\frac{f_k(t)\,dt}{t-z_k},\\ \Omega_k^0(\zeta_k) &= \frac{1}{2\pi(p_k-r_k)}\int_{\ell_k}\frac{f_k(t)\,dt}{t-\zeta_k}.\end{aligned} \tag{11}$$

On the line of interface $x=0$, the conditions of equilibrium and continuity are of the form

$$\begin{gathered}\sigma_x^{(1)} = \sigma_x^{(2)}, \quad \tau_{xy}^{(1)} = \tau_{xy}^{(2)},\\ \frac{\partial u_1(0,y)}{\partial y} = \frac{\partial u_2(0,y)}{\partial y}, \quad \frac{\partial v_1(0,y)}{\partial y} = \frac{\partial v_2(0,y)}{\partial y}.\end{gathered} \tag{12}$$

On the basis of formulas (2), (3) and (4), the conditions (12) can be rewritten as

$$
\begin{aligned}
&\operatorname{Re}\big[\beta_1^2\,\Phi_1(t_1)+\gamma_1^2\,\Psi_1(\sigma_1)\big]=\operatorname{Re}\big[\beta_2^2\,\Phi_2(t_2)+\gamma_2^2\,\Psi_2(\sigma_2)\big],\\
&\operatorname{Im}\big[\beta_1\,\Phi_1(t_1)+\gamma_1\,\Psi_1(\sigma_1)\big]=\operatorname{Im}\big[\beta_2\,\Phi_2(t_2)+\gamma_2\,\Psi_2(\sigma_2)\big],\\
&\operatorname{Im}\big[\beta_1\,p_1\,\Phi_1(t_1)+\gamma_1\,r_1\,\Psi_1(\sigma_1)\big]=\operatorname{Im}\big[\beta_2\,p_2\,\Phi_2(t_2)+\gamma_2\,r_2\,\Psi_2(\sigma_2)\big],\\
&\operatorname{Re}\big[\beta_1^2\,r_1\,\Phi_1(t_1)+\gamma_1^2\,p_1\,\Psi_1(\sigma_1)\big]=\operatorname{Re}\big[\beta_2^2\,r_2\,\Phi_2(t_2)+\gamma_2^2\,p_2\,\Psi_2(\sigma_2)\big],
\end{aligned}
\tag{13}
$$

$t_k=i\beta_k\,y$, $\sigma_k=i\gamma_k\,y$, $k=1,2$.

If now we substitute the boundary values of formulas (10) and (11) into equalities (13), multiply the obtained expressions by $\frac{1}{2\pi i}\frac{dt}{t-z}$, integrate along the imaginary axis and use the fact that if $\Phi(z)$ is holomorphic in the half-plane $\operatorname{Re}z>0$ ($\operatorname{Re}z<0$), then $\overline{\Phi(iy)}$ is the boundary value of the function $\overline{\Phi(-\overline{z})}$, holomorphic in the half-plane $\operatorname{Re}z<0$ ($\operatorname{Re}z>0$), by means of the Cauchy theorems and formulas we will get the system

$$
\left.
\begin{aligned}
&\beta_1^2W_1(\beta_1z)+\gamma_1^2\Omega_1(\gamma_1z)-\beta_2^2\overline{W_2(-\beta_2\overline{z})}-\gamma_2^2\overline{\Omega_2(-\gamma_2\overline{z})}=\\
&\quad=-\beta_1\overline{W_1^0(-\beta_1\overline{z})}+\gamma_1\overline{\Omega_1^0(-\gamma_1\overline{z})}+\beta_2W_2^0(\beta_2z)-\gamma_2\Omega_2^0(\gamma_2z),\\
&\beta_1W_1(\beta_1z)+\gamma_1\Omega_1(\gamma_1z)+\beta_2\overline{W_2(-\beta_2\overline{z})}+\gamma_2\overline{\Omega_2(-\gamma_2\overline{z})}=\\
&\quad=-\overline{W_1^0(-\beta_1\overline{z})}-\overline{\Omega_1^0(-\gamma_1\overline{z})}+W_2^0(\beta_2z)-\Omega_2^0(\gamma_2z),\\
&p_1\beta_1W_1(\beta_1z)+r_1\gamma_1\Omega_1(\gamma_1z)+p_2\beta_2\overline{W_2(-\beta_2\overline{z})}+r_2\gamma_2\overline{\Omega_2(-\gamma_2\overline{z})}=\\
&\quad=p_1\overline{W_1^0(-\beta_1\overline{z})}-r_1\overline{\Omega_1^0(-\gamma_1\overline{z})}+p_2W_2^0(\beta_2z)-r_2\Omega_2^0(\gamma_2z),\\
&\beta_1^2r_1W_1(\beta_1z)+p_1\gamma_1^2\Omega_1(\gamma_1z)-\beta_2^2r_2\overline{W_2(-\beta_2\overline{z})}-p_2\gamma_2^2\overline{\Omega_2(-\gamma_2\overline{z})}=\\
&\quad=\beta_1r_1\overline{W_1^0(-\beta_1\overline{z})}+\gamma_1p_1\overline{\Omega_1^0(-\gamma_1\overline{z})}+r_2\beta_2W_2^0(\beta_2z)-p_2\gamma_2\Omega_2^0(\gamma_2z).
\end{aligned}
\right\}
\tag{14}
$$

Solving the above system with respect to the functions $W_1(\beta_1z)$ and $\Omega_1(\gamma_1z)$, we find that

$$
\begin{aligned}
W_1(\beta_1z)=&\frac{\Delta_{11}}{\beta_1\Delta}\,\overline{W_1^0(-\beta_1\overline{z})}-\frac{\Delta_{12}}{\gamma_1\Delta}\,\overline{\Omega_1^0(-\gamma_1\overline{z})}+\\
&+\frac{\Delta_{13}}{\beta_2\Delta}\,W_2^0(\beta_2z)+\frac{\Delta_{14}}{\gamma_2\Delta}\,\Omega_2^0(\gamma_2z),
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\Omega_1(\gamma_1z)=&-\frac{\Delta_{21}}{\beta_1\Delta}\,\overline{W_1^0(-\beta_1\overline{z})}+\frac{\Delta_{23}}{\gamma_1\Delta}\,\overline{\Omega_1^0(-\gamma_1\overline{z})}+\\
&+\frac{\Delta_{23}}{\beta_2\Delta}\,W_2^0(\beta_2z)+\frac{\Delta_{24}}{\gamma_2\Delta}\,\Omega_2^0(\gamma_2z),
\end{aligned}
\tag{16}
$$

Δ_{ij} is obtained from Δ by replacing its i-column by the coefficient of the jth summand in the right-hand side.

Replacing in the equality (15) z by z_1/β_1 and in (16) $\frac{\zeta_1}{\gamma_1}$ by ζ_1/γ_1, we obtain

$$\begin{aligned} W_1(z_1) =& \frac{\Delta_{11}}{\beta_1\Delta}\,\overline{W_1^0(-\overline{z}_1)} - \frac{\Delta_{12}}{\gamma_1\Delta}\,\overline{\Omega_1^0\Big(-\frac{\gamma_1}{\beta_1}\,\overline{z}_1\Big)} + \\ &+ \frac{\Delta_{13}}{\beta_2\Delta}\,W_2^0\Big(\frac{\beta_2}{\beta_1}\,z_1\Big) - \frac{\Delta_{14}}{\gamma_2\Delta}\,\Omega_2^0\Big(\frac{\gamma_2 z_1}{\beta_1}\Big), \end{aligned} \tag{17}$$

$$\begin{aligned} \Omega_1(\zeta_1) =& \frac{\Delta_{21}}{\beta_1\Delta}\,\overline{W_1^0\Big(-\frac{\beta_1}{\gamma_1}\,\overline{\zeta}_1\Big)} - \frac{\Delta_{22}}{\gamma_1\Delta}\,\overline{\Omega_1^0(-\overline{\zeta}_1)} + \\ &+ \frac{\Delta_{23}}{\beta_2\Delta}\,W_2^0\Big(\frac{\beta_2}{\gamma_1}\,\zeta_1\Big) - \frac{\Delta_{24}}{\gamma_2\Delta}\,\Omega_2^0\Big(\frac{\gamma_2}{\gamma_1}\,\zeta_1\Big). \end{aligned} \tag{18}$$

If in the equality (9) we insert the boundary values of the functions $\Phi_1(z)$ and $\Psi_1(\zeta_1)$ and take into account the relations

$$E_1(r_1 - p_1) = \beta_1^2 - \gamma_1^2,$$
$$f_1(t) = f(t) - 2hx,$$

we will get the equation

$$\begin{aligned} \int_0^1 \frac{f(t)\,dt}{t-x} - K_1 \int_0^1 \frac{f(t)\,dt}{t+x} - K_2\gamma_1 \int_0^1 \frac{f(t)\,dt}{\beta_1\,t + \gamma_1 x} - K_2\beta_1 \int_0^1 \frac{f(t)\,dt}{\gamma_1\,t + \beta_1 x} = \\ = K_3\pi\,\sigma_y^{(1)}(x)x + f_3(x) + f_4(x), \quad 0 < x < 1, \end{aligned} \tag{19}$$

where

$$\begin{aligned} f_3(x) =& \frac{\Delta_{13}}{\beta_2\Delta}\,W_2^0\Big(\frac{\beta_2}{\beta_1}\,x\Big) - \frac{\Delta_{14}}{\gamma_2\Delta}\,\Omega_2^0\Big(\frac{\gamma_2}{\beta_1}\,x\Big) + \\ &+ \frac{\Delta_{23}}{\beta_2\Delta}\,W_2^0\Big(\frac{\beta_2}{\gamma_1}\,x\Big) - \frac{\Delta_{21}}{\gamma_2\Delta}\,\Omega_2^0\Big(\frac{\gamma_2}{\gamma_1}\,x\Big); \\ f_4(x) =& 2h\bigg[1 + K_1 + K_2\frac{\beta_1^2+\gamma_1^2}{\beta_1\gamma_1} + 2x\Big(\ln\frac{1-x}{x} + K_1\ln\frac{1+x}{x} + \\ &+ \frac{\gamma_1^2}{\beta_1^2}\,\ln\frac{\beta_1+\gamma_1 x}{\gamma_1 x} + \frac{\beta_1^2}{\gamma_1^2}\,\ln\frac{\gamma_1+\beta_1 x}{\beta_1 x}\Big)\bigg]; \\ K_1 =& (\Delta_{12}\gamma_1 + \Delta_{12}\beta_1)/\Delta(\beta_1 - \gamma_1); \\ K_2 =& \gamma_1^2\Delta_{21}/\beta_1\Delta(\gamma_1 - \beta_1); \\ K_3 =& (\gamma_1 + \beta_1)\gamma_1\beta_1/E_1. \end{aligned}$$

If $0 < x < 1$, then (19) represents the singular integral equation which at the point $x = 0$ has likewise a fixed singularity.

Below, we will show that at the point $x = 0$ an order of singularity of a solution of the obtained equation may be of any number, lesser than 1.

Taking into account that the displacement should be bounded at the point $x=0$, the unknown function $f(x)$ is required to satisfy the conditions

$$x\,f(x)\to 0 \quad \text{as} \quad x\to 0.$$

Multiplying equation (19) by x and using the equality

$$\int\limits_0^1 f(x)dx=\int\limits_0^1 \frac{\partial v_1}{\partial x}\,dx+h=v^+(1)-v^+(0)=0,$$

we obtain

$$\int\limits_0^1 \frac{t\,f(t)dt}{t-x}+\int\limits_0^1 Q\Big(\frac{x}{t}\Big)f(t)dt=f^*(x)x, \tag{20}$$

where

$$Q(x)=K_1(1+x)^{-1}+K_2\beta_1(\beta_1+\gamma_1x)^{-1}+K_2\gamma_1(\gamma_1+\beta_1x)^{-1},$$
$$f^*(x)=K_3\pi\,\sigma_y^{(1)}(x)+f_3(x)+f_4(x).$$

In formula (20) we make substitution $x=e^{\xi_0}$, $y=e^{\xi}$ and get

$$\int\limits_{-\infty}^0 \frac{f(e^{\xi})e^{\xi}d\xi}{1-\exp(\xi_0-\xi)}+\int\limits_{-\infty}^0 Q(e^{\xi_0-\xi})f(e^{\xi})e^{\xi}d\xi=f^*(e^{\xi_0})e^{\xi_0},\ \ \xi_0\in(-\infty,0). \tag{21}$$

We now rewrite equation (21) in the form

$$\int\limits_{-\infty}^{\infty}\bigg(\frac{1}{1-\exp(\xi_0-\xi)}+Q(\xi_0-\xi)\bigg)\varphi_-(\xi)\,d\xi=f_-(\xi_0)+\varphi_+(\xi_0), \tag{21*}$$
$$-\infty<\xi_0<\infty,$$

where

$$\varphi_-(\xi)=\begin{cases} f(e^{\xi})e^{\xi}, & \xi<0,\\ 0, & \xi>0,\end{cases}$$
$$\varphi_+(\xi_0)=\begin{cases} \displaystyle\int\limits_{-\infty}^{\infty}\bigg(\frac{1}{1-\exp(\xi_0-\xi)}+Q(\xi_0-\xi)\bigg)\varphi_-(\xi)\,d\xi, & \xi>0,\\ 0, & \xi<0,\end{cases}$$
$$f_-(\xi_0)=\begin{cases} f^*(e^{\xi_0})e^{\xi_0}, & \xi<0,\\ 0, & \xi>0.\end{cases}$$

Applying the Fourier transformations to equation (21*) and taking into account the last equalities, we obtain

$$\begin{aligned} \Phi^+(t) &= G(t)\,\Phi^-(t) - iF^*(t), \quad -\infty < t < \infty, \\ G(t) &= \frac{\operatorname{ch}\pi t + K_1 + 2K_2\cos\mu t}{\operatorname{sh}\pi t}\,, \quad \mu = \ln\frac{\beta_1}{\gamma_1}\,, \end{aligned} \tag{22}$$

where

$$\Phi^+(t) = -\frac{i}{\sqrt{2\pi}}\int\limits_0^\infty \varphi_+(t)\,e^{it\xi}d\xi,$$

$$\Phi^-(t) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^0 f(e^\xi)\,e^{\xi(1+it)}d\xi,$$

$$F^*(t) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^0 f^*(e^\xi)\,e^{\xi(1+it)}d\xi.$$

Since the function $f^*(e^\xi)e^\xi$ vanishes exponentially as $\xi \to -\infty$, the function $F^*(w)$, where $w = t + i\tau$, will be analytic in the half-plane $\operatorname{Im} w < 1$.

Note also that

$$\Phi^-(0) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^0 f(e^\xi)e^\xi d\xi = \frac{1}{\sqrt{2\pi}}\int\limits_0^1 f(t)dt = 0.$$

Consider now the function

$$G_1(t) = \operatorname{ch}\pi t + K_1 + 2K_2\cos\mu t.$$

We prove that if the condition

$$\nu_k < \sqrt{E_k/E_k^*}\,,$$

is fulfilled, then $G_1(0) > 0$, $G_1''(0) > 0$.

Obviously, if $K_2 < 0$, then $G_1(t) > G_1(0)$ and $G_1(t) > 0$ on the whole axis, and since $G_1''(0) > 0$, $G_1(0) > 0$ for $K_2 > 0$ we have

$$G_1''(t) = \pi^2\operatorname{ch}\pi\,t - 2K_2\mu^2\cos\mu\,t > G_1''(0),$$

i.e., in this case the function $G'(t)$ increases and at the point $t = 0$ it reaches its minimum. This implies that the function $G_1(t)$ is likewise increases and reaches its minimum at the point $t = 0$. Since $G_1(0) > 0$, therefore $G_1(t) > 0$.

The function $G(t)$ at the point $t = 0$ has the first order pole, and at infinity it has the first order discontinuity, since $G(\infty) = -G(-\infty) = 1$.

We rewrite the boundary condition (22) in the form

$$\frac{\Phi^+(t)}{\sqrt{t+i}} = \frac{G(t)\,t}{\sqrt{1+t^2}}\,\frac{\Phi^-(t)}{t}\,\sqrt{t-i} - \frac{i\,F^*(t)}{\sqrt{t+i}}, \tag{23}$$

where under $\sqrt{w+i}$ and $\sqrt{w-i}$ we mean those branches which are analytic respectively in the planes cut along the rays drawn from the points $w=-i$ and $w=i$ to x, and have respectively positive and negative values on the upper side of the cut. Under such a choice of branches, the function $\sqrt{1+w^2}$ is analytic, and on the segment $-1 < \operatorname{Im} w < 1$ it takes positive value on the real axis.

Because of the fact that the ratio $w/\sqrt{w+i}$ is holomorphic in the half-plane $\operatorname{Im} w < 1$, the ratio $\Phi^+(w)/\sqrt{w+i}$ is holomorphic in the half-plane $\operatorname{Im} w > 0$, $G(t) \neq 0$ and $\Phi^-(0)$=0, the function $\Phi^-(w)\sqrt{w-i}/w$ will be holomorphic everywhere in the half-plane $\operatorname{Im} w < 1$, except the points which are zeroes of the function $G(w)$ and lie in the upper half-plane.

Thus the above problem can be formulated as follows: Find both the function $\Phi^+(w)$, holomorphic in the upper half-plane $\operatorname{Im} w > 0$ and vanishing at infinity, and the function $\Phi^-(w)$, holomorphic in the half-plane $\operatorname{Im} w < 1$, except the points w_n which are the roots of the function $G(w)$, vanishing at infinity and continuous on the real axis $w = t$, by the condition (23).

The function $G_0(t) = G(t)t(1+t^2)^{-\frac{1}{2}}$ is positive and continuous on the whole real axis and $G_0(\infty) = G_0(-\infty) = 1$, hence Ind $G_0(t) = 0$.

The solution of the problem (22) is given by formulas (24), (25), (26) and (27) below ([2]):

$$\Phi^+(w) = -\frac{X(w)\sqrt{w+i}}{2\pi}\int\limits_{-\infty}^{\infty}\frac{F^*(t)dt}{X^+(t)\sqrt{1+t}\,(t-w)}\,, \quad \operatorname{Im} w > 0; \tag{24}$$

$$\Phi^-(w) = -\frac{X(w)w}{2\pi\sqrt{w-i}}\int\limits_{-\infty}^{\infty}\frac{F^*(t)dt}{X^+(t)(t-w)\sqrt{t+i}}\,, \quad \operatorname{Im} w \le 0; \tag{25}$$

$$\Phi^-(w) = \frac{\Phi^+(w) + i\,F^*(t)}{G(w)}\,, \quad 0 < \operatorname{Im} w < 1; \tag{26}$$

$$X(w) = \exp\left(\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\ln[t(t^2+1)^{-\frac{1}{2}}G(t)]}{t-w}\,dt\right), \quad \operatorname{Im} w \neq 0. \tag{27}$$

Using Sokhatskii-Plemelj formulas, it is not difficult to verify that

$$\Phi^-(t-i0) = \Phi^-(t+i0), \quad \operatorname{Im} w < 0,$$

thus the function $\Phi^-(w)$ is holomorphic in the half-plane $\operatorname{Im} w < 1$, except the points w_k, $k = 0, 1, \dots, n$, lying in the upper half-plane and being zeroes of the function $G(w)$.

We can prove that $G(i) < 0$, and since $G(0) > 0$, $G(w)$ has at least one pure imaginary zero $w_0 = i\tau_0$, $0 < \tau_0 < 1$.

We rewrite the function $\Phi^+(w)$ in the form

$$\Phi^+(w) = -\frac{X^+(w)}{2\pi\sqrt{w+i}}\left[\int_{-\infty}^{\infty} \frac{(t+i)F^*(t)dt}{X^+(t)\sqrt{t+i}(t-w)} - \int_{-\infty}^{\infty} \frac{F^*(t)dt}{X^+(t)\sqrt{t+i}}\right],$$

or as

$$\Phi^+(w) = \Phi_0^+(w) + \frac{X^+(w)}{2\pi\sqrt{w+i}} \int_{-\infty}^{\infty} \frac{F^*(t)dt}{X^+(t)\sqrt{t+i}}.$$

It is be easily shown that the boundary values $\Phi_0^+(t)$ and $\Phi_0^-(t)$ is the Fourier transformation of the bounded function, i.e.,

$$\Phi^+(t_0) = \Phi_0^+(t_0) + \frac{C}{\sqrt{t+i}} \tag{28}$$

where

$$C = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{F^*(t)dt}{X^+(t)\sqrt{t+i}}\,. \tag{29}$$

The function $F^*(t)$ is the Fourier transformation of the real function, hence $\overline{F^*(-t)} = F^*(t)$. Moreover, $X^\pm(-t) = X^\pm(t)$ and $\overline{X_0^\pm(-t)} = \pm i X_0^\pm(t)$, where $X_0^\pm(t) = \sqrt{t+i}$. On the basis of the above-said, it is easy to see that

$$\overline{\Phi^+(-t)} = -\Phi^+(t) \quad \text{and} \quad \overline{\Phi^-(-t)} = \Phi^-(t),$$

i.e., $\Phi^+(t)$ is the Fourier transformation of the pure imaginary function, and $\Phi^-(t)$ is that of the real function. Consequently, a solution of the above-formulated problem can be obtained by the inverse Fourier transformation of the functions $\Phi^+(t)$ and $\Phi^-(t)$.

We perform an inverse Fourier transformation of the equality (28) and get back to the variable x. After elementary calculations, we obtain

$$\sigma_y^{(1)}(x;0) = -\frac{c_1 \exp\left(\frac{\pi}{4}\, i\right)}{\pi x^2\sqrt{x-1}} + \varphi_0(x), \quad x > 1, \tag{30}$$

where $\varphi_0(x)$ is bounded for $x \geq 0$.

It is east to show that $c_1 e^{i\pi/4}$ is a real number, and for $0 \leq \operatorname{Im} w < 1$ we have

$$\Phi^-(w) = \frac{c_2}{\sqrt{w-i}} + \Phi_0^-(w), \tag{31}$$

where the function $\Phi_0^-(w)$ is holomorphic in the whole strip $0 < \operatorname{Im} w < 1$, except possibly at the points $w_0 = i\tau_0$, $\tau_0 < \beta < 1$, where it has the first order pole, and for sufficiently large $|w|$ it can be represented in the form

$$\Phi_0(w) = O\Big(\frac{1}{|w|}\Big).$$

We multiply the function $\Phi_0^-(w)$ by $e^{-i\xi w}$, $\xi < 0$, and integrate the obtained expression along the rectangle with vertices at the points $(-N;0)$, $(N;0)$, $(N;\beta)$, $(-N;\beta)$. Using the Cauchy theorem for a multiply connected domain, we obtain

$$\int\limits_{-N}^{N} \Phi_0^-(t)e^{-it\xi}dt = e^{\beta\xi} \int\limits_{-N}^{N} \Phi_0^-(t+i\beta)e^{-it\xi}dt + c_1 e^{\tau_0\xi} + \varepsilon(N,\xi),$$

where $\varepsilon(N,\xi) \to 0$ as $N \to \infty$. Thus we have found that as $N \to \infty$, the integrals exist in the Plancherel sense ([3]).

By means of the Fourier transformation, from formula (30) we get

$$e^{\xi} f(e^{\xi}) = \frac{M\, e^{\xi}}{\sqrt{-\xi}} + \frac{e^{\beta\xi}}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \Phi_0(t+i\beta)e^{-it\xi}dt + \frac{c_1 e^{\tau_0\xi}}{\sqrt{2\pi}}, \quad \xi < 0,$$

where M is real, and

$$c_1 = 2\pi i \lim_{r\to i\tau_0} (\tau - i\tau_0)\Phi^-(r).$$

Taking into account formula (26), we can write

$$\lim_{\tau\to i\tau_0} \Phi^-(\tau)(\tau - i\tau_0) = \frac{\Phi^+(i\tau_0) + iF^*(i\tau_0)}{\pi \sin \pi\tau_0 - 2K_2\mu \operatorname{sh} \mu\tau_0} \sin \pi\tau_0,$$

and consequently,

$$c_1 = -\frac{1}{\sqrt{2\pi}} \frac{\int\limits_0^\infty f^*(x)x^{-\tau_0}dx}{\pi \sin \pi\tau_0 - 2K_2\mu \operatorname{sh} \pi\tau_0}\,.$$

Thus we have

$$f(x) = \frac{N}{\sqrt{\ln \frac{1}{x}}} + x^{\beta-1}\varphi_0(x) + \frac{c_1}{\sqrt{2\pi}}\, x^{\tau_0-1} = O\big(x^{\tau_0-1}\big).$$

Using the property of the Cauchy type integrals in the neighborhood of the ends of unclosed contour [2], we can show that the functions $\Phi_1(z_1)$ and $\Psi_1(\xi_1)$ near the points $x = 0$ and $x = 1$ are of the same character as the function $f(x)$. Moreover, we can show that analytic character possess the functions $\Phi_2(z_2)$ and $\Psi_2(\xi_2)$ and also the stress components $\sigma_y^{(2)}$, $\sigma_x^{(2)}$ and $\tau_{xy}^{(2)}$ near the point $x = 0$.

In a particular case of the problem under consideration, we convince ourselves that τ_0 may take any value from the interval $(0,1)$.

Assume that the domain S_2 is formed from S_1 by turning the elastic axis around 90^o. Then we will have

$$E_2 = E_1^*, \quad E_2^* = E_1, \quad \nu_2 = \nu_1^* = \frac{E_1^*}{E_1}\,\nu_1, \quad G_2 = G_1.$$

The characteristic equation of the body S_2 takes the form

$$\mu^4 + \Big(\frac{E_1^*}{G} - 2\frac{E_1^*}{E_1}\,\nu_1\Big)\mu^2 + \frac{E_1^*}{E_1} = 0,$$

or

$$\Big(\frac{1}{\mu}\Big)^4 + \Big(\frac{E_1}{G_1} - 2\nu_1\Big)\Big(\frac{1}{\mu}\Big)^2 + \frac{E_1}{E_1^*} = 0.$$

$\pm i/\gamma_1$ and $\pm i/\beta_1$ are the roots of the above equation, i.e. $\beta_2 = 1/\gamma_1$ and $\gamma_2 = 1/\beta_1$. Next,

$$p_2 = -\left(\frac{\beta_2^2+\nu_2}{E_2}\right) = -\left(\frac{1}{\gamma_1^2} + \frac{E_1^*}{E_1}\,\nu\right)\frac{1}{E_1^*} = -\frac{1}{E_1}\left(\frac{E_1}{\gamma_1^2 E_1^*} + \nu_1\right) = -\frac{\beta_1^2+\gamma_1}{E_1}\,,$$

that is, $p_2 = p_1$. Analogously, we obtain

$$\Delta = -(p_1 - r_1)^2\frac{(\gamma_1\beta_1+1)^2}{\gamma_1\beta_1}\,, \quad \Delta_{11} = 0;$$

$$\Delta_{12} = -\Delta_{21} = (p_1 - r_1)^2\frac{\gamma_1^2\beta_1^2 - 1}{\gamma_1\beta_1}\,;$$

$$K_1 = (\Delta_{12}\gamma_1 + \Delta_{21}\beta_1)(\beta_1 - \gamma_1)\Delta = \frac{\beta_1^2\gamma_1^2 - 1}{(\gamma_1\beta_1+1)^2} =$$

$$= \frac{\beta_1\gamma_1 - 1}{\beta_1\gamma_1 + 1} = \frac{\sqrt{E_1} - \sqrt{E_1^*}}{\sqrt{E_1} + \sqrt{E_1^*}}\,;$$

$$G(t) = \left(\operatorname{ch}\pi t + \frac{\sqrt{E_1} - \sqrt{E_1^*}}{\sqrt{E_1} + \sqrt{E_1^*}}\right)\cdot\frac{1}{\operatorname{sh}\pi t}\,,$$

whence it follows that: if $E_1 < E_1^*$, then

$$\tau_0 = \frac{1}{\pi}\,\arccos\left(\frac{\sqrt{E_1} - \sqrt{E_1^*}}{\sqrt{E_1} + \sqrt{E_1^*}}\right) < \frac{1}{2}\,;$$

if $E_1^* < E_1$, then

$$\tau_0 = 1 - \frac{1}{\pi}\,\arccos\left(\frac{\sqrt{E_1} - \sqrt{E_1^*}}{\sqrt{E_1} + \sqrt{E_1^*}}\right) > \frac{1}{2}\,;$$

if $E_1 = E_1^*$, then $\tau_0 = \frac{1}{2}$.

The last proposition corresponds to the case, where the cut plane is homogeneous one.

The above-considered example shows that the more rigid is the left half-plane, the lesser is degree of stress concentration near the point lying on the line of intersection.

The first basic problem for a piecewise-homogeneous isotropic plane when a crack of finite length reaches the boundary of intersection of two bodies at right angles, has been solved in [4] and [5]. In [6], the author solved the same problem for a piecewise-homogeneous orthotropic plane.

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