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GENERAL SOLUTION OF DIFFERENTIAL SCHWARTZ EQUATION FOR CONFORMALLY MAPPING FUNCTIONS OF CIRCULAR POLUGONSM THEIR CONNECTION WITH BOUNDARY VALUE PROBLEMS OF FILTRATION AND OF AXIALLY SYMMETRIC FLOWS

Abstract. A single-valued analytic function of general type is constructed which maps a half-plane onto a circular polygon with a finite number of vertices and with arbitrary finite angles at those vertices. It is proved that this function is a general solution of the Schwartz equation.

Transcendent equations of higher order connecting geometrical characteristics of circular polygons with unknown parameters of Schwartz equation, are investigated. Possible intervals of variation of unknown accessory parameters are established.

A general mathematical method of constructing solutions of spatial axially symmetric stationary with partially unknown boundaries problems of the theory of get flows and filtration is given.

რეზიუმე. აგებულია ზოგადი სახის ცალსახა ანალიზური ფუნქცია, რომელიც ნახევარსიბრტყეს გადასახავს წრიულ მრავალკუთხედზე წვეროების სასრული რაოდენობით და ამ წვეროებთან ნებისმიერი სასრული კუთხეებით. დამტკიცებულია, რომ ეს ფუნქცია შვარცის განტოლების ზოგადი ამონახსნია.

გამოკვლეულია მაღალი რიგის ტრანსცენდენტული განტოლებები, რომლებიც აკავშირებს წრიულ მრავალკუთხედების გეომეტრიულ მახასიათებლებს შვარცის განტოლების უცნობ პარამეტრებთან. დადგენილია უცნობი აქსესორული პარამეტრების ცვლილებების შესაძლო ინტერვალები.

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Contents

Preface

The present monograph includes chapters from some papers reflecting the most important earlier and the latest results of investigations carried out by the author.

A brief list of problems: :

1. Assume that an upper half-plane of the plane $\zeta = t + i\tau$ is mapped conformally by the function $z = z(\zeta)$ onto circular polygons on the plane $z = x + iy$. Moreover, let the points $t, a_1, a_2, \ldots, a_m$ (where $-\infty < a_1 < a_2 < \cdots < a_m < +\infty$) of the real axis turn into the corresponding vertices of a circular (or linear) polygon $b_1, b_2, \ldots, b_m$. An unknown function $z = z(\zeta)$ satisfies the known Schwarts equation

$$\begin{aligned} &z'''(\zeta)/z'(\zeta) - 3\big[z'''(\zeta)/z'(\zeta)\big]^2/2 = R(\zeta), \\ &R(\zeta) = \sum_{k=1}^{m} \big[(1 - v_k^2)(\zeta - a_k)^{-2}/2 + c_k(\zeta - a_k)^{-1}\big], \end{aligned} \tag{1}$$

where c_k are unknown accessory parameters, πv_k are the given interior angles of the circular polygon at the vertices b_k. Equation (1) depends on $2(m-3)$ unknown parameters a_k, c_k, $\sum\limits_{k=1}^{m} c_k = 0$,,

$$\sum_{k=1}^{m} \big[a_k c_k + 0,5(1 - v_k^2)\big] = 0, \quad \sum_{k=1}^{m} \big[a_k^2 c_k + a_k(1 - v_k^2)\big] = 0. \tag{2}$$

An unknown function $z = z(\zeta)$, being a solution of equation (1), must satisfy the linear boundary condition

$$A(t)\, z(t)\, \overline{z(t)} - i\, \overline{B(t)}\, z(t) + i\, B\, \overline{z(t)} + D(t) = 0, \tag{3}$$

where $A(t), B(t), D(t)$ are the given piecewise-constant functions satisfying the condition $B(t)\overline{B(t)} - A(t)D(t) = 1$; $\overline{z(t)}\overline{B}(t)$ are the complex-conjugate functions, respectively, of $z(t)$ and $B(t)$.

2. A plane of stationary motion of incompressible liquid in a porous medium, subject to the Darcy law, coincides with the plane of a complex variable $z = x + iy$. The porous medium is assumed to be isotropic, homogeneous and undeformable. The boundary $\ell(z)$ of the domain $S(z)$ of liquid motion consists of a depression curve to be determined and of the known segments of straight, semidirect and direct lines. In the domain $S(z)$ with the boundary $\ell(z)$ we seek for a reduced complex potential $\omega(z) = \varphi(x, y) + i\psi(x, y)$, where $\varphi(x, y)$ is the velocity potential, $\psi(x, y)$ is a stream function satisfying the Cauchy-Riemann conditions and the conditions

$$a_{k1}\varphi(x,y) + a_{k2}\psi(x,y) + a_{k3}x + a_{k4}y = f_k, \quad k = 1,2, \quad (x,y) \in \ell(z), \quad (0.1)$$

where a_k, f_k, $k = 1,2$, $j = \overline{1,4}$ are the known piecewise-constant real functions; f_k, $k = 1,2$, dapend on the parameter Q, where Q is the liquid discharge per filtration.

3. Solutions of spatial axially symmetric problems with partially unknown boundaries become more complicated as compared with analogous plane problems 1 and 2.

Such kind of problems are encountered in the theory of filtration, in get flows theory, and also in various parts of mathematical physics.

In the present monograph we investigate the above-mentioned problems and these which are tightly connected with them.

The monograph consists of five chapters.

Each chapter is supplied with an abstract, introduction, sections and references.

Chapter I

Solution of the Schwarz Differential Equation

Abstract. A circular polygon of general form with a finite number of vertices and arbitrary angles at these vertices is given. A single-valued analytic function mapping conformally a half-plane onto the given circular polygon is constructed in a general form. The function is proved to be a general solution of the Schwarz equation. First we construct functional series convergent uniformly and rapidly near all singular points and then fundamental local matrices which are connected by analytic continuation. The constructed analytic function satisfies nonlinear boundary conditions. In a general form, we compose and investigate all higher transcendental equations connecting geometric characteristics of circular polygons with unknown parameters of the Schwarz equation. Possible intervals of variation of unknown accessory parameters are established.

1. Introduction

Let on a complex plane w be given a simply connected domain $S(w)$ with the boundary l consisting of a finite number $m+1$ of circular arcs or linear segments; note that the latter are regarded as degenerated circular arcs. The vertices of circular polygons are denoted by $b_1, b_2, \ldots, b_{m+1}$, while the sizes of inward with respect to the domain $S(w)$ angles are denoted by $\pi\nu_1, \pi\nu_2, \ldots, \pi\nu_{m+1}$. The domain $S(w)$ may be assumed to be bounded. This can always be achieved by a suitable linear-fractional mapping.

Without restriction of generality, one can by means of a linear-fractional transformation, combine one of the sides of circular polygons, say the side (b_m, b_{m+1}), with a segment of abscissa axis, the origin coinciding with the vertex b_m. For $\nu_m \neq n$, $n = 0, 1, 2$, and the side (b_{m-1}, b_m) becomes a segment of the straight line forming with the abscissa axis the angle $\pi\nu_m$. This remark will be used in the sequel.

Find and investigate the function $w(\zeta)$ which conformally maps the half-plane $\Im(\zeta) > 0$ (or $\Im(\zeta) < 0$) of the plane $\zeta = t + i\tau$ onto the domain $S(w)$. Using the theorem on the correspondence of boundaries of the domains $\Im(\zeta) > 0$ and $S(w)$, we denote by a_k, $k = 1, 2, \ldots, m+1$, the points of the real axis of the plane $\zeta = t + i\tau$ (in this case $-\infty < a_1 < a_2 < \cdots < a_m < +\infty$) to which on the plane w there correspond the vertices of circular polygons b_k, $k = 1, 2, \ldots, m, m+1$. Suppose that the point $a_{m+1} = \infty$ is mapped into the point $w = b_{m+1}$. On every interval of the t-axis, the unknown function $w = w(\zeta)$ takes between neighboring points a_k, a_{k+1} the values which lie on the corresponding circular arc [5,6].

A not complete bibliography dealing with those problems can be found in [1–27].

The function $w = w(\zeta)$ is the solution of the Schwarz equation [5–7, 9–11]

$$w'''(\zeta)/w'(\zeta) - 1,5[w''(\zeta)/w'(\zeta)]^2 = R(\zeta), \tag{1.1}$$

$$R(\zeta) = \sum_{k=1}^{m}[0,5(1-\nu_k^2)/(\zeta - a_k)^2 + c_k/(\zeta - a_k)], \tag{1.2}$$

where c_k, $k = 1, 2, \ldots, m$ are unknown real accessory parameters satisfying the conditions

$$\sum_{k=1}^{m} c_k = 0, \quad \sum_{k=1}^{m}[a_k c_k + 0,5(1-\nu_k^2)] = 0,5(1-\nu_{m+1}^2). \tag{1.3}$$

By b_k, b'_k, $k = 1, 2, \ldots, m+1$ we denote the complex coordinates of the vertices of a circular polygon at which two neighboring circumferences may intersect; but if the neighboring circumferences are tangents at the vertex $w = b_k$, then $b_k = b'_k$.

The function $w = w(\zeta)$ on the boundary l of $S(w)$ must satisfy the nonlinear boundary condition [19, 20]

$$iA(t)w(t)\overline{w(t)} + \overline{B(t)}w(t) - B(t)\overline{w(t)} + iD(t) = 0, \quad -\infty < t < +\infty, \tag{1.4}$$

$$B(t)\overline{B(t)} - A(t)D(t) = 1, \tag{1.5}$$

where $A(t)$, $B(t)$, $\overline{B(t)}$, $D(t)$ are the given piecewise constant functions; $A(t)$, $D(t)$ are real, while $B(t)$ and $\overline{B(t)}$, $w(t)$ and $\overline{w(t)}$ are mutually complex conjugate.

It should be noted that (1.4) is the equation of the contour of the circular polygon.

It is known that every function $w(\zeta)$, conformally mapping $\Im(\zeta) > 0$ onto a circular polygon, satisfies (1.1), and vice versa, every solution of (1.1) conformally maps the domain $\Im(\zeta) > 0$ onto a some circular polygon [10, p. 137]. Moreover, due to the boundary correspondence under conformal mapping, every solution of (1.1), $w = w(\zeta)$, will satisfy the boundary condition (1.4). Note hereat that when passing in (1.4) to complex conjugate values, the equation (1.4) remains unchanged.

If $w = w_1(\zeta)$ is a particular solution of (1.1), then the general solution of (1.1) is given by

$$w(\zeta) = [pw_1(\zeta) + q]/[rw_1(\zeta) + S], \quad ps - rq = 1, \tag{1.6}$$

where p, q, r, s are arbitrary, parameters of integration of the equation (1.1), connected by the condition $ps - rq = 1$.

Equation (1.1) is invariant with respect to a linear-fractional transformation of the independent ζ and dependent w variable; given ζ, the coefficients of the linear-fractional transformation are real, but given w, they are complex. Therefore we can fix arbitrarily three of the parameters a_k, $k = 1, 2, \dots, m, m+1$ one of which, $a_{m+1} = \infty$, is already fixed. It remains to fix the rest two parameters by taking, e.g., $a_1 = -m$, $a_m = m$.

Thus it becomes evident that the equation (1.1) depends on $2(m-2)$ unknown parameters a_k, c_k, $k = 1, 2, \dots, m$ and a number of singular points $\zeta = a_k$ equals $m+1$.

The contour of the circular polygon l consists of arcs of $m+1$ circumferences. For their definition, we need $3(m+1)$ real parameters. As it will be seen, there are exactly $3(m+1)$ parameters at our disposal. Indeed, the equation (1.1) depends both on $2(m-2)$ unknown parameters a_k, c_k and on $m+1$ known parameters ν_k, $k = 1, 2, \dots, m+1$. In defining a general solution of (1.1), there appear six more additional parameters of integration (see (1.6)). Thus we have $2(m-2) + m + 1 + 6 = 3(m+1)$ parameters [7].

If we assume that $w' = 1/u^2(\zeta)$, then the solution of (1.1) reduces to that of the Fuchs class differential equation [5–13]

$$u''(\zeta) + 0,5R(\zeta)u(\zeta) = 0. \tag{1.7}$$

If we find linear independent partial $v_1(\zeta)$, $v_2(\zeta)$ solutions of (1.7), then a general solution of (1.1) can be obtained by the formula (1.6) assuming $w_1(\zeta) = v_1(\zeta)/v_2(\zeta)$.

Below we will consider the Fuchs class equation of the kind

$$v''(\zeta) + p(\zeta)v'(\zeta) + q(\zeta)v(\zeta) = 0, \tag{1.8}$$

where

$$p(\zeta) = \sum_{k=1}^{m} \beta_k/(\zeta - a_k), \quad q(\zeta) = \sum_{k=1}^{m} [\sigma_k/(\zeta - a_k)^2 + c_k/(\zeta - a_k)]; \tag{1.9}$$

β_k, σ_k are the given constants and c_k are unknown $p'(s)$ accessory parameters.

Substituting

$$v(\zeta) = u(zt) \exp\Big[-\frac{1}{2}\int_0^{\zeta} p(\zeta)d\zeta\Big], \tag{1.10}$$

the equation (1.8) reduces to the equation (1.7), where

$$0,5R(\zeta) = q(\zeta) - 0,5(p'(\zeta))^2 - 0,25(p(\zeta))^2. \tag{1.11}$$

One frequently uses equations of the type (1.8) in which $p(\zeta)$ and $q(\zeta)$ are of the form [4, 15]

$$p(\zeta) = \sum_{k=1}^{m} (1-\nu_k)/(\zeta - a_k),$$
$$q(s) = \alpha'\alpha'' \prod_{k=1}^{m-2} (\zeta - \lambda_k) / \prod_{k=1}^{m} (\zeta - a_k), \tag{1.12}$$

where

$$\sum_{k=1}^{m} \nu_k + \alpha' + \alpha'' = m-1, \quad \alpha' - \alpha'' = \nu_{m+1}, \tag{1.13}$$

and $\lambda_1, \lambda_2, \ldots, \lambda_{m-2}$ are accessory parameters.

If we consider a circular polygon with equal angles $\pi\nu_j = \pi$, $j = 1, 2, \ldots, m+1$, then $\alpha' = 0$, and hence in this case it is necessary to consider the limits $\lim(\alpha'\alpha''\lambda_k)$, $k = 1, 2, \ldots, m-2$ as $\alpha' \to 0$. Therefore it is better to write $q(\zeta)$ in the form [7]

$$q(\zeta) = \frac{\left[\alpha'\alpha''\zeta^{m-2} + \delta_1\zeta^{m-3} + \delta_2\zeta^{m-4} + \cdots + \delta_{m-3}\zeta + \delta_{m-2}\right]}{\prod_{k=1}^{m} (\zeta - a_k)}, \tag{1.14}$$

where δ_k, $k = 1, 2, \ldots, m-2$ are unknown accessory parameters.

The Fuchs class equations are solved by means of the power series, hence we represent (1.14) as a sum of partial fractions

$$q(\zeta) = \sum_{j=1}^{m} c_j/(\zeta - a_j), \tag{1.15}$$

where

$$\sum_{j=1}^{m} c_j = 0, \quad \sum_{j=1}^{m} c_j a_j = \alpha'\alpha'', \tag{1.16}$$

$$c_k = \frac{\left[\alpha'\alpha'' a_k^{m-2} + \delta_1 a_k^{m-3} + \cdots + \delta_{m-3} a_k + \delta_{m-2}\right]}{\prod_{j=1, j\neq k}^{m} (a_k - a_j)}. \tag{1.17}$$

The equation (1.1), as well as the method of constructing $w(\zeta)$ for $m = 2$, was obtained by H. A. Schwarz in 1873.

The equation (1.8) for $m = 3$ was considered by K. Heun in 1889 and by Ch. Snow in 1952. But they failed in connecting the constructed local solutions [3]. G.N. Goluzin [6] constructed $w(\zeta)$ for equilateral and equiangular circular polygons. V. Koppenfels and F. Stallmann constructed $w(\zeta)$ for some particular cases of circular polygons with angles, multiple of $\frac{\pi}{2}$ [10]. Approximate methods for finding the parameters a_k, c_k can be found in [2].

P. Ya. Polubarinova-Kochina has obtained important results in constructing $w(\zeta)$ and in its application to the problems of filtration, when a finite number of new singular points, the so-called removable points, are added to the points $\zeta = a_k$.

A general analytic solution of the equation (1.1) for circular polygons with a finite number of vertices b_k $k = 1, 2, \dots, m+1$ is given in [19–26]. In other works, one can get systems of equations for finding the parameters a_j, c_j, p, q, r, s, $j = 1, 2, \dots, m$. The method making it possible to construct explicitly a solution of (1.1) for circular polygons with angles, multiple of $\pi/2$, is described in [22].

Below we present our new, not published yet, results as well as those published earlier [19–26].

2. Application of Matrix Calculus to Determination of the Fundamental System of Solutions

Denote linearly independent local solutions of (1.8) near singular points $\zeta = a_k$, $k = 1, 2, \dots, m+1$, by $v_{kj}(\zeta)$, $k = 1, 2$; $j = 1, \dots, m+1$, while the solutions containing integration constants p, q, r, s satisfying $ps - rq = 1$

$$u_{1j}(\zeta) = pv_{1j}(\zeta) + qv_{2j}(\zeta), \quad u_{2j}(\zeta) = rv_{1j}(\zeta) + sv_{2j}(\zeta). \tag{2.1}$$

The ratios u_{1j}/u_{2j} are local solutions of (1.1) (see (1.6))

Linear independent local solutions of (1.8) are proved to be suitable only near the points $\zeta = a_k$, $k = 1, 2, \dots, m+1$.

The equation (1.8) can be written in the form of a system

$$\chi'(\zeta) = \chi(\zeta)\mathcal{P}(\zeta), \tag{2.2}$$

where

$$\chi(\zeta) = \begin{pmatrix} u_{1j}(\zeta), & u'_{1j}(\zeta) \\ u_{2j}(\zeta), & u'_{2j}(\zeta) \end{pmatrix}, \mathcal{P}(\zeta) = \begin{pmatrix} 0, & -q(\zeta) \\ 1, & -p(\zeta) \end{pmatrix}, \tag{2.3}$$

$$\chi'(\zeta) = \frac{d}{d\zeta}\chi(\zeta), u'_{kj}(\zeta) = \frac{d}{d\zeta}u_{kj}(\zeta), \tag{2.4}$$

and $u_1(\zeta)$, $u_2(\zeta)$ are the linear independent solutions of (1.8).

Note that since the coefficients of (1.1) and (1.8) are real, it becomes obvious that if $w(\zeta)$ and $u_{kj}(\zeta)$, $k = 1, 2$, are solutions of (1.1) and (1.8), respectively then $\overline{w}(\overline{\zeta})$ and $\overline{u}_{kj}(\overline{\zeta})$ are likewise the solutions of (1.1) and (1.8) respectively.

In [26] we proved the basic

Theorem 2.1. *If $w(\zeta) = u_1(\zeta)/u_2(\zeta)$, where $u_1(\zeta)$ and $u_2(\zeta)$ are linearly independent solutions of* (1.8)*, then the linear boundary condition* (1.4) *is* *equivalent to the conditions* [19, 20]

$$u_1(t) = \lambda[B(t)\overline{u}_1(t) - iD(t)\overline{u}_2(t)], \quad -\infty < t < +\infty, \tag{2.5}$$

$$u_2(t) = \lambda[iA(t)\overline{u}_1(t) + \overline{B}(t)\overline{u}_2(t)], \quad -\infty < t < +\infty, \tag{2.6}$$

where $\lambda = \lambda(t)$ *takes on the intervals* a_j, a_{j+1} *constant values equal to* $+1$ *or* -1*;* $u_k(\zeta)$*,* $\overline{u}_k(\zeta)$ *are complex conjugates.*

Proof. Assume $\lambda = \lambda(t)$. We rewrite (2.5) and (2.6) as

$$u_1(t) = \lambda(t)u_1^*(t), \quad u_2(t) = \lambda(t)u_2^*(t), \quad -\infty < t < +\infty, \tag{2.7}$$

where

$$u_1^*(t) = B(t)\overline{u}_1(t) - iD(t)\overline{u}_2(t), \tag{2.8}$$

$$u_2^*(t) = iA(t)\overline{u}_1(t) + \overline{B}(t)\overline{u}_2(t) \tag{2.9}$$

are linearly independent solutions of (1.8).

Substituting (2.7) in (1.8), we obtain

$$\lambda''(t)u_1^*(t) + \lambda'(t)[2(u_1^*(t))' + p(t)u_1^*(t)] = 0, \quad -\infty < t < +\infty, \tag{2.10}$$

$$\lambda''(t)u_2^*(t) + \lambda'(t)[2(u_2^*(t))' + p(t)u_2^*(t)] = 0, \quad -\infty < t < +\infty, \tag{2.11}$$

Multiplying (2.10) by $u_2^*(t)$ and (2.11) by $u_1^*(t)$ and then subtracting the first equality from the second one, we get

$$2\lambda'(t)\big[[u_1^*(t)]'u_2^*(t) - [u_2^*(t)]'u_1^*(t)\big] = 0, \tag{2.12}$$

The braces in (2.12) involve the Wronskian $w[u_1^*(t), u_2^*(t)] \neq 0$ for all ζ, with the exception of $\zeta = a_k$, $k = 1, 2, \ldots, m$. Hence (2.12) implies

$$\lambda(t) = \text{const}, \quad t \in (a_j, a_{j+1}), \quad j = 1, 2, \ldots, m. \tag{2.13}$$

From its side, (2.13) implies

$$\lambda'(t) = 0, \quad t \in (a_j, a_{j+1}), \quad j = 1, 2, \ldots, m. \tag{2.14}$$

If we calculate the Wronskian for (2.7) and take into account (2.14), we will obtain $\lambda^2 = 1$, and hence $\lambda = \pm 1$. □

In §9, we will show which of the intervals (a_j, a_{j+1}), $j = 1, 2, \ldots, m$ requires $\lambda = 1$ and which one $\lambda = -1$.

As for the matrix $\chi(\zeta)$ defined by (2.3), we can write the conditions (2.5) and (2.6) as:

$$\chi(t) = 6(t)\overline{\chi}(t), \quad -\infty < t < +\infty, \tag{2.15}$$

where

$$G(t) = \begin{pmatrix} B(t), & -iD(t) \\ iA(t), & \overline{B}(t) \end{pmatrix}, -\infty < t < +\infty, \tag{2.16}$$

is a given piecewise constant matrix; by (1.5) $\det G(t) = 1$, and $G(t)\overline{G(t)} = E$, where E is the unit matrix and $\overline{\chi}(t)$ is a matrix, complex conjugate to the matrix $\chi(t)$.

For the intervals of the axis $\zeta = t$, the matrix $G(t)$ can be defined as follows:

$$G(t) = G_j = \begin{pmatrix} B_j & -iD_j \\ iA_j & \overline{B}_j \end{pmatrix}, \quad a_j < t < a_{j+1}, \quad j = 1, 2, \dots, m+1, \quad (2.17)$$

where $a_{j+1} = a_{m+2} = a_1$ when $j = m+1$.

As it has been said above, without restriction of generality, we may assume that $G_m = E$. Due to this fact, we can extend the matrix $\chi(\zeta)$ analytically through the interval (a_m, a_{m+1}) to the lower half-plane, or vice versa.

The matrix $\chi(\zeta)$ defined by (2.3) is a solution of (2.2). Since $\det \chi(\zeta) \neq 0$ for all ζ with the exception of the points $\zeta = a_k$, $k = 1, 2, \dots, m+1$, we can see that $\chi(\zeta)$ is likewise a fundamental matrix [8]. It is also known that if the matrix $\chi(\zeta)$ is a solution of (2.2), then the matrix $C \cdot \chi(\zeta)$ is likewise a solution of (2.2), where C is a nonsingular constant matrix.

Below we will construct locally linearly independent solutions of (1.8), $V_{kj}(\zeta)$, $\varphi_{kj}(\zeta)$ respectively for the points $\zeta = a_j$, $j = 1, 2, \dots, m, m+1$, $\zeta = e_j = (a_j + a_{j+1})/2$, $j = 1, 2, \dots, m-1$, where $k = 1, 2$, and then by means of these solutions we will construct for (2.2) the corresponding locally fundamental matrices:

$$\theta_j(\zeta) = \begin{pmatrix} V_{1j}(\zeta) & V'_{1j}(\zeta) \\ V_{2j}(\zeta) & V'_{2j}(\zeta) \end{pmatrix}, \qquad H_j(\zeta) = \begin{pmatrix} \varphi_{1j}(\zeta) & \varphi'_{1j}(\zeta) \\ \varphi_{2j}(\zeta) & \varphi'_{2j}(\zeta) \end{pmatrix}, \quad (2.18)$$
$$j = 1, 2, 3, \dots, m, m+1, \qquad j = 1, 2, 3, \dots, m-1.$$

3. Local Solutions Near Singular Points, When the Difference of Characteristic Numbers is not an Integer

Equation (1.8) near $\zeta = a_j$ can be rewritten as

$$(\zeta - a_j)^2 V''(\zeta) + (\zeta - a_j) p_j(\zeta) V'(\zeta) + q_j(\zeta) V(\zeta) = 0, \quad (3.1)$$

where

$$p_j(\zeta) = \sum_{k=0}^{\infty} p_{kj} (\zeta - a_j)^k, \quad q_j(\zeta) = \sum_{k=0}^{\infty} q_{kj} (\zeta - a_j)^k. \quad (3.2)$$

For the point $\zeta = a_{m+1} = \infty$, by means of the transformation $\zeta = 1/x$, we can write the equation (1.8) as follows [1, 7, 13]:

$$x^2 V''(x) + x[2 - \sum_{k=0}^{\infty} p_k^{\infty} x^k] V'(x) + [\sum_{k=0}^{\infty} q_k^{\infty} x^k] V(x) = 0, \quad (3.3)$$

where

$$p(1/x) = x \sum_{k=0}^{\infty} p_k^{\infty} x^k, \quad q(1/x) x^2 \sum_{k=0}^{\infty} q_k^{\infty} x^k. \quad (3.4)$$

A solution of (3.1) respectively for the points $\zeta = a_i$, $\zeta = \infty$, $j = 1, 2, \dots, m$, is sought in the form [1, 7, 8, 12, 13]

$$V_j(\zeta) = (\zeta - a_j)^{\alpha_j}\widetilde{V}_j(\zeta), \quad \widetilde{V}_j(\zeta) = \sum_{n=0}^{\infty}\gamma_{nj}(\zeta - a_j)^n, \tag{3.5}$$

$$V_\infty(\zeta) = \zeta^{-\alpha_\infty}\widetilde{V}_\infty(\zeta), \quad \widetilde{V}_\infty(\zeta) = \sum_{n=0}^{\infty}\gamma_{n\infty}(\zeta)^{-n}. \tag{3.6}$$

Theorem 3.1. *If near the point* $t = a_j$ *the equation* (3.1) *has a solution of the type* (3.5)*, then after its substitution in* (3.1) *the equality*

$$(\zeta - a_i)^{\alpha_j}\left[\sum_{k=0}^{\infty}M_{kj}(\zeta - a_j)^k\right] = 0 \tag{3.7}$$

should identically be fulfilled.

From this equality we obtain an infinite recursion system of equations for determination of γ_{nj}, $n = 1, 2, \dots$.

$$M_{0j}(\alpha_j) = \gamma_{0j}f_{0j}(\alpha_j), \quad f_{0j}(\alpha_j) = \alpha_j(\alpha_j - 1) + \alpha_j p_{0j} + q_{0j} = 0, \tag{3.8}$$

$$M_{1j}(\alpha_j) = \gamma_{1j}(\alpha_j)\cdot f_{0j}(\alpha_j + 1) + \gamma_{0j}f_{1j}(\alpha_j) = 0, \tag{3.9}$$

$$\begin{gathered} M_{2j}(\alpha_i) = \gamma_{2j}(\alpha_j)f_{0j}(\alpha_j + 2) + \\ \gamma_{1j}(\alpha_j)f_{1j}(\alpha_j + 1) + \gamma_{0j}f_{2j}(\alpha_j) = 0, \end{gathered} \tag{3.10}$$

....

$$\begin{aligned} M_{nj}(\alpha_j) = \gamma_{nj}(\alpha_j)f_{0j}(\alpha_j + n) + \gamma_{(n-1)j}(\alpha_i)f_{1j}(\alpha_j + n - 1) + \cdots + \\ +\gamma_{[n-(k-2)]j}(\alpha_j)f_{(k-2)j}(\alpha_j + n - k + 2) + \cdots + \\ +\gamma_{1j}(\alpha_j)f_{(n-1)j}(\alpha_j + 1) + \gamma_{0j}f_{nj}(\alpha_j) = 0, \end{aligned} \tag{3.11}$$

....

$$f_{kj}(\alpha_j) = \alpha_j p_{kj} + q_{kj} \tag{3.12}$$

Theorem 3.2. *If for the point* $\zeta = a_j$ *the determining equation* (3.8) *has the roots* α_{1j}, α_{2j} $(\alpha_{1j} > \alpha_{2j})$ *such that* $\alpha_{1j} - \alpha_{2j} \neq n$, $n = 0, 1, 2$, *then for the equation* (3.1) *we construct by formulas* (3.9)–(3.11)*, two local linearly independent solutions of the type*

$$\begin{gathered} V_{kj}(\zeta) = (\zeta - a_j)^{\alpha_{kj}}\gamma_{0j}\widetilde{V}_{kj}(\zeta), \\ \widetilde{V}_{kj}(\zeta) = 1 + \sum_{n=1}^{\infty}\gamma_{nj}^{k}(\zeta - a_j)^n, \quad k = 1, 2. \end{gathered} \tag{3.13}$$

In a complete analogy with the above theorem, we can formulate and prove the theorem for the point $\zeta = a_{m+1} = \infty$ [1, 7–13].

The convergence radius of the series $\widetilde{V}_{kj}(\zeta)$ is bounded by the distance from the point $\zeta = a_j$ to the nearest of the points $\zeta = a_{j-1}$, $\zeta = a_{j+1}$ [1, 7,8].

The coefficient $\gamma_{0j} \neq 0$ will be defined below.

4. Construction of the Second Solution by Means of the Frobenius Method, When the Difference of Characteristic Numbers is Equal to an Integer

As is known, when $\alpha_{1j} - \alpha_{2j} = n$, $n = 0, 1, 2$, using the formulas (3.9)–(3.11), one can construct at the point $\zeta = a_j$ only one solution $V_{1j}(\zeta)$ corresponding to the root $\alpha_j = \alpha_{1j}$.

In such cases, there exist two methods to construction the second solution $V_{2j}(\zeta)$: the Frobenius method and the method of lowering the order of the equation (1.8).

By the Frobenius method, $V_{2j}(\zeta)$ is sought as follows [8].

Consider the case where $\alpha_{1j} - \alpha_{2j} = 0$. In this case, for the point $\zeta = a_j$ we seek for the second solution of (3.1). First we differentiate (3.5) with respect to α_j and then calculate the limit $\alpha_j \to \alpha_{2j}$ and obtain $V_{2j}(\zeta)$. Thus we have

$$\begin{aligned} V_{2j}(\zeta) = V_{1j}(\zeta)\ln(\zeta - a_j) + (\zeta - a_j)^{\alpha_{2j}}\gamma_{0j}\times \\ \times \sum_{n=0}^{\infty}\Big\{\frac{d}{d\alpha_j}\gamma^2_{nj}(\alpha_j)\Big\}_{\alpha_j=\alpha_{2j}} \times (\zeta - a_j)^n. \end{aligned} \tag{4.1}$$

Consequently, the following theorem is valid.

Theorem 4.1. *If for the point $\zeta = a_j$ the determining equation* (3.8) *has the roots such that $\alpha_{1j} - \alpha_{2j} = 0$ (at the point $w = b_j$, the two neighboring arcs are tangent, $\nu_j = 0$), then for the point $\zeta = a_j$ there exists the second solution $V_{2j}(\zeta)$ of the form* (4.1).

If for the point $\zeta = a_j$ the roots of (3.8) satisfy the condition $\alpha_{1j} - \alpha_{2j} = s$, $s \in \{1, 2\}$, then the second linearly independent solution of (3.1) is sought in the form [8]

$$V_j(\zeta, \alpha) = \gamma_{0j}(\zeta - a_j)\alpha_j\Big[\alpha_j - \alpha_{2j} + \sum_{n=1}^{\infty}\gamma_{nj}(\alpha_j)(\zeta - a_j)^n\Big]. \tag{4.2}$$

Substituting (4.2) in (3.1), we obtain for determination of $\gamma^2_n(\alpha_j)$, $n = 1, 2, \ldots$, a recursion system of equations. This system can also be obtained from (3.8)-(3.11), if instead of γ^2_{0j} $(\alpha_j - \alpha_{2j})$ we substitute γ^2_{nj} (α_j), $n = 1, 2, \ldots$. From this system we determine γ^2_{nj} (α_j), $n = 1, 2, \ldots$, and substitute them in (4.2). Then we differentiate (4.2) with respect to α_j and finally calculate the limits as $\alpha_j \to \alpha_{2j}$. As a result, we get the solution $V_{2j}(\zeta)$,

$$V_{2j}(\zeta) = \lim_{\alpha_j - \alpha_{2j}} \gamma_{0j} \Big\{ (\zeta - a_j)^{\alpha_j} \big[\alpha_j - \alpha_{2j} + \sum_{n=1}^{\infty} \gamma_{nj}(\alpha_j)(\zeta - a_j)^n \big] \times$$
$$\times \ln(\zeta - a_j) + (\zeta - a_j)^{\alpha_j} \Big[1 + \sum_{n=1}^{\infty} \frac{d}{d\alpha_j} [\gamma_{nj}^2(\alpha_j)](\zeta - a_j)^n \Big] \Big\} \qquad (4.3)$$

Reasoning as above, we have proved the following

Theorem 4.2. *If for the point* $\zeta = a_j$ *the equation* (3.8) *has the roots such that* $\alpha_{1j} - \alpha_{2j} = s$, $s = \{1, 2\}$ (*two neighboring circular arcs are tangent and* $\nu_j = 1$ *and* $\nu_j = 2$), *then for the point* $\zeta = a_j$ *the second linearly independent solution of* (3.1) *is of the form* (4.3).

5. Conditions for the Absence of the Logarithmic Term in the Solution $V_{vj}(\zeta)$

The boundary l of the domain $s(w)$ may contain circular or rectilinear cuts of $s(w)$. For the cut end $w = b_j$, equation (3.8) possesses the roots such that $\alpha_{1j} - \alpha_{2j} = 2$. For the points $\zeta = a_j$, P. Ya. Polubarinova–Kochina has proved that solutions $V_{2j}(\zeta)$ contain no logarithmic terms. Moreover, for these points she has obtained the equation connecting the parameters a_j, c_j, ν of some circular polygons.

Below, using the method different from that used in [15], we derive for the end of the cut of the angle 2π an equation connecting parameters a_j, c_j, ν_j for any circular polygons and then prove that the second solution $V_{2j}(\zeta)$ constructed for this end should not contain a logarithmic term.

Denoting the first summand in formula (4.3) by $V_{2j}^1(\zeta)$, we have

$$V_{2j}^1(\zeta) = \gamma_{0j}(\zeta - a_j)^{\alpha_j} \times$$
$$\times \Big[\alpha_j - \alpha_{2j} + \sum_{k=1}^{\infty} \gamma_{nj}^2(\alpha_j)(\zeta - a_j)^n \Big] \ln(\zeta - a_j). \qquad (5.1)$$

For determination of the coefficients $\gamma_{nj}^2(\alpha_{2j})$, we need the formulas (3.9)–(3.12) in which we replace γ_{0j} by γ_{0j} $(\alpha_j - \alpha_{2j})$. Having defined $\gamma_{nj}(\alpha_j)$ and passing to the limit in $\gamma_{nj}(\alpha_{2j})$, as $\alpha_j \to \alpha_{2j}$, we obtain from (5.1) the equality

$$v_{2j}^{1*}(\zeta) = \lim_{\alpha_j \to \alpha_{2j}} V_{2j}^1(\zeta) = \gamma_{2j}^2(\alpha_{2j}) \cdot V_{1j}(\zeta) \ln(\zeta - a_j), \qquad (5.2)$$

where $v_{1j}(\zeta)$ is the solution of (3.1) for $\alpha_j = \alpha_{1j}$.

Now we prove

Theorem 5.1. *A necessary and sufficient condition for the absence of a logarithmic term in the solution $v_{2j(\zeta)}$ constructed for the cut end is of the form*

$$\gamma_{2j}^2(\alpha_{2j}) = \frac{\gamma_{0j}}{2}\times$$
$$\times\{-f_{1j}(\alpha_{2j})\cdot f_{1j}(\alpha_{2j}+1)/f_{0j}(\alpha_{2j+1}) + f_{2j}(\alpha_{2j})\} = 0, \tag{5.3}$$

where $f_{kj}(\alpha)$, $k=0,1,2$, are defined by (3.8) *and* (3.12).

Proof. Let us prove the sufficiency of (5.3). From (5.2) it is obvious that if (5.3) holds, then $v_{2j}^{1*}(\zeta)=0$ which proves the sufficiency of the condition (5.3).

Let us prove now the necessity of the condition (5.3). As far as the equation (3.1) for the cut end $\zeta = a_j$ must have two locally independent solutions containing no logarithmic terms, we take this fact into account and construct the solution $v_{2j}(\zeta)$ by using the formulas (3.9)–(3.11) for, only the solutions of (3.1) constructed by (3.9)–(3.12) contain no logarithmic terms.

Really, all γ_{nj}^2, $n=1,3,4,\dots$, with the exception of $\gamma_{2j}^2(\alpha_{2j})$, are defined from the system (3.9)–(3.11). For definition of γ_{2j}^2 we have equation (3.10) in which the first term $\gamma_{2j}^2(\alpha_j)f_{0j}(\alpha_j+2)=0$ for $\alpha_j=\alpha_{2j}$. Hence the sum of the last two summands in (3.10) must vanish,

$$\gamma_{1j}^2(\alpha_{2j})f_{1j}(\alpha_{2j}+1) + \gamma_{0j}f_{2j}(\alpha_{2j}) = 0; \tag{5.4}$$

moreover, the equation (5.4) coincides with (5.3) if we substitute in it $\gamma_{1j}^2(\alpha_{2j})$ defined by (3.9).

From (5.4), we have

$$q_{2j} + q_{1j}^2 + q_{1j}p_{1j} = 0, \tag{5.5}$$

where q_{2j}, q_{1j}, p_{1j} are defined from the corresponding coefficients of (3.2).

Finally, define $\gamma_{2j}^2(\alpha_{2j})$ uniquely. To this end, from (3.10) we define $\gamma_{2j}(\alpha_j)$ for $\alpha_j \neq \alpha_{2j}$. We have

$$\gamma_{2j}(\alpha_j) = -\frac{\gamma_{1j}(\alpha_j)f_{1j}(\alpha_j+1) + \gamma_{0j}f_{2j}(\alpha_j)}{f_{0j}(\alpha_j+2)} \tag{5.6}$$

□

For $\alpha_j=\alpha_{2j}$, the numerator and the denominator in (5.6) vanish. Thus we have indeterminacy 0/0. If we develop it by means of the de L'Hospital rule, we will arrive at

$$\gamma_{2j}^{2*}(\alpha_{2j}) = -0,5\gamma_{0j}[p_{1j}(p_{1j}+2q_{1j}) + p_{2j}]. \tag{5.7}$$

Thus by formulas (3.9)–(3.11), we define $v_{2j}(\zeta)$ uniquely and complete the proof of the necessity of the condition (5.3).

For the cut end $\zeta=a_j$, one can construct $v_{2j}(\zeta)$ by means of the Frobenius method under the condition (5.3). Indeed, if the condition (5.3) is fulfilled, then the first summand in (4.3) vanishes, while the second one takes the form

$$V_{2j}(\zeta) = (\zeta - a_j)^{\alpha_{2j}} \gamma_{0j} \left[1 + \sum_{n=1}^{\infty} \gamma_{nj}^{2*} (\zeta - a_j)^n \right], \tag{5.8}$$

where all the coefficients γ_{nj}^{2*}, $n = 1, 2, \ldots$, are defined by

$$\lim_{\alpha_j \to \alpha_{2j}} \frac{d}{d\alpha_j} [\gamma_{nj}(\alpha_j)] = \gamma_{nj}^{2*} \quad n = 1, 2, 3, \ldots. \tag{5.9}$$

Among them γ_{2j}^{2*} is defined by

$$\gamma_{2j}^{2*} = -0,5[p_{1j}(p_{1j} + 2q_{1j}) + p_{2j}], \tag{5.10}$$

which coincides with (5.7) since γ_{0j} in (5.8) is a factor standing out of brackets.

6. Searching for the Second Solution $v_{2j}(\zeta)$ by the Method of Lowering the Order of (1.8) when $\alpha_{1j} - \alpha_{2j} = s$, $s = 0, 1, 2$

There naturally arises the question whether there is a more simple way of constructing $v_{2j}(\zeta)$ than that indicated by Frobenius. They may say that there is a second method, that is the method of lowering the order of equation (1.8) [7, 9, 10, 11, 12].

Using this method, one can get the well-known Liouville formula which in turn results in the following expression for $v_{2j}(\zeta)$:

$$v_{2j}(\zeta) = A_{0j} v_{1j}(\zeta) \ln(\zeta - a_j) + v_{2j}^2(\zeta), \tag{6.1}$$

where $v_{1j}(\zeta)$ is the solution corresponding to the root α_{1j}, A_{0j} is an unknown constant, and $v_{2j}^2(\zeta)$ for the case $\alpha_{1j} - \alpha_{2j} = 0$ takes the form

$$v_{2j}^2(\zeta) = (\zeta - a_j)^{\alpha_{2j}} \gamma_{0j} \sum_{n=1}^{\infty} h_{nj}(t - a_j)^n, \quad h_{1j} = 1. \tag{6.2}$$

For the cases $\alpha_{1j} - \alpha_{2j} = s$, $s = 1, 2$, the solution $v_{2j}^2(\zeta)$ is defined as follows:

$$v_{2j}^2(\zeta) = (\zeta - a_j)^{\alpha_{2j}} \gamma_{0j} \sum_{n=0}^{\infty} h_{nj}(\zeta - a_j)^n, \quad h_{0j} = 1, \tag{6.3}$$

where the coefficients h_{nj} $n = 1, 2, \ldots$, can be defined theoretically by the Liouville formula. but practically they cannot be defined in such a way.

Some well-known authors [9, 10, 12] recommend to substitute (6.1) in (3.1) and to obtain the recursion formulas which no longer has those defects we spoke about. Unfortunately, these statements are not true for $\alpha_{1j} - \alpha_{2j} = s$, $s = 1, 2$. Such an approach leaves again the coefficients h_{1j}, h_{2j} for $f_{0j}(\alpha_{2j} + s)$, where $f_{0j}(\alpha_{2j} + s) = 0$, $s = 1, 2$, undefined.

Indeed, the substitution of (6.1) in (3.1) results in

$$(\zeta-a_j)^{\alpha_{1j}-\alpha_{2j}}A_j\big\{2\widetilde{v}'_{1j}(\zeta)+\widetilde{v}_{1j}(\zeta)(p_{1j}(\zeta)-1)\big\}+ \\ +\big\{(\widetilde{v}^2_{2j}(\zeta))''+p_{1j}(\zeta)(\widetilde{v}^2_{2j}(\zeta))'+q_{1j}(\zeta)\widetilde{v}^2_{2j}(\zeta)\big\}=0, \tag{6.4}$$

where

$$v_{1j}(\zeta)=\gamma_{0j}(\zeta-a_j)^{\alpha_{1j}}\widetilde{v}_{1j}(\zeta),\quad \widetilde{v}_{1j}(\zeta)=1+\sum_{n=1}^{\infty}\gamma^1_{nj}(\zeta-a_j)^n, \tag{6.5}$$

$$v'_{1j}(\zeta)=\gamma_{0j}(\zeta-a_j)^{\alpha_{1j}-1}\widetilde{v}^1_{1j}(\zeta)$$

$$\widetilde{v}^1_{1j}(\zeta)=\alpha_{1j}+\sum_{n=1}^{\infty}\gamma^1_{nj}(\alpha_{1j}+n)(\zeta-a_j)^n. \tag{6.6}$$

Formulas for $\widetilde{v}^2_{2j}(\zeta)$, $(\widetilde{v}^2_{2j}(\zeta))'$, $(\widetilde{v}^2_{2j}(\zeta))''$ are defined similarly.

After the substitution of $\widetilde{v}_{kj}(\zeta)$, $k=1,2$, in (6.4), we obtain

$$\sum_{k=0}^{\infty}Q_{kj}(\zeta-a_j)^n=0, \tag{6.7}$$

The equation (6.7) implies

$$Q_{kj}=A_{0j}l_{(k-s)j}+M_{kj}=0. \tag{6.8}$$

For $k=0$, we have

$$Q_{0j}=A_{0j}l_{(0-s)j}+M_{0j}=0,\quad s=0,1,2; \tag{6.9}$$

moreover,

$$l_{(k-s)j}=0,\quad k-s<0.$$

The coefficients M_{kj}, $k=0,1,2,\dots$, can be defined by the formulas (3.8)–(3.11), while coefficients $l_{(k-s)j}$ are defined by

$$l_{0j}=2\alpha_{1j}+p_{0j}-1=\alpha_{1j}-\alpha_{2j}, \tag{6.10}$$

$$l_{1j}=\gamma^1_{1j}[2(\alpha_{1j}+1)+p_{0j}-1]+p_{1j}, \tag{6.11}$$

$$l_{2j}=\gamma^1_{2j}[2(\alpha_{1j}+2)+\alpha_{1j}(p_{0j}-1)]+\gamma^1_{1j}p_{1j}+p_{2j}, \tag{6.12}$$

$$\dots\dots\dots\dots\dots\dots$$

$$l_{nj}=\gamma^1_{nj}[2(\alpha_{1j}+n)+\alpha_{1j}(p_{0j}-1)]+\gamma^1_{(n-1)j}\alpha_{2j}p_{nj}+\cdots+ \\ +\gamma^1_{2j}\alpha_{1j}p_{(n-2)j}+\gamma^1_{1j}\alpha_{1j}p_{(n-1)j}+p_{nj}, \tag{6.13}$$

$$\dots\dots\dots\dots\dots\dots$$

According to (6.8), in order to define the parameter A_{0j} for the cases $s=1$ and $s=2$, respectively, we have the following equations:

$$A_{0j}+h_{1j}f_{0j}(\alpha_{2j}+1)+f_{1j}(\alpha_{2j})=0 \tag{6.14}$$

$$2A_{0j}+h_{2j}f_{0j}(\alpha_{2j}+1)+h_{1j}\cdot f_{1j}(\alpha_{2j}+1)+f_{2j}(\alpha_{2j})=0. \tag{6.15}$$

From (6.14) and (6.15) we can see that the recursion formulas (6.8) do not permit one to define $v_{2j}(\zeta)$ in the cases $\alpha_{1j} - \alpha_{2j} = s$, $s = 1, 2$. Hence it remains to use the Frobenius method. But one can act differently: first calculate the coefficients h_{sj}, $s = 1, 2$, by the Frobenius method and then the rest coefficients h_{nj}, $n \geq 3$, by the formula (6.8). The parameter A_{0j} can be defined as

$$A_{0j} = -f_{1j}(\alpha_{1j}), \quad s = 1. \tag{6.16}$$

$$A_{0j} = -h_{1j} f_{0j}(\alpha_{2j} + 1) - f_{2j}(\alpha_{2j}), \quad s = 2. \tag{6.17}$$

If we use the above-indicated method, then in the solution $v_{1j}(\zeta)$ instead of γ_{0j} we have to take $\gamma_{0j} A_{0j}$ and instead of $v_{2j}(\zeta)$ (formula (6.1)) the formula

$$v_{2j}(\zeta) = v_{1j}(\zeta) \ln(\zeta - a_j) + \gamma_{0j} v_{2j}^2(\zeta). \tag{6.18}$$

7. Local Matrices

For multi-valued functions $\exp[\alpha_{kj} \ln(\zeta - a_j)]$ encountered in local solutions, we select single-valued branches such as

$$\exp[\alpha_{kj} \ln(t - a_j)] > 0, \quad t > a_j;$$

$$\exp[\alpha_{kj} \ln(t - a_j)]^{\pm} = \exp[\pm i\pi\alpha_{kj}] \exp[\alpha_{kj} \ln(a_j - t)], \quad t < a_j;$$

$$\exp[-\alpha_{k\infty} \ln(-t)]\big]^{\pm} > 0, \quad -\infty < t < a_1;$$

$$\big[\exp[-\alpha_{k\infty} \ln t]\big]^{\pm} = \exp[\pm i\pi(-\alpha_{k\infty})] \exp[-\alpha_{k\infty} \ln t]. \quad a_m < t < +\infty.$$

Besides the matrix (2.18), we introduce the matrices

$$\theta_j^*(t) = \begin{pmatrix} v_{1j}^*(t), & v_{1j}^{'*}(t) \\ v_{2j}^*(t), & v_{2j}^{'*}(t) \end{pmatrix}, \quad a_{j-1} < t < a_j, \tag{7.1}$$

where

$$v_{kj}^*(t) = (a_j - t)^{\alpha_{kj}} \gamma_{0j} \widetilde{v}_{kj}(t), \tag{7.2}$$

$$v_{kj}^{'*}(t) = -(a_j - t)^{\alpha_{kj}} \gamma_{0j} \widetilde{v}^{1*}(t) \tag{7.3}$$

$$v_{kj}'(t) = d[u_{kj}(t)]/dt,$$

$$\widetilde{v}_{kj}^{1*}(t) = \alpha_{kj} + \sum_{n=1}^{\infty} \gamma_{nj}^k (\alpha_{kj} + n)(t - a_j)^n,$$

Between the matrices $\theta_j(t)$ and $\theta_j^*(t)$, there is a relation

$$\theta_j^{\pm}(t) = \vartheta_j^{\pm} \theta_j^*(t), \quad a_{j-1} < t < a_j, \tag{7.4}$$

$$\theta_{\infty}^{\pm}(t) = \vartheta_{\infty}^{\pm} \theta_{\infty}^*(t), \quad a_m < t < \infty \tag{7.5}$$

Matrices $\vartheta_j^{\pm}$ for $\alpha_{1j}-\alpha_{2j}\neq s$, $s=0,1,2$, are defined by

$$\vartheta_j^{\pm}=\begin{pmatrix}\exp(\pm i\pi\alpha_{1j}) & 0\\ 0 & \exp(\pm i\pi\alpha_{2j})\end{pmatrix}. \tag{7.6}$$

For $\alpha_{1j}-\alpha_{2j}=s$, $s=0,1,2$, they are defined by the equality

$$\vartheta_j^{\pm}=e^{\pm i\pi\alpha_{2j}}\begin{pmatrix}1 & 0\\ \pm\pi i & 1\end{pmatrix}. \tag{7.7}$$

Matrices $\vartheta_j^{\pm}$ for the cut end $w=b_j$ are defined as follows: if the use is made of the equation (1.7), then the characteristic numbers can be defined as $\alpha_{1j}=3/2$ and $\alpha_{2j}=-1/2$. To this case there correspond matrices $\vartheta_j^{\pm}=\mp iE$; however if we use the equation (1.8), then characteristic numbers are defined as $\alpha_{1j}=2$, $\alpha_{2j}=0$ with the corresponding matrices $\vartheta_j^{\pm}=E$.

The elements of the matrix $\theta_j^*(t)$ involving logarithmic terms are defined by the formulas

$$v_{2j}^*(t)=\gamma_{0j}\big\{(a_j-t)^{\alpha_{2j}}\big[(t-a_j)^s\widetilde{v}_{1j}(t)\ln(t-a_j)+\widetilde{v}_{2j}^2(t)\big]\big\}, \tag{7.8}$$

$$v_{2j}^{'*}(t)=-\gamma_{0j}(a_j-t)^{\alpha_{2j}-1}\times$$
$$\times\big\{\big[(a_j-t)^s e^{i\pi s}\widetilde{v}_{1j}^1(t)\ln(a_j-t)+\widetilde{v}_{1j}(t)\big]+\widetilde{v}_{2j}^2(t)\big\}, \tag{7.9}$$

In the local solutions $v_{kj}(\zeta)$ and $\varphi_{kj}(\zeta)$, there respectively appear the constants γ_{0j} and φ_{0j} defined with the help of the Liouville formula

$$\gamma_{0j}=\Big\{\prod_{k=1,k\neq j}^{m}|a_j-a_k|^{\beta_k}\Big\}^{1/2}, \tag{7.10}$$

$$\varphi_{0j}=\Big\{\prod_{k=1}^{m}|e_j-a_k|^{\beta_k}\Big\}^{1/2} \tag{7.11}$$

8. Construction of the Fundamental Matrix

Construct the matrix

$$\chi(\zeta)=\begin{pmatrix}u_1(\zeta) & u_1'(\zeta)\\ u_2(\zeta) & u_2'(\zeta)\end{pmatrix}, \tag{8.1}$$

where $u_1(\zeta)$ and $u_2(\zeta)$ are linearly independent solutions of (1.8); moreover, $u_1'(\zeta)=du_1(\zeta)/d\zeta$ and $u_2'(\zeta)=du_2(\zeta)/d\zeta$.

The domain of convergence of the matrices $\theta_j(t)$, $H_j(t)$ has always a general part in which we can write the equalities

$$\theta_j^*(t)=T^*H_j(t),\quad H_j(t)=T_{0j}\theta_{j-1}(t),\quad a_{j-1}<t<a_j, \tag{8.2}$$

$$\theta_1^*(t)=T_{-\infty}\theta_\infty(t),\quad -\infty<t<a_1,$$
$$\theta_\infty^*(t)=T_\infty\theta_m(t),\quad a_m<t<+\infty, \tag{8.3}$$

where T_j^*, T_{0j}, $T_{-\infty}$, T_∞ are the real constant matrices defined by the equalities (8.2) and (8.3); in this case, we have to fix t in the domain where the two local matrices converge.

Define the matrix (8.1) along the axis t of the plane ζ:

$$\chi^\pm(t) = T\theta_m^\pm(t), \quad \theta_m^+(t) = \theta_m^-(t), \quad a_m < t < +\infty \tag{8.4}$$

$$\chi^\pm(t) = T\vartheta_m^\pm \vartheta_m^*(t), \quad a_{m-1} < t < a_m; \tag{8.5}$$

$$\chi^\pm(t) = T\vartheta_m^\pm T_m \theta_{m-1}(t), \quad T_m = T_m^* \cdot T_{0m}, \quad a_{m-1} < t < a_m; \tag{8.6}$$

$$\chi^\pm(t) = T\vartheta_m^\pm T_m \vartheta_{m-1}^\pm \theta_{m-1}^*(t), \quad a_{m-2} < t < a_{m-1}; \tag{8.7}$$

....

$$\chi^\pm(t) = T\vartheta_m^\pm T_m \dots T_1 \vartheta_1^\pm \theta_1^*(t), \quad -\infty < t < a_1; \tag{8.8}$$

$$\chi^\pm(t) = T\vartheta_m^\pm T_m \dots \vartheta_1^\pm T_{-\infty} \theta_\infty(t), \quad -\infty < t < a_1; \tag{8.9}$$

$$\chi^\pm(t) = T\vartheta_m^\pm T_m \dots \vartheta_\infty^\pm T_\infty \vartheta_\infty^\pm(t), \quad a_m < t < \infty. \tag{8.10}$$

The upper signs $(\pm)$ in the matrices (8.4)–(8.10) denote the limiting values of the matrix $\chi(\zeta)$ from the upper and lower half-planes, respectively. The matrix T is defined by the equality

$$T = \begin{pmatrix} p & q \\ r & s \end{pmatrix}. \tag{8.11}$$

Obviously, the matrices (8.4)–(8.10) are the solutions of (2.2).

9. Solution of the Boundary Value Problem

Theorem 9.1. *The solution of the equation* (2.2) *satisfying the boundary condition* (2.15) *is given by formulas* (8.4)–(8.10).

Proof. We begin with the interval $(a_m, +\infty)$. We have

$$\begin{gathered} T\theta_m^+(t) = G_m T\theta_m^-(t), \quad \theta_m^+(t) = \theta_m^-(t), \\ G_m = E, \quad T = \overline{T}, \quad a_m < t < +\infty, \end{gathered} \tag{9.1}$$

For the interval (a_{m-1}, a_m), there takes place the equality

$$T\vartheta_m^+ \theta_m^*(t) = G_{m-1} T\vartheta_m^- \theta_m^*(t), \quad a_{m-1} < t < a_m, \tag{9.2}$$

The equalities (9.1) and (9.2) result in the matrix equation

$$(\vartheta_m^+)^2 = TG_m^{-1} G_{m-1} T \tag{9.3}$$

It is seen from (9.3) that the matrices $(\vartheta_m^+)^2$ and $G_m^{-1}G_{m-1}$ are similar.

In a fashion analogous to the matrix equation (9.3), we find the corresponding matrix equations for the remaining points $\zeta = a_j$, $j = 1, 2, \dots, m$,

$m+1$. We have

$$T\vartheta_m^+ T_m \vartheta_{m-1}^+ = G_{m-2} T\vartheta_m^- T_m \vartheta_{m-1}^-, \tag{9.4}$$

$$T\vartheta_m^+ T_m \vartheta_{m-1}^+ T_{m-1}\vartheta_{m-2}^+ = G_{m-3} T\vartheta_m^- T_m \vartheta_{m-1}^- T_{m-1}\theta_{m-2}^-, \tag{9.5}$$

....

$$T\vartheta_m^+ T_m \vartheta_{m-1}^+ T_{m-1}\vartheta_{m-2}^+ T_{m-2}\dots T_1\vartheta_1^+ =$$
$$= G_{m+1} T\vartheta_m^- T_m \vartheta_{m-1}^- T_{m-1}\theta_{m-2}^- T_{m-2}\dots T_1\vartheta_1^-, \tag{9.6}$$

$$T\vartheta_m^+ T_m \vartheta_{m-1}^+ T_{m-1}\dots T_{-\infty}\vartheta_\infty^+ =$$
$$= G_m T\vartheta_m^- T_m \vartheta_{m-1}^- T_{m-1}\dots T_{-\infty}\vartheta_\infty^-. \tag{9.7}$$

These equations can be written in terms of the equation (9.3), for example, the equation (9.4) can be written in the form

$$(\vartheta_{m-1}^+)^2 = T_m^{-1}(\vartheta_m^-)^{-1} T^{-1} G_{m-1}^{-1} G_{m-2} T\vartheta_m^- T_m.$$

As is said above, the matrices G_k can be defined first to within the factor $\lambda = \pm 1$, and then exactly. To define G_k exactly, we proceed from equation (3.8). Having defined χ_{kj}, it is necessary to construct the equation

$$\det(G_j^{-1} G_{j-1} - \lambda E) = 0. \tag{9.8}$$

Denote the roots of (9.8) by λ_{kj} and consider the equality

$$\alpha_{kj} = (2\pi i)^{-1} \ln \lambda_{kj} \tag{9.9}$$

The right-hand side of (9.9) is defined to within an integer summand. A suitable choice of $\lambda = \pm 1$ makes it always possible to fulfill the equation (9.9) and to define the matrices G_j, $j = 1, 2, \dots, m, m+1$, exactly. But this operation should be done successively beginning, for example, with the matrix G_{m-1}.

It should be noted at this point that two neighboring circular arcs forming a cut with the end $w = b_j$ (in particular, segments of straight lines) belong to the same circumference. This implies that $G(t) = G_j$ for $\zeta > a_j$ and $G(t) = \lambda G_j$ for $\zeta < a_j$, where $\lambda = \pm 1$. If the use is made of the equation (1.7), then the equation (3.8) has the roots $3/2$ and $-1/2$, but if we use the equation (1.8), then the equation (3.8) has the roots 2 and 0. In the first case $\lambda = -1$, while in the second one $\lambda = 1$.

We rewrite the matrix equation (9.3) as follows:

$$T\vartheta_m^+ = G_{m-1} T\vartheta_m^- \tag{9.10}$$

From (9.10), we have

$$p \exp(i\pi\alpha_{1m}) = B_{m-1} p \exp(-i\pi\alpha_{1m}) - iD_{m-1} r \exp(-i\pi\alpha_{1m}), \quad (9.11)$$

$$r \exp(i\pi\alpha_{1m}) = iA_{m-1} p \exp(-i\pi\alpha_{1m}) + \overline{B}_{m-1} r \exp(-i\pi\alpha_{1m}), \quad (9.12)$$

$$q \exp(i\pi\alpha_{2m}) = B_{m-1} q \exp(-i\pi\alpha_{2m}) - iD_{m-1} s \exp(-i\pi\alpha_{2m}), \quad (9.13)$$

$$s \exp(i\pi\alpha_{2m}) = iA_{m-1} q \exp(-i\pi\alpha_{2m}) + \overline{B}_{m-1} s \exp(-i\pi\alpha_{2m}). \quad (9.14)$$

If we divide the corresponding parts of (9.11) and (9.12), (9.13) and (9.14), then we can see that the ratios p/r and q/s on the interval (a_{m-1}, a_m) satisfy the boundary condition (1.4):

$$\frac{p}{r} = \frac{B_{m-1} p/r - iD_{m-1}}{iA_{m-1} p/r + \overline{B}_{m-1}}, \quad \frac{q}{s} = \frac{B_{m-1} q/s - iD_{m-1}}{iA_{m-1} q/s + \overline{B}_{m-1}}. \quad (9.15)$$

The same boundary condition is satisfied by the coordinates of the points $w = b_m$, $w = b'_m$. Hence

$$p/r = b_m, \quad q/s = b'_m. \quad (9.16)$$

On the plane w, the origin of coordinates coincides with the point b_m, therefore $b_m = 0$, $b'_m = \infty$, and hence

$$p = 0, \quad s = 0. \quad (9.17)$$

If the determining equation (3.8) has for the point $\zeta = a_m$ the roots such that $\alpha_{1j} - \alpha_{2j} \neq n$, $n = 0, 1, 2$, then we can define the matrix G_{m-1}:

$$G_{m-1} = \begin{pmatrix} B_{m-1} & 0 \\ 0 & \overline{B}_{m-1} \end{pmatrix} \quad (9.18)$$

Consider the matrix equation (9.4):

$$T_{*m} \vartheta^{+}_{m-1} = G_{m-2} \overline{T}_{*m} \vartheta^{-}_{m-1}, \quad T_{*m} = T \vartheta^{+}_{m} T_m. \quad (9.19)$$

Reasoning as above, from (9.19) we have the following system of equations:

$$p_{*m}/r_{*m} = b_{m-1}, \quad q_{*m}/s_{*m} = b'_{m-1}, \quad (9.20)$$

where p_{*m}, q_{*m}, r_{*m}, s_{*m} are the elements of the matrix T_{*m}.

The equalities (9.20) can be rewritten as

$$\frac{p_* p_m + q_* r_m}{r_* p_m + s_* r_m} = b_{m-1}, \quad \frac{p_* p_m + q_* s_m}{r_* q_m + s_* s_m} = b'_{m-1}, \quad (9.21)$$

where p_*, q_*, r_*, s_* are the elements of the matrix $T_* = T\vartheta^{+}_{m}$.

Taking (9.16) into account, we can rewrite (9.21) as

$$\frac{r_* p_m b_m + s_* r_m b'_m}{r_* p_m + s_* r_m} = b_{m-1}, \quad \frac{r_* q_m b_m + s_* s_m b'_m}{r_* q_m + s_* s_m} = b'_{m-1}. \quad (9.22)$$

We rewrite (9.22) as

$$r_*p_m(b_m - b_{m-1}) + s_*r_m(b'_m - b_{m-1}) = 0, \tag{9.23}$$

$$r_*q_m(b_m - b'_{m-1}) + s_*s_m(b'_m - b'_{m-1}) = 0. \tag{9.24}$$

The condition of compatibility of the system of equations (9.23) and (9.24) with respect to r_* and s_* has the form

$$\frac{p_m s_m}{r_m q_m} = \frac{b'_m - b_{m-1}}{b_m - b_{m-1}} \cdot \frac{b_m - b'_{m-1}}{b'_m - b'_{m-1}}. \tag{9.25}$$

Exactly in the same way as above, from the matrix equation (9.5) we obtain a system of equations:

$$\begin{aligned} \frac{p_{*(m-1)}p_{m-1} + q_{*(m-1)}r_{m-1}}{r_{*(m-1)}p_{m-1} + s_{*(m-1)}r_{m-1}} &= b_{m-2}, \\ \frac{p_{*(m-1)}q_{m-1} + q_{*(m-1)}s_{m-1}}{r_{*(m-1)}q_{m-1} + s_{*(m-1)}s_{m-1}} &= b'_{m-2} \end{aligned} \tag{9.26}$$

Taking into consideration (9.20), after certain transformations we rewrite (9.26) as:

$$r_{*(m-1)}p_{m-1}(b_{m-1} - b_{m-2}) + s_{*(m-1)}r_{m-1}(b'_{m-1} - b_{m-2}) = 0, \tag{9.27}$$

$$r_{*(m-1)}q_{m-1}(b_{m-1} - b'_{m-2}) + s_{*(m-1)}s_{m-1}(b'_{m-1} - b'_{m-2}) = 0. \tag{9.28}$$

The condition of compatibility of the system of equations (9.27) and (9.28) with respect to $r_{*(m-1)}$ and $s_{*(m-1)}$ is of the form

$$\frac{p_{m-1}s_{m-1}}{r_{m-1}q_{m-1}} = \frac{b'_{m-1} - b_{m-2}}{b_{m-1} - b_{m-2}} \cdot \frac{b_{m-1} - b'_{m-2}}{b'_{m-1} - b'_{m-2}}. \tag{9.29}$$

Reasoning analogously we can successively consider all matrix equations (9.6) and (9.7).

The equations (9.25) and (9.29) represent invariant cross-ratios of four points belonging to the same circumference at which the latter intersects two neighboring circumferences.

From the matrix equations (9.3)–(9.7), we get all needed equations with respect to a_k, c_k and to the integration parameters p, q, r, s, as well. For every point $\zeta = a_j$, the obtained system of two equations is homogeneous with respect to the elements of the matrix T_k. Its compatibility conditions, for example, for the points $\zeta = a_m$ and $\zeta = a_{m-1}$, are of the form (9.25) and (9.29). These equations have been obtained under the assumption $\alpha_{1j} - \alpha_{2j} \neq n$, $n = 0, 1, 2$.

Consider the case where $\alpha_{1j} - \alpha_{2j} = n$, $n = 0, 1, 2$.

Using the representation (8.4)–(8.10) for the interval (a_{j-1}, a_j), the unknown matrices $\chi^+(t)$, $\chi^-(t)$ must satisfy the boundary condition

$$\begin{pmatrix} p_{*j} & q_{*j} \\ r_{*j} & s_{*j} \end{pmatrix} e^{i\pi\alpha_{2j}} \begin{pmatrix} 1 & 0 \\ \pi i & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} B_{j-1} & -iD_{j-1} \\ iA_{j-1} & \overline{B}_{j-1} \end{pmatrix} \begin{pmatrix} \overline{p}_{*j} & \overline{q}_{*j} \\ \overline{r}_{*j} & \overline{s}_{*j} \end{pmatrix} e^{-i\pi\alpha_{2j}} \begin{pmatrix} 1 & 0 \\ -\pi i & 1 \end{pmatrix}, \tag{9.30}$$

where p_{*j}, q_{*j}, r_{*j}, s_{*j} are defined by (8.4)–(8.10).

Reasoning in the same way as in deducing (9.11)–(9.14), we can see that the ratios

$$\frac{p_{*j} + \pi i q_{*j}}{r_{*j} + \pi i s_{*j}}, \quad \frac{q_{*j}}{s_{*j}} \tag{9.31}$$

satisfy the boundary condition (2.15). The same condition will likewise be satisfied by the coordinates of the point $w = b_j$ as well as by those of the points b_{j-1} or b'_{j-1}. Thus we obtain the following system of equations:

$$\frac{p_{*j} + \pi i q_{*j}}{r_{*j} + \pi i s_{*j}} = b_j, \quad \frac{q_{*j}}{s_{*j}} = b_j^*, \tag{9.32}$$

where b_j^* are equal either to b_{j-1} or to b_{j-1}^*.

The system (9.32) is also homogeneous with respect to the elements of the corresponding matrices T_j whose compatibility conditions by this time does not provide the relations similar to (9.25)–(9.29).

As is said above, matrix equations similar to (9.3)–(9.7) can be obtained for all points, with the exception of the points $\zeta = a_k$. To these points there correspond the ends of the cuts $w = b_j$ for which $\nu_j = 2$. For such points we have conditions of the absence of logarithmic terms in the solutions $v_{2j}(\zeta)$, for example, the equation (5.5); the second equation will be given below.

From the matrix representations of $\chi^+(t)$ we first define $u_1^+(t)$, $u_2^+(t)$ and then compose the relation $w^+(t) = u_1^+(t)/u_2^+(t)$.

Suppose that the function $w^+(t)$ on the interval (a_k, a_{k+1}) is defined by

$$w^+(t) = [A_j^* v_{1j}^+(t) + B_j^* v_{2j}^+(t)]/[c_j^* v_{1j}(t) + D_j^* v_{2j}^+(t)], \tag{9.33}$$

Using the formula (9.33) and calculating the limit as $\zeta \to a_j$, we get the equation

$$b_j = B_j^*/D_j^*. \tag{9.34}$$

The corresponding equations for another points $\zeta = a_k$ can be obtained analogously.

Finally, for every point $t = a_j$ we obtain two real homogeneous equations with respect to p_j, q_j, r_j, s_j, for instance, the equations (9.11)–(9.14). From the conditions of compatibility of homogeneous equations for $\nu_j \neq 0, 1, 2$, we obtain invariant cross-ratios for four points of one and the same circle, for example, equations (9.25)–(9.29). In the case $\nu_j = 0, 1, 2$, the condition of compatibility of two equations provides certain condition rather than a cross-ratio.

Thus we can take from each system one equation and the compatibility condition, i.e, two equations for each point $\zeta = a_j$. The number of equations equals $2(m+1)$, and the number of unknown parameters a_k, c_k, p, q, r, s $(ps - rq = 1)$ will be $2m - 1$. Consequently, the number of equations is greater by three than the number of unknown parameters. This is connected with the fact that the bypass of all singular points a_k, $k = 1, 2, \dots, m$, is equivalent to going around the point $\zeta = \infty$. This yields one matrix equation. Therefore these three equations are consequences of the remaining ones. This means that if we find all a_k, c_k and p, q, r, s and substitute them in the remaining system of equations, then they will identically be equal to zero. The appearance of three superfluous equations can be explained exactly in the same way as in the case of linear polygons. □

Having found the system of equations for definition of a_k, c_k, p, q, r, s, we have to define the intervals of variation of the parameters c_k, $k = 1, 2, \dots, m$, then to solve the system with respect to a_k, c_k, $k = 1, 2, \dots, m$, and finally to specify p, q, r, s. Recall that p_j, q_j, r_j, s_j, $j = 1, 2, \dots, m+1$, depend implicitly on the parameters a_k, c_k, $k = 1, 2, \dots, m$.

Theorem 9.2. *If the contoure of the domain $s(w)$ of a circular polygon contains a cut with the end $w = b_j$ $\alpha_{1j} - \alpha_{2j} = 2$, then the second linearly independent solution $v_{2j}(\zeta)$ of* (3.1) *at the point $\zeta = a_j$ does not contain the logarithmic term.*

Proof. Suppose the contrary. Let $v_{2j(\zeta)}$ contain a logarithmic term. For the point $\zeta = a_j$, we construct first a local fundamental matrix $\theta_j(\zeta)$ and then the matrices $\chi^+(t) = B_{0j}\theta_j^+(t)$, $\chi^-(t) = \overline{B}_{0j}\overline{\theta}_j(t)$, where B_{0j}, $\overline{B}_{0j}$ are the constants of the matrix constructed by (8.4)–(8.10). The matrices $\chi^+(t)$, $\chi^-(t)$ must satisfy the boundary conditions

$$B_{0j}\theta_j^+(t) = G_j\overline{B}_{0j}\theta_j^-(t), \quad \theta_j^+(t) = \theta_j^-(t), \quad t > a_j, \tag{9.35}$$

$$B_{0j}\vartheta_j^+(t) = G_j\overline{B}_{0j}\vartheta_j^-\theta_j^*(t), \quad t < a_j. \tag{9.36}$$

The equalities (9.35) and (9.36) imply that

$$\vartheta_j^+ = \lambda\vartheta_j \quad \text{either} \quad \lambda = 1 \quad \text{or either} \quad \lambda = -1. \tag{9.37}$$

When $\alpha_{1j} = 3/2$, $\alpha_{2j} = -1/2$, and $\lambda = -1$, the equality (2.37) yields

$$i\begin{pmatrix} 1 & 0 \\ \pi i & 1 \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ -\pi i & 1 \end{pmatrix}. \tag{9.38}$$

It follows from (9.38) that $\pi = 0$, which is not true. In case $\alpha_{1j} = 2$, $\alpha_{2j} = 0$ and $\lambda = 1$, the equality (9.37) implies

$$\begin{pmatrix} 1 & 0 \\ \pi i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\pi i & 1 \end{pmatrix}. \tag{9.39}$$

It again follows from (9.39) that $\pi = 0$, which is not true. Hence our supposition is invalid and the theorem is complete. □

Theorem 9.2 has been proved in somewhat different way by P.Ya. Polubarinova-Kochina.

10. Representation of the Solutions $v_{kj}(\zeta)$, $j = \overline{1, m+1}$, by Means of Functional Series

It is known that the series $v_{kj}(\zeta)$, $k = 1, 2$, $j = 1, 2, \dots, m, m+1$ converge near the points $\zeta = a_j$, $j = 1, 2, \dots, m+1$, while the series $\varphi_{kj}(\zeta)$ converge near the points $\zeta = e_j = (a_j + a_{j+1})/2$. The radii of convergence of these series are bounded by the distance from the given point $t = a_j$ (or from the point $\zeta = e_j$) to the nearest points $\zeta = a_{j-1}$, $\zeta = a_{j+1}$.

The constructed series $v_{kj}(\zeta)$, $\varphi_{kj}(\zeta)$ converge slowly thereby making numerical calculations more complicated. As n increases, the coefficients γ^k_{nj} sometimes increase strongly, although their factor $(\zeta - a_j)^n$, on the contrary, strongly decrease as n increases. Electronic computers are unable to multiply γ^k_{nj} by $(t - a_j)^n$ despite the fact that these series converge. To remove this deficiency we suggest to represent these series as rapidly and uniformly convergent functional series.

Theorem 10.1. *If one considers the Fuchs class equation* (1.8)*, with* $p(\zeta)$*,* $q(\zeta)$ *defined by* (1.9) (*or by* (1.12))*, and represent it near the points* $\zeta = a_j$ *and* $\zeta = \infty$ *in terms of the series* (3.2) *and* (3.4)*, respectively, then the local solutions* $v_{kj}(\zeta)$*,* $j = 1, 2, \dots, m+1$*, can be represented as rapidly and uniformly convergent functional series, the formulas* (3.9)–(3.11) *remaining valid.*

Proof. Consider the structure of the recursion formulas (3.9)–(3.11). The sum of the first subscripts for the expression $\gamma_{(k-n)j} \cdot f_{nj}(\alpha_j + k - n)$ is always equal to k, that is, to the exponent $(t - a_j)^k$. Consider instead of the series (3.5) the functional series

$$v_j(t) = (t - a_j)^{\alpha_j} \widetilde{v}_j(t - a_j), \quad \widetilde{v}_j(t - a_j) = \sum_{n=0}^{\infty} \gamma_{nj}(t - a_j), \tag{10.1}$$

where, owing to (3.9)–(3.11), γ_{nj} is defined in terms of $\gamma_{1j}, \gamma_{2j}, \dots, \gamma_{(n-1)j}$, and the latters in terms of $f_{kj}(\alpha_j)$, where

$$f_{kj}(t-a_j, \alpha_j) = \alpha_j p_{kj}(t-a_j) + q_{kj}(t-a_j), \tag{10.2}$$

$$p_{nj}(t-a_j) = \sum_{k=1,k=j}^{m} (-1)^{n-1}(1-\nu_k)\Big(\frac{t-a_j}{a_j-a_k}\Big)^n,$$
$$p_{0j} = 1-\nu_j, \tag{10.3}$$

$$q_{nj}(t-a_j) = \sum_{k=1,k\neq j}^{m} (-1)^{n-2}\times$$
$$\times\{\sigma_k(n-1)+c_k(a_j-a_k)\}\Big(\frac{t-a_j}{a_j-a_k}\Big)^n \tag{10.4}$$
$$n = 2,3,\dots,$$

$$q_{0j} = \sigma_j, \quad q_{1j} = c_j \tag{10.5}$$

$$\Big|\frac{t-a_j}{a_j-a_k}\Big| < 1 \quad k \neq j, \tag{10.6}$$

$$|t-a_j| < M_{in}\{|a_j-a_{j-1}|,\ |a_j-a_{j+1}|\}. \tag{10.7}$$

It is seen from (10.6) that the functional series (10.1) converges uniformly near the point $\zeta = a_j$ and rapidly in comparison with the series (3.5).

The functional series for the point $\zeta = a_{m+1} = \infty$ can be constructed analogously.

In all subsequent formulas instead of the solution $v_{kj}(\zeta)$ we will represent the functional series (10.1).

Obviously, the functional series for regular points $t = e_j$, $e_j = (a_j + a_{j+1})/2$, $j = 1, 2, \dots, m-1$, converge likewise uniformly and rapidly. □

11. Determination of Intervals of Variation of Accessory Parameters

It was proved in [26] that $v_{kj}(\zeta)$, $k = 1,2$, $j = 1, 2, \dots, m+1$, were entire functions of the accessory parameters, c_k, $k = 1, 2, \dots, m$, and in [23] we determined possible intervals of variation of these parameters.

Consider two cases: 1. There is a circular polygon with the angles $\nu_j = 1$, $j = 1, 2, \dots, m+1$. We pass to that consisting of one circle. In this case, the equation (1.1) takes the form

$$w(\zeta) = (A\zeta + B)/(C\zeta + D), \tag{11.1}$$

where A, B, C, D are unknown integration constants of (1.1).

Substitution of (11.1) in (1.1) results in the identity

$$R(\zeta) = \sum_{k=1}^{m} \frac{C_k}{\zeta - a_j} = 0. \tag{11.2}$$

From (11.2) follows

$$C_k = 0, \quad k = 1, 2, \dots, m.$$

2. On the plane w, there is a linear polygon. The accessory parameters vanish for this case and the solution of (1.1) is given by the Christofel–Schwarz's formula

$$w(\zeta) = M \int_0^{\zeta} \prod_{j=1}^{m} (\zeta - a_j)^{\nu_j - 1} d\zeta + N. \tag{11.3}$$

Substituting (11.3) in (1.1), we get

$$C_j^* = -(\nu_j - 1) \sum \sum_{k=1, k \neq j}^{m} (\nu_k - 1)/(a_j - a_k) \tag{11.4}$$

It follows from this reasoning that

$$\text{either} \quad c_j^* \leq c_j \leq 0 \quad \text{or} \quad c_j^* \geq c_j \geq 0. \tag{11.5}$$

To the equation (1.8), there corresponds the following Schwarz's equation:

$$\frac{w''(\zeta)}{w'(\zeta)} - \frac{3}{2}\Big(\frac{w''(\zeta)}{w'(\zeta)}\Big)^2 = 2q(\zeta) - p'(\zeta) - 0,5[p(\zeta)]^2, \tag{11.6}$$

where $p(\zeta)$ and $q(\zeta)$ are defined by (1.9) or (1.14).

For the equation (11.6) we consider the same two cases as above.

1. For this case, we have

$$\alpha'' = 0, \quad c_k = 0. \tag{11.7}$$

Thus with respect to δ_k, $k = 1, 2, \dots, m-3$, we have obtained the following homogeneous system:

$$\delta_1 a_k^{m-2} + \cdots + \delta_{m-3} a_k^{m-3} + \delta_{m-2} = 0, \quad k = 1, 2, \dots, m. \tag{11.8}$$

The equation (11.8) implies

$$\delta_k = 0, \quad k = 1, 2, \dots, m-3. \tag{11.9}$$

2. In this case, we arrive at

$$\alpha' \neq 0, \quad \alpha'' \neq 0, \quad c_k = 0. \tag{11.10}$$

It follows from (11.10) that we get

$$\alpha' \alpha'' a_k^{m-2} + \delta_1 a_k^{m-3} + \cdots + \delta_{m-3} a_k + \delta_{m-2} = 0 \tag{11.11}$$

The system which this time is inhomogeneous with respect to δ_k, $k = 1, 2, \dots, m-2$ (11.11) is solved with respect to δ_k, $k = 1, 2, \dots, m-3$, hence in this case too one can determine possible intervals of variation of the accessory parameters.

12. Conclusion

Having known $w(\zeta)$ along the whole real axis t of the plane ζ, one can find $w = w(\zeta)$ for all $\Im(\zeta) > 0$ by the well-known formula [10, p. 152, formula (12.5.10)]

$$w(\zeta) = \frac{1}{\pi} \int_{-\infty}^{\infty} w^+(x) \frac{\tau dx}{(x-t)^2 + \tau^2}. \tag{12.1}$$

Along the whole real axis, $w = w^+(t)$ is defined by (8.1):

$$w^+(t) = u_1^+(t)/u_2^+(t), \quad -\infty < t < +\infty, \tag{12.2}$$

where $u_1^+(t)$, and $u_2^+(t)$, as linearly independent solutions of (1.8), are defined uniquely by (8.4)–(8.10).

As is seen from the above-said, an algorithm for constructing the single-valued analytic functions $w = w(\zeta)$ is given in a general form. These functions represent general solutions of (1.1) and map conformally the half-plane $\zeta = t + i\tau$ onto circular polygons with a finite number of vertices and any angles at those vertices. At those vertices the system of equations is composed which connects geometrical characteristics of circular polygons with unknown parameters of the Schwarz's equation. Rapidly and uniformly convergent functional series are also constructed.

Possible intervals of variation of the accessory parameters are defined. Consequently, the solution of (1.1) and the construction of $w = w(\zeta)$ are reduced, with regard for the boundary conditions (1.4), to the solution of a system of higher transcendental equations with respect to the parameters a_k, c_k, $k = 1, 2, \dots, m$.

References

1. E. L. Ince, Ordinary differential equations. (Translated from English) *Kharkov,* 1939.
2. I. A. Alexandrov, Parametric continuations in the theory of one-sheeted functions. *Nauka Publishing House, Moscow,* 1976.
3. G. Bateman and A. Erdelyi, Higher transcendental functions. *McGrow-Hill Book Company, New York-Toronto-London* 1955.
4. J. Bear, D. Zaslavsky, and S. Irmay, Physical and mathematical foundations of water filtration. (Translated from English) *Mir Publishing House, Moscow,* 1971.
5. A. Hurvitz and R. Courant, Vorlesungen über Allgemeine Funktionentherie und Elliptische Funktionen, Geometrische Funktionentheorie. *Springer-Verlag, Berlin-Göttingen-Heidelberg-New York,* 1964.
6. G. N. Goluzin, Geometrical theory of functions of a complex variable, 2nd ed. *Nauka, Moscow,* 1966.
7. V. V. Golubev, Lectures in analytic theory of differential equations, 2nd ed. *Moscow–Leningrad,* 1950.
8. E. A. Koddington and N. Levinson, Theory of ordinary differential equations. (Translated from English) *Moscow,* 1958.

9. G. Korn and T. Korn, Mathematical Handbook for Scientists and Engineers *New York-Toronto-London,* 1961.
10. V. Koppenfels and F. Stallmann, (Translated from German) *Springer-Verlag, Berlin-Gottingen-Heidelberg,* 1959.
11. E. Kamke, Differentialgleichungen. Lösungsmethoden and Lösungen. I Gewöhnliche Differentialglei chungen, Leipzig, 1959. 2nd Edition. 1961.
12. E. T. Whittaker and G. N. Watson, A course of Modern Analysis. *Cambridge University Press,* 1927.
13. J. Sansone, Equazioni Differentiali nel Campo Reale, Parte Prima. *Bologna,* 1948.
14. P. Ya. Polubarinova–Kochina, Application of the theory of linear differential equations to the problems of motion of ground water. (Russian) *Izv. Akad. Nauk SSSR, Ser. Mat.* **3**(1939), Nos. 5–6, 576–602.
15. P. Ya. Polubarinova–Kochina, Theory of a motion of ground water, 2nd ed. *Nauka, Moscow,* 1977.
16. P. Ya. Polubarinova–Kochina, On circular polygons in the theory of filtration. *Problems of Mathematics and Mechanics, Novosibirsk, Nauka,* 1983, 166–177.
17. P. Ya. Polubarinova–Kochina, On additional parameters on the examples of circular quadrangles. (Russian) *App. Math. Mech.* **55**(1991), No. 2, 222–227.
18. P. Ya. Polubarinova–Kochina, Analytic theory of linear differential equations in the theory of filtration. Mathematics and problems of water handling facilities. *Naukova Dumka, Kiev,* 1986, 19–36.
19. A. R. Tsitskishvili, on the construction of analytic functions mapping conformally a half-plane on circular polygons. *Differential Equations* **XXI**(1985), No. 4, 646–656.
20. A. R. Tsitskishvili, On conformal mapping of a half-plane onto circular polygons. *Trudy Tbiliss. Gos. Univ., Ser. Mat., Mekh., Astr.* **185**(1977), 65–89.
21. A. R. Tsitskishvili, on conformal mapping of a half-plane onto circular pentagons with cuts. *Differential Equations* **XII**(1976), No. 1, 2044–2051.
22. A. R. Tsitskishvili, The method of explicit solution of one class of plane problems of the theory of filtration. *Soobshch. Akad. Nauk Gruzii,* **142**(1991), No. 2, 285–288.
23. A. R. Tsitskishvili, Application of the theory of linear differential equations to the solution of some plane problems of the theory of filtration. *Trudy Tbiliss. Gos. Univ.* **259**(1986), Nos. 19–20, 295–329.
24. A. R. Tsitskishvili, On the filtration in trapezoidal plane earth dams. *Trudy Tbiliss. Gos. Univ.* **210**(1980), No. 8, 12–39.
25. A. R. Tsitskishvili, Application of the methods of complex analysis to the solution of one class of two-dimensional problems of the theory of filtration. *Boundary value problems of filtration of ground water. Thesises of reports of Republican scientific and technical seminar, Kazan Univ. Press,* 1988, 72–73.
26. A. R. Tsitskishvili, Application of I. A. Lappo–Danilevsly to finding the functions mapping conformally a half-plane on circular polygons. *Differential Equations* **X**(1974), No. 3, 458–469.
27. M. A. Lavrentyev and B. V. Shabat, Methods of the theory of functions of a complex variable, 2nd ed. *Moscow,* 1958.

Chapter II

Solution of Some Plane Filtration Problems with Partially Unknown Boundaries

Abstract. Plane problems of the stationary filtration theory with partially unknown boundaries are considered. The porous medium is assumed to be homogeneous, isotropic and non-deformable. The motion of the fluid obeys the Darcy law. The simply connected domain occupied by the moving fluid is bounded by a simple sectionally analytic contour consisting of unknown depression curves, line segments, half-lines and straight lines. The paper describes mathematical methods allowing one to find unknown parts of the boundary of the fluid motion domain and determine geometric, kinematic and physical characteristics of a moving fluid. In solving the corresponding mathematical problem, the use is made of a general solution of the non-linear Schwarz differential equation. The general solution is constructed in the paper.

1. Introduction

In this chapter we consider some plane problems of the theory of filtration theory for stationary motion of an incompressible fluid in a porous medium obeying the Darcy law. The porous medium is assumed to be non-deformable, isotropic and homogeneous. The formulation and fundamental investigation of these problems belongs to P. Ya. Polubarinova-Kochina [1–5].

The plane of motion of the fluid coincides with that of the complex variable $z = x + iy$. We introduce the complex potential $\omega(z) = \varphi(x, y) + i\psi(x, y)$, where $\varphi(x, y)$ and $\psi(x, y)$ are the velocity potential and the flow function, respectively. The functions $\varphi(x, y)$, $\psi(x, y)$ are connected by the Cauchy–Riemann conditions.

If the analytic function $\omega(z)$ is known, then by virtue of the equalities

$$\varphi(x,y) = -k(p/\gamma + y) + c, \quad \omega'(z) = u - iv, \quad u = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

we can find all characteristics of the filtration flow, i.e., filtration velocity, pressure, stress, discharge of the fluid upon filtration, etc. Here k is the filtration coefficient, c is an arbitrary constant, p is the hydrodynamic pressure, γ is the specific weight of the fluid, u, v are the components of the vector of filtration velocity, $\omega'(z)$ is the complex velocity.

The boundary of the domain of the flow involves unknown parts, and depression curves with equations to be found. We denote the simply connected domains of fluid, of complex potential and of complex velocity respectively by $S(z)$, $S(\omega)$ and $S(w)$, and their boundaries respectively by $l(z)$, $l(\omega)$ and $l(w)$. Here $w = \omega'(z)$. Below the boundary $l(z)$ of the domain $S(z)$ will be assumed to be a simple, sectionally analytic, closed contour consisting of a finite number of unknown depression curves, line segments, half-lines and straight lines. The domain $S(z)$ may be bounded or unbounded. In the particular case where all parts of the boundary $l(z)$ are known, the domain $S(z)$ is a linear polygon.

In the domain $S(z)$, we seek for an analytic function $\omega(z) = \varphi(x,y) + i\psi(x,y)$ satisfying two linearly independent boundary conditions of the type [2]

$$a_{11}\varphi(x,y) + a_{12}\psi(x,y) + a_{13}x + a_{14}y = f_1, \quad (x,y) \in l(z), \tag{1.1}$$

$$a_{21}\varphi(x,y) + a_{22}\psi(x,y) + a_{23}x + a_{24}y = f_2, \quad (x,y) \in l(z), \tag{1.2}$$

where a_{ik}, f_i, $i = 1,2$, $k = 1,2,3,4$, are the known piecewise-constant real functions, which are constant on every part of the boundary, and the rank of the matrix

$$a = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

is equal to two.

If a part of the boundary $l(z)$ of $S(z)$ is known, then in one of the conditions (1.1) or (1.2) the coefficients at the functions $\varphi(x,y)$, $\psi(x,y)$ for the known part of the boundary $l(z)$ turn out to be equal to zero.

There is a theory [1–6] which allows one to determine the boundary $l(w)$ of $S(w)$ and a part of the boundary $l(\omega)$ of $S(\omega)$ without solving the basic problem. Moreover, one can determine the coordinates of those vertices of the domain $S(w)$ to which there correspond angular points on the boundary $l(z)$ of $S(z)$. As for the vertices of the domain $S(w)$ (the cut ends with the angles 2π) to which there correspond ordinary (non-angular) points on the boundary $l(z)$ of $S(z)$, the coordinates of these vertices remain undetermined until the problem is solved completely.

When determining the boundary $l(w)$ of the domain $S(w)$, we have used some known results from the complex analysis [2, 21, 30, 31, 32].

Under the conditions imposed on the domains $S(z)$ and on the corresponding boundaries $l(z)$, one can claim that the function $\omega(z)$ is analytic in $S(z)$, continuous in the closed domain $S(z)$, satisfies $\omega'(z) \neq 0$ everywhere including the boundary $l(z)$ except its angular points, and is analytically continuable across any part of the boundary $l(z)$ not containing angular points.

As far as the functions $\omega(z)$ and $\omega'(z)$ map conformally the domain $S(z)$ and its boundary $l(z)$ (the conformity is violated at the angular points of

$l(z)$) respectively onto the domains $S(\omega)$ and $S(w)$ with the boundaries $l(\omega)$ and $l(w)$; these functions are analytically continuable across the parts of the boundaries not containing angular points [30, Ch. II, §28–29].

In the sequel, for the complex-conjugate functions we will use the notation $f(z) = f_1(x,y) + if_2(x,y)$, $\overline{f(z)} = f_1(x,y) - if_2(x,y)$, while for the derivatives of functions and matrices, the notation $f'(z) = \frac{d}{dz} f(z)$.

Theorem. *If an analytic function $\omega(z)$ satisfies in the domain $S(z)$ two linearly independent boundary conditions* (1.1)–(1.2)*, then the function $w(z) = \omega'(z)$ maps the boundary $l(z)$ of $S(z)$ into the boundary of the domain $S(w)$ consisting of a finite number of circular arcs, line segments, half-lines and straight lines, that is, to the domain $S(z)$ with the boundary $l(z)$ there corresponds a circular polygon on the plane $w(z)$.*

Proof. If we take arbitrarily a part of the boundary $l(z)$ of $S(z)$ and differentiate the conditions (1.1)-(1.2) along that part with respect to the real parameter s, then we obtain

$$(a_{11}u - a_{12}v + a_{13})\cos(x,s) + (a_{11}v + a_{12}u + a_{14})\cos(y,s) = 0, \qquad (1.3)$$

$$(a_{21}u - a_{22}v + a_{23})\cos(x,s) + (a_{21}v + a_{22}u + a_{24})\cos(y,s) = 0, \qquad (1.4)$$

where s is the arc length of the arbitrarily taken part of the boundary of $S(z)$, $\cos(x,s) = dx/ds$, $\cos(y,s) = dy/ds$.

In order for the system of equations (1.3), (1.4) to have a nonzero solution with respect to $\cos(x,s)$ and $\cos(y,s)$, it is necessary and sufficient that the determinant of this system be equal to zero,

$$\begin{aligned}\Delta_0 = (a_{11}u - a_{12}v + a_{13})(a_{21}v + a_{22}u + a_{24}) - \\ -(a_{21}u - a_{22}v + a_{23})(a_{11}v + a_{12}u + a_{14}).\end{aligned} \qquad (1.5)$$

From (1.5) we obtain

$$A_0(u^2 + v^2) + B_1^* u + B_2^* v + D_0 = 0, \qquad (1.6)$$

where

$$A_0 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad D_0 = \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}, \qquad (1.7)$$

$$B_1^* = \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{12} \\ a_{23} & a_{22} \end{vmatrix}, \qquad (1.8)$$

$$B_2^* = \begin{vmatrix} a_{14} & a_{12} \\ a_{24} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix}. \qquad (1.9)$$

The second order curve decomposes into two straight (real or imaginary) lines if and only if $A_0^* = -A_0\Delta/4 = 0$, where $\Delta = (B_1^*)^2 + (B_2^*)^2 - 4A_0D_0$.

If $A_0^* \neq 0$, $A_0^2 > 0$, and A_0^* and A_0 are of the same sign, then we have an imaginary circle; if $A_0^2 > 0$, A_0^* and A_0 are of different signs, we have a circle [7, 8].

The center coordinates (u_0, v_0) of the circle (1.6) and its radius R are defined by

$$u_0 = -B_1^*/[2a_0], \quad v_0 = -B_2^*/[2A_0], \quad R = \sqrt{\Delta}/[2A_0].$$

The circle (1.6) will be tangent to the axis of abscissas ou if $(B_1^*)^2 = 4A_0D_0$ and tangent to the axis of ordinates ov if $(B_2^*)^2 = 4A_0D_0$.

In deducing (1.6), a part of the boundary of $S(z)$ has been taken arbitrarily. To some other parts of the boundary of $S(z)$, on w there correspond arcs of the circles, i.e., the domain $S(w)$ is a circular polygon. In the case where $A_0 = 0$ along the whole contour $l(z)$, we have a linear polygon.

The equation (1.6) can be written as follows:

$$i2A_0 w\overline{w} - B_0 w + \overline{B}_0 \overline{w} + i2D_0 = 0, \tag{1.10}$$

where

$$w = u - iv, \quad \overline{w} = u + iv, \quad B_0 = B_2^* - iB_1^*, \tag{1.11}$$

From (1.10) we find that $w = \frac{-\overline{B}_0\overline{w} - i2D_0}{i2A_0\overline{w} - B_0}$, where $\Delta = B_0\overline{B}_0 - 4A_0D_0 = (B_1^*)^2 + (B_1^*)^2 - 4A_0D_0 \neq 0$.

Note that in the general case the equality

$$\Delta = 4A_0^2R^2 = 1 \tag{1.12}$$

is not valid.

We will get back to the equality (1.12) later on.

Here we make the following remark. Using a linear-fractional transformation, one can always transform the domain $S(w)$ in such way that a part of the boundary $l(w)$ on the plane w will coincide with the abscissae axis along which $w = \overline{w}$, i.e., $v = 0$. This remark will be used later on.

Below we will come across the class of matrices G_j, $j = 1, 2, \ldots, n, \ldots$, satisfying the following conditions: $G_j\overline{G}_j = \overline{G}_jG_j = E$, $\det G_j = 1$, $G_jG_k \neq G_kG_j$, $k \neq j$, $(G_jG_k)(\overline{G_jG_k}) \neq E$, $k \neq j$ where $\overline{G}_j$ is a matrix which is complex conjugate to the matrix G_j, and E is the unit matrix. The properties of the matrices G_j, $j = 1, 2, \ldots$ are very close to those of the complex-orthogonal ones [32].

The matrices G_j can be represented as

$$G_j = \begin{pmatrix} \overline{B}_j & -iD_j \\ iA_j & B_j \end{pmatrix}, \quad j = 1, 2, \ldots,$$

where A_j, D_j are real and B_j, $\overline{B}_j$ are complex-conjugate numbers.

Denoting characteristic numbers of the matrix G_j by λ_{kj}, $k = 1, 2$, we obtain $\lambda_{1j} + \lambda_{2j} = \overline{B}_j + B_j$, $\lambda_{1j}\lambda_{2j} = 1$.

It follows from the property of the matrix G_j that $\lambda_{2j} = \overline{\lambda}_{1j}$. Therefore $\lambda_{1j}\overline{\lambda}_{1j} = 1$, $|\lambda_{1j}| = 1$ and hence $\lambda_{kj} = \exp[i2\pi\alpha_{kj}]$, where α_{kj} are real numbers.

If we take two arbitrary matrices G_j and G_k from the above-mentioned class and consider the matrix $g_{jk} = G_j G_k$, then we can see that the characteristic numbers μ_{kj} of the matrices g_{jk} satisfy the conditions $\mu_{kj} = \exp[i2\pi\beta_{kj}]$, where β_{kj} are real numbers.

2. Statement of the Boundary Value Problem

Let a moving fluid occupy a simply connected domain $S(z)$ with the boundary $l(z)$ consisting of a finite number of known and unknown simple analytic Jordan arcs.

An analytic function $\omega(z)$ maps conformally the domain $S(z)$ onto a domain $S(\omega)$, and its boundary $l(z)$ into the boundary $l(\omega)$ of $S(\omega)$. Note that a part of angular points of the boundary $l(z)$ are mapped by the function $\omega(z)$ into angular points of $l(\omega)$, while the remaining angular points are mapped into non-angular points of the boundary $l(\omega)$ [1–6].

Analogously, the analytic function $w(z) = \omega'(z) = u(x,y) - iv(x,y)$ maps conformally the domain $S(z)$ onto a domain $S(w)$, and its boundary $l(z)$ into the boundary $l(w)$ of $S(w)$. Moreover, the function $w(z)$ maps a part of angular points of the boundary $l(z)$ into those of $l(w)$, and the remaining angular points are mapped into ordinary non-angular points of the boundary $l(w)$. The function $w(z)$ can map some non-angular points of the boundary $l(z)$ into angular points of the boundary $l(w)$ with interior (with respect to the domain $S(w)$) angles 2π [1–6].

Below, the points of the boundaries $l(z)$, $l(\omega)$, $l(w)$ will be assumed to be singular if to these points on either of the boundaries $l(z)$, $l(\omega)$, $l(w)$ there correspond angular points.

Let us take arbitrarily a singular point on the boundary $l(z)$ of the domain $S(z)$, for example, $l(z, E_1)$. Let to a point $l(z, E_1)$ on the boundaries $l(\omega)$, $l(w)$ there correspond the points $l(\omega, E_1')$, $l(w, E_1'')$. When the domain $S(z)$ goes around in the positive direction starting from the point $l(z, E_1)$, then the boundaries $l(\omega)$, $l(w)$ go around in the positive direction starting from the points $l(\omega, E_1')$, $l(w, E_1'')$. We enumerate all the singular points on the boundaries $l(z)$, $l(\omega)$, $l(w)$ as follows: $l(z, E_k)$, $l(\omega, E_k')$, $l(w, E_k'')$, $k = 1, 2, \dots, n, n+1$.

Of all singular points $l(z.E_k)$, $l(\omega, E_k')$, $k = 1, 2, \dots, n, n+1$, we select such ones to which on the boundary $l(w)$ of the domain $S(w)$ there correspond ordinary non-angular points. Such singular points are commonly called removable singularities. Let the number of such points be equal to m_1. When the boundary $l(z)$ goes around in the positive direction, we enumerate the removable singular points as $\varepsilon_1^*, \varepsilon_2^*, \dots, \varepsilon_{m_1}^*, \varepsilon_{m_1}^*$. The interior angles on $l(z)$ and $l(\omega)$ at the removable singular points are equal to $\pi/2$ [1–6].

In tracing $l(\omega)$ we enumerate all angular points: $l(\omega, \omega_k)$, $k = 1, 2, \dots, m_2 + 1$, while in tracing $l(w)$ we enumerate them as follows: $l(w, w_k) = b_k$, $k = 1, 2, \dots, m, m+1$, where b_k, $k = 1, 2, \dots, m, m+1$, are the complex coordinates of the vertices of the domain $S(w)$.

The equation (1.6) determines completely the circle. Two circles pass through every vertex of the domain $S(w)$ (two straight lines upon degeneration), and each of them forms four angles. We have to choose one of them. To this end we, first of all, use the equation (1.6) and then the value of the corresponding angles of the domains $S(z)$, $S(\omega)$. By means of these angles we can determine the angles at the vertices of the domains $S(w)$, $S(\omega)$ [1–6]. Despite the fact that some of the interior angles of $S(\omega)$ are unknown, we have at hand the angle values for the corresponding vertices of the domain $S(w)$. By means of the latter we can determine the unknown interior angle at the vertex of the domain $S(\omega)$ [2]. This henceforth allows us to take for granted that all the interior angles and the coordinates of the vertices of $S(w)$, excluding the cut ends with interior angles 2π, are determined.

Denote the interior angles at the vertices b_j, $j = 1, 2, \dots, m, m+1$, of the domain $S(w)$ by $\pi\nu_j$, $j = 1, 2, \dots, m, m+1$, respectively.

Note that the two neighboring circles passing through the point b_k intersect at the point b'_k which in the general case is beyond the boundary $l(w)$. If these circles are tangent, then $b_k = b'_k$.

In general it is quite difficult to find an analytic function $\omega(z) = \varphi(x, y) + i\psi(x, y)$ by the boundary conditions (1.1)–(1.2). Therefore one introduces an auxiliary complex plane $\zeta = t + i\tau$. The half-plane $\mathrm{Im}(\zeta) > 0$ of this plane is mapped conformally onto the domains $S(z)$, $S(\omega)$, $S(w)$. Denote the domain $\mathrm{Im}(\zeta) > 0$ and its boundary respectively by $S(\zeta)$ and $l(\zeta)$.

In what follows, we will need the following result from the papers [21, 30, 32].

If D and D^* are simply connected domains whose boundaries consist of a finite number of analytic Jordan arcs, then there exists a unique conformal mapping $w = f(z)$ of the domain D onto the domain D^*, transferring three boundary points z_k, $k = 1, 2, 3$, of D into three boundary points W_k, $k = 1, 2, 3$ of D^*. The points z_k and w_k are given arbitrarily, their order in tracing the boundaries of the domains being preserved.

Let the analytic functions $z(\zeta)$, $\omega(\zeta)$, $w(\zeta) = \omega'(\zeta)/z'(\zeta)$ map conformally the domain $S(\zeta)$ ($\mathrm{Im}(\zeta) > 0$) onto the domains $S(z)$, $S(\omega)$, $S(w)$, respectively. Moreover, let the points of the boundary $l(\zeta)$ of $S(\zeta)$, that is the points of the real axis t of the plane ζ, $t = e_k$, $k = 1, 2, \dots, n, n+1$ ($-\infty < e_1 < e_2 < \cdots < e_{n+1} = \infty$), be respectively mapped into the points $l(z, E_k)$, $l(\omega, E'_k)$, $l(w, E''_k)$, $k = 1, 2, \dots, n, n+1$, of the boundaries $l(z)$, $l(\omega)$, $l(w)$ of the domains $S(z)$, $S(\omega)$, $S(w)$.

The boundary values of the functions $z(\zeta)$, $\omega(\zeta)$, $w(\zeta)$, as $\zeta \to t$, are denoted by $z(t) = x(t) + iy(t)$, $\omega(t) = \varphi(t) + i\psi(t)$, $w(t) = u(t) - iv(t)$, and by $\overline{z(t)}$, $\overline{\omega(t)}$, $\overline{w(t)}$ we denote the complex functions, conjugate to the functions $z(t)$, $\omega(t)$, $w(t)$.

The boundary conditions (1.1)–(1.2) with respect to the analytic functions $\omega(\zeta)$ and $z(\zeta)$ can be written in the form [2]

$$\operatorname{Im}[m_{11}(t)\omega(t)+m_{12}(t)z(t)]=f_1(t),\quad -\infty<t<+\infty, \tag{2.1}$$

$$\operatorname{Im}[m_{21}(t)\omega(t)+m_{22}(t)z(t)]=f_2(t),\quad -\infty<t<+\infty, \tag{2.2}$$

where $m_{k1}(t)=a_{k2}(t)+ia_{k1}(t)$, $m_{k2}(t)=a_{k4}(t)+ia_{k3}(t)$, $f_k(t)$, $k=1,2$, are piecewise constant functions with the discontinuity points $t=e_k$, $k=1,2,\dots,n,n+1$.

In the domain $S(\zeta)$ we have to find analytic functions $z(\zeta)$, $\omega(\zeta)$ satisfying the boundary conditions (2.1)–(2.2). By means of these functions, the points $z(\zeta)$, $\omega(\zeta)$ are mapped respectively into the points $t=e_k$, $k=1,2,\dots,n,n+1$. Moreover, each part of the boundary must necessarily be mapped into the corresponding parts of the boundaries $l(z,E_k)$, $l(\omega,E'_k)$, $k=1,2,\dots,n+1$. The unknown parts of the boundaries $l(\zeta)$, $-\infty<t<e_1$, $e_k<t<e_{k+1}$, $k=1,2,\dots,n$ and the parameters $t=e_k$, $k=1,2,\dots,n$, are to be determined.

If we succeed in constructing analytic functions $z(\zeta)$, $\omega(\zeta)$, which map conformally the domain $S(\zeta)$ respectively onto the domains $S(z)$, $S(\omega)$, then the boundary values $z(t)$, $\omega(t)$ of these functions will satisfy the conditions (2.1)–(2.2). Moreover, if the functions $z(\zeta)$, $\omega(\zeta)$ are known, then we are able to construct the function $w(\zeta)=\omega'(\zeta)/z'(\zeta)$.

If one or several coefficients m_{kj}, $k=1,2$; $j=1,2$, are equal to zero, and $m_{11}(t)m_{22}(t)-m_{12}(t)m_{21}(t)\neq 0$, then by the conditions (2.1)–(2.2) the functions $\omega(\zeta)$, $z(\zeta)$ can be constructed by means of the Cauchy type integrals. There are particular cases where all $m_{kj}(t)\neq 0$, $k=1,2$, $j=1,2$, but, nevertheless, one manages to construct the functions $\omega(\zeta)$, $z(\zeta)$ in an elementary way [12].

As we will see below, in a general case we have succeeded in constructing first the analytic function $w(\zeta)$, then, by means of this function, we have constructed analytic functions $\omega'(\zeta)$, $z'(\zeta)$ and, finally, we have found the functions $\omega(\zeta)$ and $z(\zeta)$.

The notion of singular and removable singular points of the boundary $l(z)$ has been introduced above. As is said, to singular points of the boundary $l(z)$ there correspond singular points $t=e_k$, $k=1,2,\dots,n,n+1$, of the boundary $l(\zeta)$. They can be divided into two groups: removable and unremovable. We have enumerated the removable points by $t=\varepsilon_k$, $n=1,2,\dots,m_1$, and the unremovable ones by $t=a_k$, $k=1,2,\dots,m,m+1$. To the points $t=a_k$, $k=1,2,\dots,m+1$, on the boundary $l(w)$ there correspond the points $l(w,w_k)=b_k$ while to the points $t=\varepsilon_k$, $k=1,2,\dots,m$, there correspond the points $l(z,z_k)=\varepsilon^*_k$, $k=1,2,\dots,m_1$. By our choice, the point $t=e_{n+1}=a_{m+1}=\infty$ is an unremovable singular point. Among the points $t = a_k$, $k = 1, 2, \dots, m$, we select and fix arbitrarily two points, because one point $t = a_{m+1} = \infty$ is already fixed.

An investigation of the problem (2.1)–(2.2) from the point of view of the Riemann-Hilbert problem can be found in [17, 18].

Introduce an analytic vector $\Phi(\zeta)$ and a vector $f(t)$ as follows:

$$\Phi(\zeta) = [\omega(\zeta), z(\zeta)], \quad \mathrm{Im}(\zeta) > 0; \quad \overline{\Phi(\overline{\zeta})} = [\overline{\omega}(\overline{\zeta}), \overline{z(\overline{\zeta})}], \quad \mathrm{Im}(\zeta) < 0,$$
$$f(t) = [f_1(t), f_2(t)], \quad -\infty < t < +\infty.$$

The conditions (2.1)–(2.2) with respect to the vector $\Phi(\zeta)$ can be written as

$$\Phi(t) = A_*^{-1}(t)\overline{A}_*(t)\overline{\Phi(t)} + 2iA_*^{-1}(t)f(t), \quad -\infty < t < +\infty, \tag{2.3}$$

where

$$A_*(t) = \begin{pmatrix} m_{11}(t) & m_{12}(t) \\ m_{21}(t) & m_{22}(t) \end{pmatrix}, \quad -\infty < t < +\infty,$$

is a non-singular piecewise-constant matrix, A_*^{-1} is the inverse to A_*

$$A_*^{-1}(t) = \frac{1}{\det A_*(t)} \begin{pmatrix} m_{22}(t) & m_{12}(t) \\ m_{21}(t) & m_{11}(t) \end{pmatrix}, \quad -\infty < t < +\infty,$$

and $\overline{A_*(t)}$ is the complex conjugate to $A_*(t)$.

It can be easily verified that

$$A_*^{-1}(t)\overline{A_*(t)} = \frac{1}{\det A_*(t)} \begin{pmatrix} \overline{-B_0(t)} & -i2D_0(t) \\ i2A_0(t) & -B_0(t) \end{pmatrix}, \quad -\infty < t < +\infty,$$

where $A_0(t)$, $B_0(t)$, $D_0(t)$ are defined by (1.7)–(1.9) and (1.11).

We can directly verify that the equalities

$$\Delta(t) = B_0(t)\overline{B_0(t)} - 4A_0(t)D_0(t) = \det A_*(t) \cdot \det \overline{A_*(t)},$$
$$\det[A_*^{-1}(t) \cdot \overline{A_*(t)}] = [\det \overline{A_*(t)}]/[\det A_*(t)]$$

are also valid.

Differentiating the equality (2.3) along the t, u-axis and writing it in terms of projections, we obtain

$$\omega'(t) = [-\overline{B_0(t)}\,\overline{\omega'(t)} - i2D_0(t)\overline{z'(t)}]/\det A_*(t), \quad -\infty < t < +\infty, \tag{2.4}$$
$$z'(t) = [i2a_0(t)\overline{\omega'(t)} - B_0(t)\overline{z'(t)}]/\det A_*(t), \quad -\infty < t < +\infty. \tag{2.5}$$

After division, from (2.4) and (2.5) we get

$$\frac{\omega'(t)}{z'(t)} = \frac{-\overline{B_0(t)}\,\overline{\omega'(t)} - i2D_0(t)\overline{z'(t)}}{i2A_0(t)\overline{\omega'(t)} - B_0(t)\overline{z'(t)}}, \quad -\infty < t < +\infty. \tag{2.6}$$

The equality (2.6) can also be written as

$$w(t)=\frac{-\overline{B_0(t)}\,\overline{w(t)}-i2D_0(t)}{i2A_0\overline{w(t)}-\overline{B_0(t)}},\quad -\infty<t<+\infty, \tag{2.7}$$

where $w(t)=\omega'(t)/z'(t)$.

As we will see below, by means of the solution of the well-known Schwarz differential equation we can find an analytic function satisfying (2.7) on the t-axis, provided the condition

$$\Delta(t)=B_0(t)\overline{B_0(t)}-4A_0(t)D_0(t)=1,\quad -\infty<t<+\infty, \tag{2.8}$$

is fulfilled.

However, the condition (2.8) may not be fulfilled. If we divide the numerator and the denominator in (2.7) by $\sqrt{\Delta(t)}$ and introduce the notation

$$B(t)=-B_0(t)/\sqrt{\Delta(t)},\ \ \overline{B(t)}=-\overline{B_0(t)}/\sqrt{\Delta(t)},\ \ -\infty<t<+\infty, \tag{2.9}$$

$$A(t)=2A_0(t)/\sqrt{\Delta(t)},\ \ \overline{D(t)}=2D_0(t)/\sqrt{\Delta(t)},\ \ -\infty<t<+\infty, \tag{2.10}$$

then the condition

$$\Delta_1(t)=\overline{B(t)}B(t)-A(t)D(t)=1,\quad -\infty<t<+\infty, \tag{2.11}$$

will be fulfilled.

With regard for (2.9) and (2.10), we can rewrite (2.7) as

$$w(t)=\frac{\overline{B(t)}\,\overline{w(t)}-iD(t)}{iA(t)\overline{w(t)}+B(t)},\quad -\infty<t<+\infty, \tag{2.12}$$

and (2.4) and (2.5) as

$$\omega'(t)=\sqrt{\det\overline{A_*(t)}/\det A_*(t)}\,\big[\,\overline{B(t)}\,\overline{\omega'(t)}-iD(t)\overline{z'(t)}\,\big],\ -\infty<t<+\infty, \tag{2.13}$$

$$z'(t)=\sqrt{\det\overline{A_*(t)}/\det A_*(t)}\,\big[\,iA(t)\overline{\omega'(t)}+B(t)\overline{z'(t)}\,\big],\ -\infty<t<+\infty. \tag{2.14}$$

A solution of the system (2.13)-(2.14) will be sought in the form

$$\omega'(t)=\gamma(t)\omega_1(t),\quad z'(t)=\gamma(t)z_1(t),\quad -\infty<t<+\infty, \tag{2.15}$$

where $\omega_1(t)$, $z_1(t)$ and $\gamma(t)$ must satisfy the boundary conditions

$$\omega_1(t)=\overline{B(t)}\,\overline{\omega_1(t)}-iD(t)\overline{z_1(t)},\quad -\infty<t<+\infty, \tag{2.16}$$

$$z_1(t)=iA(t)\overline{\omega_1(t)}+B(t)\overline{z_1(t)},\quad -\infty<t<+\infty, \tag{2.17}$$

$$\gamma(t)=\sqrt{\det\overline{A_*(t)}/\det A_*(t)}\ \overline{\gamma(t)},\quad -\infty<t<+\infty. \tag{2.18}$$

Note that the value of the function $w(t)=\omega'(t)/z'(t)$ does not change after the representation (2.15), and hence so does (2.12). A little later we will prove that (2.12) implies (2.16)–(2.17) [13–16].

If we denote the values of the matrix $A_*(t)$ for the intervals $-\infty < t < e_1$, $e_j < t < e_{j+1}$, $j = 1, 2, \dots, n$, respectively by $A_{*(n+1)}$, A_{*j}, $j = 1, 2, \dots, n$, then we can write

$$\det A_{*j} = |\det A_{*j}| \exp[i\varphi_j], \quad \det \overline{A_{*j}} = |\det \overline{A_{*j}}| \exp[-i\varphi_j],$$
$$|\det \overline{A_{*j}}|/|\det A_{*j}| = 1, \quad j = 1, 2, \dots, n, n+1,$$
$$\sqrt{\det \overline{A_{*j}}/\det A_{*j}} = \sqrt{\exp[-i2\varphi_j]} = \exp[-i\varphi_j], \quad j = 1, 2, \dots, n+1. \qquad (2.19)$$

Taking into account (2.19), we rewrite (2.18) as

$$\gamma(t) = \exp[i\varphi_0(t)]\overline{\gamma(t)}, \quad -\infty < t < +\infty, \qquad (2.20)$$

where $\varphi_0(t)$ is a piecewise constant function defined by

$$\sqrt{\det \overline{A_*(t)}/\det A_*(t)} = \exp[-i\varphi_0(t)], \quad -\infty < t < +\infty.$$

After taking the logarithm of (2.20), we get

$$\ln \gamma(t) - \ln \overline{\gamma(t)} = -i\varphi_0(t), \quad -\infty < t < +\infty. \qquad (2.21)$$

We will not introduce here the notion of index but will act formally and will find from (2.21) a particular solution belonging to some class, and then we will define more exactly which solution out of all possible solutions of (2.21) is just needed.

The particular solution of the boundary value problem (2.21) can be obtained by the formula [17]

$$\ln \gamma(\zeta) = \frac{(-1)}{2\pi} \int_{-\infty}^{+\infty} \frac{\zeta + i}{t + i} \, \frac{\varphi_0(t)dt}{t - \zeta}. \qquad (2.22)$$

From (2.22) we find that

$$\gamma(\zeta) = \text{const}(\zeta - e_1)^{\beta_1}(\zeta - e_2)^{\beta_2} \cdots (\zeta - e_n)^{\beta_n}, \qquad (2.23)$$

where $\beta_1 = (\varphi_{n+1} - \varphi_1)/2\pi$, $\beta_j = (\varphi_{j-1} - \varphi_j)/2\pi$, $j = 2, 3, \dots, n$, and φ_j, $j = 1, 2, \dots, n+1$ are the values of the function $\varphi_0(t)$ on the intervals $e_j < t < e_{j+1}$, $j = 1, 2, \dots, n$, $-\infty < t < e_1$, respectively.

The numbers φ_j, $j = 1, 2, \dots, n+1$, in (2.23) will be chosen appropriately after finding the functions $\omega'(\zeta)$ and $z'(\zeta)$.

It follows from the above-said that to construct in the domain $S(\zeta)$ the analytic functions $\omega(\zeta)$ and $z(\zeta)$ satisfying the boundary conditions (2.1)–(2.2), it is necessary first to construct in the domain $S(\zeta)$ the functions $\omega_1(\zeta)$, $z_1(\zeta)$ satisfying the conditions (2.16)–(2.17). And, as we will see below, to construct the functions $\omega_1(\zeta)$ and $z_1(\zeta)$, it is necessary first to construct in the domain $S(\zeta)$ the function $w(\zeta) = \omega'(\zeta)/z'(\zeta) = \omega_1(\zeta)/z_1(\zeta)$ satisfying the boundary condition (2.12).

3. Investigation of the Value Problem (2.16)–(2.17)

We write the boundary value problem (2.16)–(2.17) in the vector form:

$$\Phi_1(t) = g(t)\overline{\Phi_1(t)}, \quad -\infty < t < +\infty, \tag{3.1}$$

where

$$\Phi_1(\zeta) = [\omega_1(\zeta), z_1(\zeta)], \ \mathrm{Im}(\zeta) > 0; \quad \overline{\Phi_1(\overline{\zeta})} = [\overline{\omega_1(\overline{\zeta})}, \overline{z_1(\overline{\zeta})}], \ \mathrm{Im}(\zeta) < 0,$$

$$g(t) = \begin{pmatrix} \overline{B(t)} & -i2D(t) \\ iA(t) & B(t) \end{pmatrix}, \quad -\infty < t < +\infty.$$

For the intervals $a_j < t < a_{j+1}$, $j = 1, 2, \dots, n$, $-\infty < t < a_1$, denote the values of the matrix $g(t)$ respectively by g_j, $j = 1, 2, \dots, n, n+1$. There is a close connection between the characteristic numbers of the matrices $g_j^{-1} g_{j-1}$, $j = 1, 2, \dots, n+1$, and the interior angles at the vertices of the circular polygon $S(\omega)$. Indeed, consider the characteristic equation for the point $t = e_j$ [2, 17, 18]:

$$\det(g_j^{-1}(t) g_{j-1}(t) - \lambda_j E) = 0, \tag{3.2}$$

where λ_j is a parameter and E is the unit matrix.

The equation (3.2) can be also written as $\det(g_{j-1}(t) - \lambda_j g_j(t)) = 0$. Hence, taking into account the fact that $\det g_j = 1$, $j = 1, 2, \dots, n+1$, we obtain $\lambda_j^2 - a_0 \lambda_j + 1 = 0$, $a_{0j} = \overline{B}_{j-1} B_j + \overline{B}_j B_{j-1} - A_{j-1} D_j - A_j D_{j-1}$, which implies that $\lambda_{1j} \lambda_{2j} = 1$, $\quad \lambda_{1j} + \lambda_{2j} = a_{0j}$, where λ_{1j} and λ_{2j} are the characteristic roots of (3.2).

Consider the numbers $\alpha_{kj} = \frac{1}{2\pi i} \ln \lambda_{kj}$, which are defined to within integer summands.

It has been proved in [2] that α_{kj} are real numbers satisfying $\alpha_{1j} - \alpha_{2j} = \nu_j$.

Let us get back to the removable singular points $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m_1}$ For these points the neighboring matrices g_{j-1} and g_j are diagonal, and $t = e_j = \varepsilon_j$, $\lambda_{kj} = -1$, $k = 1, 2$, $\alpha_{1j} = -1/2$, $\alpha_{2j} = -1/2$ [1–5]. Moreover, $A_0(t) = 0$, $B_1(t) = 0$, $B_2(t) \neq 0$, $D_0(t) = 0$, $v(t) = 0$, $u(t) \neq 0$ or $A_0(t) = 0$, $B_2(t) = 0$, $B_1(t) \neq 0$, $D_0(t) = 0$, $u(t) = 0$, $v(t) \neq 0$, where $t \in (a_{j-1}, a_{j+1})$.

Introduce a new sought for vector $\Phi_2(\zeta) = [\omega_2(\zeta), z_2(\zeta)]$ by

$$\Phi_1(\zeta) = \chi_1(\zeta) \Phi_2(\zeta), \tag{3.3}$$

where

$$\chi_1(t) = [(t - \varepsilon_1)(t - \varepsilon_2) \cdots (t - \varepsilon_{m_1})]^{-1/2}, \quad \chi_1(t) > 0, \quad t > \varepsilon_{m_1}. \tag{3.4}$$

The boundary condition (3.1) for $\Phi_2(\zeta)$ takes the form

$$\Phi_2(t) = G(t)\overline{\Phi_2(t)}, \quad -\infty < t < +\infty, \tag{3.5}$$

where $G(t) = [\chi_1(t)]^{-1} g(t) \overline{\chi_1(t)}$, $-\infty < t < +\infty$, also is a piecewise constant matrix with the discontinuity points $a_1, a_2, a_3, \dots, a_m, a_{m+1} = \infty$.

The matrix $G(t)$ differs from the matrix $g(t)$ only by the fact that some matrices g_j are multiplied by -1 and the others remain unchanged.

If some elements of the matrix $G(t)$ are equal to zero and $\det G(t) \neq 0$, then the problem (3.5) is solved completely by the Cauchy type integrals, and the equations for the determination of the unknown parameters are derived [17]. Besides these cases, there are the ones where all elements of the matrix $G(t)$ differ from zero and the problem (3.5) is solved simply. Such cases involve circular polygons, when the boundary $S_1(w)$ consists of a finite number of arcs of concentric circles with the center $M(w_0)$ and straight cuts passing through $M(w_0)$ upon their extension. By means of the logarithmic function such domains $S_1(w)$ can be transformed into linear polygons. Moreover, there exist many domains $S_2(w)$ which by the linear-fractional transformation reduce to a set of domains $S_1(w)$. Hence, using the Christoffel–Schwarz formula [12], for the domains $S_1(w)$, $S_2(w)$, we construct the functions $w(\zeta)$.

We will now proceed to the solution of (3.5). If a circular polygon is bounded, then $0 \leq \nu_k \leq 2$. Below we will consider the case where one or several vertices of the domain $S(w)$ are at the point $w = \infty$. This may happen if two neighboring circular arcs degenerate to half-lines or straight lines. Moreover, if the sides of the corresponding angle are parallel, then the vertex of the interior angle is assumed to be equal to zero. If, however, the sides at the vertex $b_k = \infty$ diverge and intersect at a finite point b_k^* upon their extention, forming the angle $\pi\nu_k^*$ turned to the vertex b_k^*, then we will assume that $\pi\nu_k = -\pi\nu_k^*$; hence, ν_k may take the values $-2 \leq \nu_k \leq 2$.

It is known that the construction of the sought for function $w(\zeta)$ is reduced to the solution of the nonlinear Schwarz equation which in its turn reduces to a Fuchs class equation. Therefore for the domain $S(w)$ we construct the Fuchs class equation

$$V''(\zeta) + P_*(\zeta)V'(\zeta) + q_*(\zeta)V(\zeta) = 0, \tag{3.6}$$

where

$$P_*(\zeta) = \sum_{j=1}^{m} \frac{1-\nu_j}{\zeta - a_j}, \quad q_*(\zeta) = \sum_{j=1}^{m} \frac{c_j}{\zeta - a_j},$$

c_j, $j = 1, 2, \dots, m$, are the unknown accessory real parameters which for the present satisfy the conditions

$$\sum_{j=1}^{m} c_j = 0, \quad \sum_{j=1}^{m} c_j a_j = \alpha_1\alpha_2, \tag{3.7}$$

$$\sum_{j=1}^{m} \nu_j + \alpha_1 + \alpha_2 = m - 1, \quad \alpha_1 - \alpha_2 = \nu_{m+1}.$$

Denote by $V_1(\zeta)$, $V_2(\zeta)$ linearly independent solutions of the equation (3.6) and construct the function $w_1(\zeta) = V_1(\zeta)/V_2(\zeta)$. The function $w_1(\zeta)$ is a particular solution of the following Schwarz equation:

$$\frac{w'''(\zeta)}{w'(\zeta)} - \frac{3}{2}\Big(\frac{w''(\zeta)}{w'(\zeta)}\Big)^2 = 2q_*(\zeta) - P'_*(\zeta) - \frac{1}{2}[P_*(\zeta)]^2, \qquad (3.8)$$

which is constructed with regard for the equation (3.6).

The general solution of the equation (3.8) is given by $w(\zeta) = \frac{pw_1(\zeta)+q}{rw_1(\zeta)+s}$, where p, q, r, s are the constants (complex in general) of integration of (3.8) satisfying $ps - rq = 1$.

The equation (3.8) is invariant under linear-fractional transformations both of the function $w(\zeta)$ and of ζ. Note that the coefficients of the transformation of $w(\zeta)$ may be either complex or real, while those of the transformation of ζ may be only real. Moreover, the equation (3.6) is also invariant under the transformations of ζ with real coefficients [19–22].

In constructing a general solution of the equation (3.8), we have already used its invariance property with respect to $w(\zeta)$. Exploit now the invariance of the equation (3.6) with respect to ζ. Using this property, we choose arbitrarily and fix three of the parameters $t = a_k$, $k = 1, 2, \dots, m+1$, while the remaining $(m-2)$ ones are to be defined. Moreover, the coefficients of the equation (3.6) involve the parameters c_j, $j = 1, 2, \dots, m$ which for the present satisfy only two conditions (3.7), so one can define only two of them. The remaining $(m-2)$ parameters are also to be defined. Consequently, the coefficients of the equation (3.6) depend on $2(m-2)$ unknown parameters. The parameters p, q, r, s are to be defined. Thus, to construct $w(\zeta)$ we must define only $2(m+1)$ parameters, while to construct the functions $\omega'(\zeta)$, $z'(\zeta)$ we must add the parameters connected with the removable singular points. Their number is m_1.

4. Solution of Equation (3.6)

Each of the Fuchs class equations (3.6) near every singular point $t = a_k$, $k = 1, 2, \dots, m+1$, and near any ordinary point, where $p_*(\zeta)$, $q_*(\zeta)$, are analytic, have two linearly independent local solutions. They are constructed by means of infinite series whose coefficients are defined in the well-known manner. The series converge in the circles with centers at the points for which they have been constructed. Radii of these circles are determined by the distances to the singular points nearest from the centers.

Denote by $V_{kj}(\zeta)$, $k = 1, 2$, $j = 1, 2, 3, \dots, m+1$, linearly independent local solutions of the equation (3.6) for the singular points $\zeta = a_k$, $k = 1, 2, \dots, m+1$, and by $\varphi_{kj}(\zeta)$, $k = 1, 2$, $j = 1, 2, \dots, m-1$, the ones for the points $t = a_j^* = (a_j + a_{j+1})/2$, $j = 1, 2, \dots, m-1$.

Assume $u_{1j}(\zeta) = pv_{1j}(\zeta) + qv_{2j}(\zeta)$, $u_{2j}(\zeta) = rv_{1j}(\zeta) + sv_{2j}(\zeta)$, where p, q, r, s are the integration constants of (3.8).

The differential equation (3.6) can be written in the form of a system

$$\chi'(\zeta) = \chi(\zeta)\mathcal{P}(\zeta), \tag{4.1}$$

where

$$\chi(\zeta) = \begin{pmatrix} u_1(\zeta) & u_1'(\zeta) \\ u_2(\zeta) & u_2'(\zeta) \end{pmatrix}, \quad \mathcal{P}(\zeta) = \begin{pmatrix} 0 & -q_*(\zeta) \\ 1 & -p_*(\zeta) \end{pmatrix}, \tag{4.2}$$

$u_1(\zeta)$ and $u_2(\zeta)$ are linearly independent solutions of (3.6).

First we find the solution of (4.1), that is, we construct the matrix $\chi(\zeta)$. Then by means of this matrix $\chi(\zeta)$ we seek for a solution of the boundary value problem (3.5).

It is known that if the matrix $\chi_*(\zeta)$ is a solution of (4.1), then the matrix $T\chi_*(\zeta)$ is also a solution of (4.1), where

$$T = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad \det T = 1. \tag{4.3}$$

If we construct the local linearly independent solutions $v_{kj}(\zeta)$ and $\varphi_{kj}(\zeta)$ of the equation (3.6) for the points $\zeta = a_j$, $j = 1, 2, \dots, m+1$, and $\zeta = a_j^* = (a_j + a_{j+1})/2$, respectively, then the local fundamental matrices for (4.1) will take the form

$$\begin{aligned} \Theta_j(\zeta) &= \begin{pmatrix} v_{1j}(\zeta) & v_{1j}'(\zeta) \\ v_{2j}(\zeta) & v_{2j}'(\zeta) \end{pmatrix}, \quad j = 1, 2, \dots, m+1, \\ H_j(\zeta) &= \begin{pmatrix} \varphi_{1j}(\zeta) & \varphi_{1j}'(\zeta) \\ \varphi_{2j}(\zeta) & \varphi_{2j}'(\zeta) \end{pmatrix}, \quad j = 1, 2, \dots, m-1. \end{aligned} \tag{4.4}$$

Assume that the inequality $|a_m| > |a_1|$ holds. Then at the point $a_m^* = -|a_m|$ we construct the local series $\varphi_{*k}(\zeta)$, $k = 1, 2$, and the corresponding local matrix $H_*(\zeta)$. Radii of convergence of these series will be determined by the distance from the point $t = a_m^*$ to the singular point $t = a_1$. Analogously, if $|a_1| > |a_m|$, then at the point $a_1^* = |a_1|$ we construct local series $\varphi_k^*(\zeta)$, $k = 1, 2$, and the matrix H. Radii of convergence of these series will be determined by the distance from the point a_1^* to the point $t = a_m$.

After this we can see that there exists a finite number of circles with the centers $\zeta = a_j$, $j = 1, 2, \dots, m+1$, $\zeta = a_j^* = (a_j + a_{j+1})/2$, $j = 1, 2, \dots, m$, covering completely the abscissae axis. Note that by the circle with the center $\zeta = \infty$ will be meant the exterior of the circle $|\zeta| < r$, where r will be assumed to be equal to the greatest of the numbers $|a_1|$, $|a_m|$.

The equation (3.6) near the point $\zeta = a_j$ can be written as

$$(\zeta - a_j)^2 v''(\zeta) + (\zeta - a_j)p_j(\zeta)v'(\zeta) + q_j(\zeta)v(\zeta) = 0, \tag{4.5}$$

where

$$p_j(\zeta)=\sum_{k=0}^{\infty}p_{kj}(\zeta-a_j)^k,\quad q_j(\zeta)=\sum_{k=1}^{\infty}q_{kj}(\zeta-a_j)^k. \tag{4.6}$$

The solutions of the equations (4.5) and (4.6) for the point $\zeta=a_{m+1}=\infty$ can be written by means of the transformation $\zeta=1/\zeta_1$, as follows [29, 27]

$$\zeta_1^2v''(\zeta_1)+\zeta_1[2-\sum_{k=0}^{\infty}p_{k\infty}\zeta_1^k]v'(\zeta_1)+\left[\sum_{k=0}^{\infty}q_{k\infty}\zeta^k\right]v(\zeta_1)=0, \tag{4.7}$$

where

$$p_*(1/\zeta_1)=\zeta_1\sum_{k=0}^{\infty}p_{k\infty}\zeta_1^k,\quad q_*(1/\zeta_1)=\zeta_1^2\sum_{k=0}^{\infty}q_{k\infty}\zeta_1^k. \tag{4.8}$$

The solutions of the equations (4.5) and (4.7) for the points $\zeta=a_j$, $j=1,2,\dots,m$, $\zeta=\infty$ are sought respectively in the form [22, 27]

$$v_j(t)=(t-a_j)^{\alpha_j}\widetilde{v}_j(t),\quad \widetilde{v}_j(t)=\sum_{n=0}^{\infty}\gamma_{nj}(t-a_j)^n, \tag{4.9}$$

$$v_\infty(t)=t^{-\alpha_\infty}\widetilde{v}_\infty(t),\quad \widetilde{v}_\infty(t)=\sum_{n=0}^{\infty}\gamma_{nj}t^{-n}. \tag{4.10}$$

Substituting (4.9) in (4.5), we obtain

$$(\zeta-a_j)^{\alpha_j}\left[\sum_{k=0}^{\infty}M_{kj}(\zeta-a_j)^k\right]=0,$$

whence there follows an infinite recursion system of equations to define γ_{nj}, $n=1,2,\dots$,

$$M_{0j}(\alpha_j)=\gamma_{0j}f_{0j}(\alpha_j)=0;\;\; f_{0j}(\alpha_j)=\alpha_j(\alpha_j-1)+\alpha_jp_{0j}+q_{0j}=0, \tag{4.11}$$

$$M_{1j}(\alpha_j)=\gamma_{1j}(\alpha_j)f_{0j}(\alpha_j+1)+\gamma_{0j}f_{1j}(\alpha_j)=0, \tag{4.12}$$

$$M_{2j}(\alpha_j)=\gamma_{2j}(\alpha_j)f_{0j}(\alpha_j+2)+\gamma_{1j}(\alpha_j)f_{1j}(\alpha_j+1)+\gamma_{0j}f_{2j}(\alpha_j)=0, \tag{4.13}$$

....

$$\begin{aligned}M_{nj}(\alpha_j)&=\gamma_{nj}(\alpha_j)f_{0j}(\alpha_j+n)+\gamma_{(n-1)j}(\alpha_j)f_{1j}(\alpha_j+n-1)+\dots+\\&+\gamma_{[n-(k-2)]j}(\alpha_j)f_{(k-2)j}(\alpha_j+n-k+2)+\dots+\\&+\gamma_{1j}(\alpha_j)f_{(n-1)j}(\alpha_j+1)+\gamma_{0j}f_{nj}(\alpha_j)=0,\end{aligned} \tag{4.14}$$

....

$$f_{kj}(\alpha_j)=\alpha_jp_{kj}+q_{kj},\quad q_{0j}=0. \tag{4.15}$$

If the determining equation (4.11) has the roots α_{1j}, α_{2j} $(\alpha_{1j}>\alpha_{2j})$ such that $\alpha_{1j}-\alpha_{2j}\neq n$, $n=0,1,2$, then by the formulas (4.12)–(4.14) we construct for the equation (4.5) two local linearly independent solutions of the form

$$v_{kj}(t) = (t-a_j)^{\alpha_{kj}}\gamma_{0j}\widetilde{v}_{kj}(t), \quad \widetilde{v}_{kj}(t) = 1 + \sum_{n=1}^{\infty}\gamma_{nj}^{k}(t-a_j)^n, \qquad (4.16)$$
$$k=1,2, \quad j=1,2,\dots,m.$$

The proof for the convergence of the series (4.16) can be found in [19,22]. The convergence radii of the series $\widetilde{v}_{kj}(\zeta)$ are determined by the distance from the point $t=a_j$ to the nearest of the points $t=a_{j-1}$, $t=a_{j+1}$.

In the case where the equation (4.11) has the roots such that $\alpha_{1j} = \alpha_{2j}$, we differentiate it with respect to α_j, calculate and obtain the second solution,

$$v_{2j}(\zeta) = v_{1j}(\zeta)\ln(\zeta-a_j) + v_{2j}^*(\zeta),$$
$$v_{2j}^*(\zeta) = (\zeta-a_j)^{\alpha_{2j}}\gamma_{0j}\sum_{k=1}^{\infty}\frac{d}{d\alpha_j}[\gamma_{kj}(\alpha_j)]_{\alpha_j=\alpha_{2j}}(\zeta-a_j)^k.$$

The first one $v_{1j}(\zeta)$ is of the form (4.16).

Finally, if the equation (4.11) has the roots such that $\alpha_{1j} - \alpha_{2j} = s$, $s=1,2$, then the first solution in these cases can again be determined by (4.16), and the second one is sought in the form [23]

$$v_j(t) = \gamma_{0j}(t-a_j)^{\alpha_j}[\alpha_j - \alpha_{2j} + \sum_{n=1}^{\infty}\gamma_{nj}(\alpha_j)(t-a_j)^n]. \qquad (4.17)$$

To calculate $\gamma_{nj}(\alpha_j)$, in (4.11)–(4.14) we substitute instead of γ_{0j} the product $\gamma_{0j}(\alpha_j-\alpha_{2j})$ and then define successively $\gamma_{nj}(\alpha_j)$, $n=1,2,\dots$. The defined in such a manner $\gamma_{nj}(\alpha_j)$ we substitute in (4.17), differentiate with respect to α_j and then calculate the limit as $\alpha_j \to \alpha_{2j}$. We obtain

$$v_{2j}(t) = \lim_{\alpha_j\to\alpha_{2j}}\gamma_{0j}\Bigg\{(t-a_j)^{\alpha_j}\left[\alpha_j-\alpha_{2j}+\sum_{n=1}^{\infty}\gamma_{nj}(\alpha_j)(t-a_j)^n\right]\ln(t-a_j) +$$
$$+(t-a_j)^{\alpha_j}\left[1+\sum_{n=1}^{\infty}\frac{d}{d\alpha_j}[\gamma_{nj}(\alpha_j)](t-a_j)^n\right]_{\alpha_j\to\alpha_{2j}}\Bigg\}.$$

To define $v_{k\infty}(t)$, $k=1,2$, we act as in defining $v_{kj}(\zeta)$ but in this case the use is made of the equation (4.7) and the representation (4.10).

Consider the case where the contour of the circular polygon contains a cut with the vertex b_j, where $\alpha_{1j}-\alpha_{2j}=2$. For this case, P.Ya. Polubarinova–Kochina has proved that the solution $v_{2j}(\zeta)$ must contain no logarithmic term. She has also obtained an equation connecting the parameters a_j, c_j [2].

The necessary and sufficient conditions for the absense of the logarithmic term in the solution $v_{2j}(\zeta)$ is of the form [9, 11]

$$\gamma^1_{1j}(\alpha_{2j})f_{1j}(\alpha_{2j}+1)+f_{2j}(\alpha_{2j})=0, \tag{4.18}$$

where $\gamma^1_{1j}(\alpha_{2j})$ is defined by (4.12), and $f_{1j}(\alpha_{2j}+1)$, $f_{2j}(\alpha_{2j})$ by (4.15).

After some transformations, (4.18) takes the form $q_{2j}+q^2_{1j}+q_{1j}p_{1j}=0$.

To construct $v_{2j}(\zeta)$ for the cut end, it suffices to calculate $\gamma^2_{2j}(\alpha_{2j})$. The other coefficients $\gamma^2_{nj}(\alpha_{2j})$, $n=1,3,4,5,\dots$ are calculated by the formulas (4.14). The equation (4.13) is fulfilled under the condition (4.18), since $f_{0j}(\alpha_{2j}+2)=f_{0j}(\alpha_{1j})=0$. To define $\gamma^2_{nj}(\alpha_{2j})$ uniquely, we have to solve (4.13) for any $\alpha_j\neq\alpha_{2j}$ with respect to $\gamma_{2j}(\alpha_j)$:

$$\gamma_{2j}(\alpha_j)=\frac{-[\gamma_{1j}(\alpha_j)f_{1j}(\alpha_j+1)+\gamma_{0j}f_{2j}(\alpha_j)]}{f_{0j}(\alpha_j+2)}. \tag{4.19}$$

In (4.19), the numerator and the denominator vanish as $\alpha_j\to\alpha_{2j}$. Hence there is an indeterminacy. Uncovering the indeterminacy by the L'Hospital rule, we get $\gamma^2_{2j}=-0,5[p_{1j}(p_{1j}+2q_{1j})+p_{2j}]$.

5. Local Matrices

From the set of branches of the functions $\exp[\alpha_{kj}\ln(t-a_j)]$ appearing in the local solutions $v_{kj}(\zeta)$ we choose as follows:

$$\exp[\alpha_{kj}\ln(t-a_j)]>0,\quad t>a_j,$$
$$[\exp[\alpha_{kj}\ln(t-a_j)]]^{\pm}=\exp[\pm i\pi\alpha_{kj}]\exp[\alpha_{kj}\ln(a_j-t)],\quad t<a_j;$$
$$\exp[-\alpha_{k\infty}\ln(-t)]^{\pm}>0,\quad -\infty<t<a_1;$$
$$[\exp[-\alpha_{k\infty}\ln(t-a_j)]]^{\pm}=\exp[\pm i\pi(-\alpha_{k\infty})]\exp[-\alpha_{k\infty}\ln t],\quad a_m<t<+\infty.$$

Besides the matrices (4.4), let us introduce the matrices

$$\Theta^*_j(t)=\begin{pmatrix} v_{1j^*(t)} & {v'_{1j}}^*(t)\\ v_{2j^*(t)} & {v'_{2j}}^*(t)\end{pmatrix},\quad a_{j-1}<t<a_j,$$

where

$$v^*_{kj}(t)=(a_j-t)^{\alpha_{kj}}\gamma_{0j}\widetilde{v}_{kj}(t),\quad v'^*_{kj}(t)=-(a_j-t)^{\alpha_{kj}}\gamma_{0j}\widetilde{v}'^*_{kj}(t),$$
$$v'_{kj}(t)=\frac{d}{dt}[v_{kj}(t)],\quad \widetilde{v}'^*_{kj}(t)=\alpha_{kj}+\sum_{n=1}^{\infty}\gamma^k_{nj}(\alpha_{kj}+n)(t-a_j)^n.$$

Between the matrices $\Theta_j(t)$ and $\Theta^*_j(t)$ there is a connection

$$\Theta^{\pm}_j(t)=\theta^{\pm}_j\Theta^*_j(t),\ \ a_{j-1}<t<a_j,\ \ \Theta^{\pm}_{\infty}(t)=\theta^{\pm}_{\infty}\Theta^*_{\infty}(t),\ \ a_m<t<+\infty.$$

Here the matrices $\theta_j^{\pm}$ for $\alpha_{1j} - \alpha_{2j} \neq s$, $s = 0, 1, 2$, are defined by

$$\theta_j^{\pm} = \begin{pmatrix} \exp(\pm i\pi\alpha_{1j}) & 0 \\ 0 & \exp(\pm i\pi\alpha_{2j}) \end{pmatrix},$$

while for $\alpha_{1j} - \alpha_{2j} = s$, $s = 0, 1, 2$, by

$$\theta_j^{\pm} = e^{\pm i\pi\alpha_{2j}} \begin{pmatrix} 1 & 0 \\ \pm\pi i & 1 \end{pmatrix}.$$

For the cut end $w = b_j$, the matrices $\theta_j^{\pm}$ are defined as follows. If the characteristic numbers are of the form $\alpha_{1j} = 3/2$, $\alpha_{2j} = -1/2$, then $\theta_j^{\pm} = \mp iE$, where E is the unit matrix, and if $\alpha_{1j} = 2$, $\alpha_{2j} = 0$ then $\theta_j^{\pm} = E$.

The elements of the matrix $\theta_j^*(t)$ containing the logarithmic terms are defined by

$$v_{2j}^*(t) = \gamma_{0j}\Big\{(a_j - t)^{\alpha_{2j}}[(t - a_j)^s \widetilde{v}_{1j}(t)\ln(a_j - t) + \widetilde{v}_{2j}^2(t)],$$

$$v_{2j}'^*(t) = -\gamma_{0j}(a_j - t)^{\alpha_{2j}-1}[(a_j - t)^s e^{i\pi s}\widetilde{v}_{2j}'(t)\ln(a_j - t) + \widetilde{v}_{1j}(t)] + \widetilde{v}_{2j}^2(t)\Big\}.$$

In the local solutions $v_{kj}(\zeta)$, $\varphi_{kj}(\zeta)$ there appear the constants γ_{0j}, φ_{0j} which are defined by means of the Liouville formula

$$\gamma_{0j} = \Big\{\prod_{k=1,k\neq j}^{m} |\nu_j|\,|a_j - a_k|^{1-\nu-k}\Big\}^{-1/2}, \quad \varphi_{0j} = \Big\{\prod_{k=1}^{m} |a_j^* - a_k|^{1-\nu_k}\Big\}^{-1/2}.$$

If $\nu_j = 0$, then we take $|\nu_j| = 1$.

6. Construction of the Fundamental Matrix

Construct the matrix

$$\chi(\zeta) = \begin{pmatrix} u_1(\zeta) & u_1'(\zeta) \\ u_2(\zeta) & u_2'(\zeta) \end{pmatrix}, \tag{6.1}$$

where $u_1(\zeta)$ and $u_2(\zeta)$ are linearly independent solutions of (3.6).

The convergence domains of the matrices $\Theta_j(t)$ and $H_j(t)$ have always a common part in which we can write the equalities

$$\Theta_j^*(t) = T_j^* H_j(t), \quad H_j(t) = T_{0j}\Theta_{j-1}(t), \quad a_{j-1} < t < a_j, \tag{6.2}$$

$$\Theta_1^*(t) = T_{-m}H_{-m}(t), \quad H_{-m}(t) = T_{-\infty}\Theta_\infty(t), \quad -\infty < t < a_1, \tag{6.3}$$

$$\Theta_\infty^*(t) = T_\infty\Theta_m(t), \quad a_m < t < +\infty,$$

where T_j^*, T_{0j}, T_{-m}, $T_{-\infty}$, T_∞ are the constant real matrices defined from the equalities (6.2) and (6.3). Note that in these equalities t may be fixed arbitrarily in the domains where both local matrices occurring in the above-mentioned equalities converge.

Define the matrix (6.1) along the t-axis of the plane ζ as follows:

$$
\begin{aligned}
&\chi^{\pm}(t) = T\Theta_m^{\pm}(t), \quad \Theta_m^{+}(t) = \Theta_m^{-}(t), \quad a_m < t < +\infty;\\
&\chi^{\pm}(t) = T\theta_m^{+}\Theta_m^{*}(t), \quad a_{m-1} < t < a_m;\\
&\chi^{\pm}(t) = T\theta_m^{\pm}T_m\Theta_{m-1}(t), \quad T_m = T_m^{*}T_{0m}, \quad a_{m-1} < t < a_m;\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\chi^{\pm}(t) = T\theta_m^{\pm}T_m \cdots T_1\theta_1^{\pm}\Theta_1^{*}(t), \quad -\infty < t < a_1;\\
&\chi^{\pm}(t) = T\theta_m^{\pm}T_m \cdots \theta_1^{\pm}\Theta_{-m}T_{-\infty}\Theta_{\infty}(t), \quad -\infty < t < a_1;\\
&\chi^{\pm}(t) = T\theta_m^{\pm}T_m \cdots \theta_{\infty}^{\pm}T_{\infty}\Theta_{\infty}^{\pm}(t), \quad a_m < t < \infty.
\end{aligned}
\tag{6.4}
$$

The signs $(\pm)$ in the matrices (6.4) denote the limiting values of the matrix $\chi(\zeta)$ in the upper and in the lower half-planes, respectively. The matrix T is defined by (4.3).

7. Solution of the Boundary Value Problem

Direct checking shows that the matrices (6.4) satisfy the equation (4.1). Therefore, the parameters a_j, c_j, $j = 1, 2, \dots, m$, p, q, r, s being chosen appropriately, the same matrices must satisfy the boundary condition (3.5). Indeed, we begin our proof with the interval $(a_m, +\infty)$. We have

$$
\begin{gathered}
T\Theta_m^{+}(t) = G_m T\Theta_m^{-}(t), \;\; \Theta_m^{+}(t) = \Theta_m^{-}(t), \;\; G_m = E, T = \overline{T},\\
a_m < t < +\infty.
\end{gathered}
\tag{7.1}
$$

For the interval (a_{m-1}, a_m) we obtain

$$
T\theta_m^{+}\Theta_m^{*}(t) = G_{m-1}T\theta_m^{-}\Theta_m^{*}(t), \quad a_{m-1} < t < a_m. \tag{7.2}
$$

The equations (7.1) and (7.2) result in

$$
(\theta_m^{+})^2 = T^{-1}G_m^{-1}G_{m-1}T \tag{7.3}
$$

from which we can see that the matrices $(\theta_m^{+})^2$, $G_m^{-1}G_{m-1}$ are similar.

In a similar way we find the matrix equations for the remaining points $\zeta = a_j$, $j = m-1, m-2, \dots, 2, 1$, $\zeta = \infty$. We have

$$
T\theta_m^{+}T_m\theta_{m-1}^{+} = G_{m-2}T\theta_m^{-}T_m\theta_{m-1}^{-}, \tag{7.4}
$$

$$
T\theta_m^{+}T_m\theta_{m-1}T_{m-1}\theta_{m-2}^{+} = G_{m-3}T\theta_m^{-}T_m\theta_{m-1}^{-}T_{m-1}\theta_{m-2}^{-}, \tag{7.5}
$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$
\begin{gathered}
T\theta_m^{+}T_m\theta_{m-1}^{+}T_{m-1}\theta_{m-2}^{+}T_{m-2}\dots T_1\theta_1^{+} =\\
= G_{m+1}T\theta_m^{-}T_m\theta_{m-1}^{-}T_{m-1}\theta_{m-2}^{-}T_{m-2}\dots T_1\theta_1^{-},
\end{gathered}
\tag{7.6}
$$

$$
T\theta_m^{+}T_m\theta_{m-1}^{+}T_{m-1}\dots T_{-\infty}\theta_m^{+} = G_mT\theta_m^{-}T_m\theta_{m-1}^{-}T_{m-1}\dots T_{-\infty}\theta_{\infty}^{-}. \tag{7.7}
$$

The matrix equations (7.4)–(7.7) can be written as

$$(\theta_{m-1}^{+})^2 = T_m^{-1}(\theta_m^{+})^{-1}T^{-1}G_{m-1}^{-1}G_{m-2}T\theta_m^{-}T_m. \tag{7.8}$$

Here we make the following remark. In composing the matrix equations, we have to into account that two neighboring circular arcs forming a cut with the end $w = b_j$ belong to one circle.

We rewrite (7.3) as $T\theta_m^{+} = G_{m-1}T\theta_m^{-}$, whence

$$p\exp(i\pi\alpha_{1m}) = \overline{B}_{m-1}p\exp(-i\pi\alpha_{1m}) - iD_{m-1}r\exp(-i\pi\alpha_{1m}), \tag{7.9}$$

$$r\exp(i\pi\alpha_{1m}) = iA_{m-1}p\exp(-i\pi\alpha_{1m}) + B_{m-1}r\exp(-i\pi\alpha_{1m}), \tag{7.10}$$

$$q\exp(i\pi\alpha_{2m}) = \overline{B}_{m-1}q\exp(-i\pi\alpha_{2m}) - iD_{m-1}s\exp(-i\pi\alpha_{2m}), \tag{7.11}$$

$$s\exp(i\pi\alpha_{2m}) = iA_{m-1}q\exp(-i\pi\alpha_{2m}) + B_{m-1}s\exp(-i\pi\alpha_{2m}). \tag{7.12}$$

If we divide the corresponding parts of the equalities (7.9), (7.10) and (7.11), (7.12), we can see that the ratios p/r and q/s in the interval (a_{m-1}, a_m) satisfy the boundary condition (3.5),

$$\frac{p}{r} = \frac{\overline{B}_{m-1}p/r - iD_{m-1}}{iA_{m-1}p/r + B_{m-1}}, \quad \frac{q}{s} = \frac{\overline{B}_{m-1}q/s - iD_{m-1}}{iA_{m-1}q/s + B_{m-1}}.$$

The coordinates of the points $w = b_m$ and $w = b'_m$ also satisfy this condition; hence

$$p/r = b_m, \quad q/s = b'_m, \tag{7.13}$$

where b'_m is the second intersection point of the two neighboring circles.

Take advantage of the remark made at the beginning of Section 1. The origin on the plane w coincides with the point $w = b_m$. Therefore $b_m = 0$ and $b'_m = \infty$. Hence $p = 0$, $s = 0$.

Remind that by b_k, b'_k, $k = 1, 2, \ldots, m+1$ we have denoted the complex coordinates of the angular points of the circular polygon at which two neighboring circles may intersect, and b'_k is more often exterior to the contour $l(w)$.

Note that if (a_{m-1}, a_m), then for the interval $\nu_m \neq 0$ we can always suppose that

$$G_{m-1} = \begin{pmatrix} \overline{B}_{m-1} & 0 \\ 0 & B_{m-1} \end{pmatrix}.$$

Consider the matrix equation (7.4),

$$T^{+}_{*m\theta_{m-1}} = G_{m-2}\overline{T}_{*m}\theta_{m-1}^{-}, \quad T_{*m} = T\theta_m^{+}T_m. \tag{7.14}$$

From (7.14) there follows the following system of equations:

$$p_{*m}/r_{*m} = b_{m-1}, \quad q_{*m}/s_{*m} = b'_{m-1}, \tag{7.15}$$

where p_{*m}, q_{*m}, r_{*m}, s_{*m} are the elements of the matrix T_{*m}.

Taking into account (7.14), we rewrite the equalities (7.15) as

$$\frac{p_*p_m + q_*r_m}{r_*p_m + s_*r_m} = b_{m-1}, \quad \frac{p_*q_m + q_*s_m}{r_*q_m + s_*s_m} = b'_{m-1}, \tag{7.16}$$

where p_*, q_*, r_*, s_* are the elements of the matrix $T_* = T\theta_m^+$.

Bearing in mind (7.13), the equalities (7.16) can be written as

$$\frac{r_*p_m b_m + s_*r_m b'_m}{r_*p_m + s_*r_m} = b_{m-1}, \quad \frac{r_*q_m b_m + s_*s_m b'_m}{r_*q_m + s_*s_m} = b'_{m-1}. \tag{7.17}$$

After simplification, the equalities (7.17) take the form

$$r_*p_m(b_m - b_{m-1}) + s_*r_m(b'_m - b_{m-1}) = 0, \tag{7.18}$$

$$r_*q_m(b_m - b'_{m-1}) + s_*s_m(b'_m - b'_{m-1}) = 0. \tag{7.19}$$

The condition of the compatibility of (7.18) and (7.19) with respect to r_*, s_* is of the form

$$\frac{p_m s_m}{r_m q_m} = \frac{b'_m - b_{m-1}}{b_m - b_{m-1}} \frac{b_m - b'_{m-1}}{b'_m - b'_{m-1}}. \tag{7.20}$$

From the matrix equation (7.5) we get the system of equations:

$$\begin{aligned} &\frac{p_{*(m-1)}p_{m-1} + q_{*(m-1)}r_{m-1}}{r_{*(m-1)}p_{m-1} + s_{*(m-1)}r_{m-1}} = b_{m-2}, \\ &\frac{p_{*(m-1)}q_{m-1} + q_{*(m-1)}s_{m-1}}{r_{*(m-1)}q_{m-1} + s_{*(m-1)}s_{m-1}} = b'_{m-2}, \end{aligned} \tag{7.21}$$

where $p_{*(m-1)}$, $q_{*(m-1)}$, $r_{*(m-1)}$, $s_{*(m-1)}$ are the elements of the matrix $T_{*(m-1)} = T\theta_m^+ T_m \theta_m \theta_{m-1}^+$.

After some transformations, (7.21) can be rewritten in the form

$$r_{*(m-1)}p_{m-1}(b_{m-1} - b_{m-2}) + s_{*(m-1)}r_{m-1}(b'_{m-1} - b_{m-2}) = 0,$$

$$.r_{*(m-1)}q_{m-1}(b_{m-1} - b'_{m-2}) + s'_{*(m-1)}s_{m-1}(b'_{m-1} - b'_{m-2}) = 0.$$

The above equalities imply

$$\frac{p_{m-1}s_{m-1}}{r_{m-1}q_{m-1}} = \frac{b'_{m-1} - b_{m-2}}{b_{m-1} - b_{m-2}} \frac{b_{m-1} - b'_{m-2}}{b'_{m-1} - b'_{m-2}}. \tag{7.22}$$

All the matrix equations can be considered analogously.

The equations (7.20) and (7.22) are exactly the invariant anharmonic ratios of the four points of the circle at which it intersects two neighboring circles.

From the matrix equations one can obtain all the required equations with respect to a_k, c_k and to the integration constants p, q, r, s. For every point $\zeta = a_j$ we obtain a system of two equations which are homogeneous with respect to the elements of the matrix T_k. Their conditions of compatibility, for example, for the points $\zeta = a_m$, $\zeta = a_{m-1}$, are of the form (7.20) and

(7.22). These conditions have been obtained under the assumption that $\alpha_{1j} - \alpha_{2j} \neq s$, $s \neq 0, 1, 2$.

Consider now the case where $\alpha_{1j} - \alpha_{2j} = s$, $s = 0, 1, 2$.

According to the representation (6.4), the unknown matrices $\chi^+(t)$, $\chi^-(t)$ for the interval (a_{j-1}, a_j) must satisfy the boundary condition

$$\begin{pmatrix} p_{*j} & q_{*j} \\ r_{*j} & s_{*j} \end{pmatrix} e^{i\pi\alpha_{2j}} \begin{pmatrix} 1 & 0 \\ \pi i & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} \overline{B_{j-1}} & -iD_{j-1} \\ iA_{j-1} & B_{j-1} \end{pmatrix} \begin{pmatrix} \overline{p}_{*j} & \overline{q}_{*j} \\ \overline{r}_{*j} & \overline{s}_{*j} \end{pmatrix} e^{-i\pi\alpha_{2j}} \begin{pmatrix} 1 & 0 \\ -\pi i & 1 \end{pmatrix},$$

where p_{*j}, q_{*j}, r_{*j}, s_{*j} are defined by (6.4).

Reasoning as when deducing (7.1)–(7.8), we see that the ratios $\frac{p_{*j}+\pi i q_{*j}}{r_{*j}+\pi i s_{*j}}$, $\frac{q_{*j}}{s_{*j}}$ satisfy the boundary condition (3.5). But the coordinates of the point $w = b_j$ and the coordinates b_{j-1} and b'_{j-1} will also satisfy (3.5). Hence we obtain the system of equations

$$\frac{p_{*j} + \pi i q_{*j}}{r_{*j} + \pi i s_{*j}} = b_j, \quad \frac{q_{*j}}{s_{*j}} = b_j^*, \tag{7.23}$$

where b_j^* is equal either to b_{j-1} or to b'_{j-1}.

The system (7.23) is also homogeneous with respect to the elements of the corresponding matrices T_{*j}, but the compatibility condition this time fails to provide us with the ratios like (7.20) and (7.22).

As is mentioned above, the matrix equations similar to (7.1)–(7.7) can be obtained for all points $\zeta = a_k$, with the exclusion of those $\zeta = a_n$ to which there correspond the cut ends $w = b_j$ with $\nu_j = 2$. For such points there are the conditions for the absence of logarithmic terms in the solutions $v_{2j}(\zeta)$, for example, the equation (4.18). This gives us one condition for one point; the second equation will be given below.

From the matrix representations $\chi^+(t)$ we define first $u_1^+(t)$, $u_2^+(t)$ and then construct the relation $w^+(t) = u_1^+(t)/u_2^+(t)$.

According to the representation (6.4), let the function (a_j, a_{j+1}) for the interval $w^+(t)$ be defined by $w^+(t) = \frac{A_j^* v_{1j}^+(t) + B_j^* v_{2j}^+(t)}{C_j^* v_{1j}^+(t) + D_j^* v_{2j}(t)}$. If, using this, we calculate the limit as $\zeta \to a_j$, then we will the equation

$$b_j = B_j^*/D_j^*. \tag{7.24}$$

The corresponding equations for the other points $\zeta = a_k$, $k = 1, 2, 3, \ldots, m, m+1$, can be obtained in a similar way.

Consequently, for every point $t = a_j$ we obtain two real equations, homogeneous with respect to p_j, q_j, $r_j s_j$, for example, (7.3)–(7.7). For $\nu_j \neq 0, 1, 2$, from the condition of compatibility of homogeneous equations there follow invariant anharmonic ratios for four points of a circle, for example, (7.20), (7.22). In the case where $\nu_j = 0, 1, 2$, the condition of compatibility of the two equations provides us with well-defined, but not anharmonic, ratios.

Finally, we can take from every system of two equations one equation and add one more equation of compatibility , i.e., we will have two equations for every point $\zeta = a_j$. The number of equations will be $2(m+1)$ and the number of unknown parameters a_k, c_k, p, q, r, s, $ps - rq = 1$, will be equal to $2m-1$. Hence the number of equations will be greater by three than that of the unknown parameters. This is connected with the fact that the going around the singular points $\zeta = a_k$, $k = 1, 2, \dots, n$ in the positive direction is equivalent to that of the point $\zeta = \infty$ in the negative direction. This provides us with one matrix equation. Therefore any three equations from the obtained system of equations are consequences of the remaining ones.

The appearance of the three additional equations can be explained as in the case of linear polygons.

Having found the system of equations to determine a_k, c_k, p, q, r, s, it is necessary to define the intervals of variation of the parameters c_k, to solve the system with respect to a_k, c_k and finally to determine p, q, r, s. Remind that p_j, q_j, r_j, s_j, $j = 1, 2, \dots, m+1$, depend implicitly on the parameters a_k, c_k, $k = 1, 2, \dots, m$ via the coefficients of the generalized hypergeometric series . The variation intervals of the parameters c_k, $k = 1, 2, \dots, m$, can be defined according to [16].

It is known that the series $v_{kj}(\zeta)$ and $j = 1, 2, \dots, m, m+1$, converge near the points $\zeta = a_j$, and $j = 1, 2, \dots, m, m+1$, respectively. The convergence radii $\varphi_{kj}(\zeta)$ of these series are determined by the distance $a_j^* = (a_j + a_{j+1})/2$ from the point $t = a_j$ (or from the point a_j^*) to the nearest points $\zeta = a_{j-1}$, $\zeta = a_{j+1}$.

The series v_{kj} are the entire functions of the parameters c_j, $j = 1, 2, \dots, m$, and converge slowly with respect to ζ. This makes numerical calculations very difficult. As n grows, the coefficients sometimes strongly increase, though their multipliers $(\zeta - a_j)^n$, on the contrary, decrease. Computers are unable to multiply γ_{nj}^k by $(t - a_j)^n$ despite the fact that these series converge. To eliminate this defect, we suggest to write these series in the form of rapidly and uniformly converging functional series.

Consider the structure of recursive formulas (4.12)–(4.14). The sum of the first lower indices in the expressions $\gamma_{(k-n)j} f_{nj}(\alpha + k - n)$ is always equal to k, i.e., to the exponent $(t - a_j)^k$. Instead of the series (4.9), let us consider the functional series of the form

$$\begin{aligned} v_j(t) &= (t - a_j)^{\alpha_j} \widetilde{v}_j(t - a_j), \\ \widetilde{v}_j(t) &= \sum_{n=0}^{\infty} = \sum_{n=0}^{\infty} \gamma_{nj}(t - a_j), \end{aligned} \tag{7.25}$$

where according to (4.12)–(4.14) γ_{nj} is defined via $\gamma_{1j}, \gamma_{2j}, \dots, \gamma_{(n-1)j}$ while the latters are defined via $f_{kj}(\alpha_j)$, where

$$f_{kj}(t-a_j, \alpha_j) = \alpha_j p_{kj}(t-a_j) + q_{kj}(t-a_j),$$

$$p_{nj}(t-a_j) = \sum_{k=1, k\neq j} (-1)^n (1-\nu_k)\Big(\frac{t-a_j}{a_j-a_k}\Big)^n, \quad p_{oj} = 1-\nu_k,$$

$$q_{nj}(t-a_j) = \sum_{k=1, k\neq j} (-1)^{n-1} c_k \Big(\frac{t-a_j}{a_j-a_k}\Big)^n, \quad q_{oj}=0, \quad q_{1j}=c_j,$$

$$|t-a_j| < \min\{|a_j-a_{j-1}|, |a_j-a_{j+1}|\}, \tag{7.26}$$

$$\Big|\frac{t-a_j}{a_j-a_k}\Big| < 1, \quad k \neq j. \tag{7.27}$$

We can see from (7.27) that the functional series (7.25) converges in the domain (7.26) more rapidly in comparison with the series (4.8).

The functional series for the point $\zeta = a_{m+1} = \infty$ is constructed analogously. In all the above formulas instead of $v_{kj}(\zeta)$ we will have to substitute the functional series (7.25). It is obvious that the functional series for the ordinary points $t = a_j^*$, $a_j^* = (a_j + a_{j+1})/2$, $j = 1, 2, \dots, m-1$, will also converge uniformly and rapidly.

8. On a Connection Between the Conditions (2.12) and (2.16)–(2.18)

We write the matrix $\chi(\zeta)$ defined by (4.2) as

$$\chi(\zeta) = T\chi_2(\zeta), \tag{8.1}$$

where the constant matrix T is defined by (4.3), and the matrix $\chi_2(\zeta)$ by

$$\chi_2(\zeta) = \begin{pmatrix} v_1(\zeta) & v_1'(\zeta) \\ v_2(\zeta) & v_2'(\zeta) \end{pmatrix},$$

$v_1(\zeta)$ and $v_2(\zeta)$ being the linearly independent solutions of (3.6) along the t-axis are defined by (6.4).

The equality (8.1) implies

$$u_1(\zeta) = pv_1(\zeta) + qv_2(\zeta), \quad u_2(\zeta) = rv_1(\zeta) + sv_2(\zeta). \tag{8.2}$$

The functions $u_1(\zeta)$, $u_2(\zeta)$ are again linearly independent solutions of the equation (3.6) provided $ps - rq \neq 0$, where p, q, r, s are arbitrary complex numbers. Below we will assume that $ps - rq = 1$.

The functions $w_1(\zeta) = v_1(\zeta)/v_2(\zeta)$ and $w(\zeta) = u_1(\zeta)/u_2(\zeta)$ satisfy Schwarz's equation (3.8), where $w_1(\zeta)$ will be its partial and $w(\zeta)$ its general solutions.

Remind also that $w(\zeta) = \omega'(\zeta)/z'(\zeta) = \omega_1(\zeta)/z_1(\zeta) = \omega_2(\zeta)/z_2(\zeta)$, where $\omega_k(s)$, $z_k(\zeta)$, $k = 1, 2$, are defined by (2.15) and (3.3).

Now we present the proof of a theorem proven by us in [13–16]. It can be formulated as follows: if the equality (2.12) holds, then so do the equalities (2.16)–(2.17), and vice versa, (2.16)–(2.17) imply (2.12).

The second part of our theorem is evident, therefore we dwell on proving the first part.

The equality (2.12) with regard for $w(\zeta)=u_1(\zeta)/u_2(\zeta)$ can be rewritten as

$$\frac{u_1(t)}{u_2(t)}=\frac{\overline{B(t)}\,\overline{u_1(t)}-iD(t)\overline{u_2(t)}}{iA(t)\overline{u_1(t)}+B(t)\overline{u_2(t)}},\quad -\infty<t<+\infty.$$

Assume that

$$u_1(t)=\lambda(t)u_1^*(t),\quad u_2(t)=\lambda(t)u_2^*(t),\quad -\infty<t<+\infty,\tag{8.3}$$

where $u_1^*(t)=\overline{B(t)}\,\overline{u_1(t)}-iD(t)\overline{u_2(t)}$, $u_2^*(t)=iA(t)\overline{u_1(t)}+B(t)\overline{u_2(t)}$, $-\infty<t<+\infty$.

If we substitute (8.3) in (3.6), we obtain

$$\lambda''(t)u_1^*(t)+\lambda'(t)[2[u_1^*(t)]'+p_*(t)u_1^*(t)]=0,\quad -\infty<t<+\infty,\tag{8.4}$$

$$\lambda''(t)u_2^*(t)+\lambda'(t)[2[u_2^*(t)]'+p_*(t)u_2^*(t)]=0,\quad -\infty<t<+\infty.\tag{8.5}$$

Multiplying (8.4) by $u_2^*(t)$ and (8.5) by $u_1^*(t)$ and then subtracting the second equality from the first one, one gets

$$2\lambda'(t)[[u_1^*(t)]'u_2^*(t)-[u_2^*(t)]'u_1^*(t)]=0.\tag{8.6}$$

In the braces of (8.6) there is the Wronskian $w^*[u_1^*(t),u_2^*(t)]\neq 0$, therefore (8.6) implies $\lambda'(t)=0$, $-\infty<t<+\infty$, which yields $\lambda(t)=\text{const}$, $t\in(a_j,a_{j+1})$.

Note that

$$w^*[u_1^*(t),u_2^*(t)]=w^*[\overline{u_1(t)},\overline{u_2(t)}]=w^*[u_1(t),u_2(t)],\tag{8.7}$$

since the equality (2.11) holds.

If for (8.3) we calculate the Wronskian with regard for (8.7), then we obtain $\lambda^2(t)=1$, $t\in(a_j,a_{j+1})$, which in its turn, implies $\lambda(t)=\pm1$, $t\in(a_j,a_{j+1})$.

But the functions $A(t)$, $B(t)$, $D(t)$ are defined uniquely from the conditions (1.1)–(1.2), hence $\lambda(t)$ is also defined uniquely.

9. Definition of the Functions $\omega(\zeta)$, $z(\zeta)$

The function $w^+(t)$ along the real t-axis is defined by $w^+(t)=u_1^+(t)/u_2^+(t)$, $-\infty<t<+\infty$, where $u_1^+(t)$, $u_2^+(t)$ are defined by (6.4).

Given $w^+(t)$, we can find $w(\zeta)$ for all $\text{Im}(\zeta)>0$ by [24, 30]

$$w(\zeta)=\frac{1}{\pi}\int\limits_{-\infty}^{+\infty}w^+(t)\frac{\eta dt}{(t-\xi)^2+\eta^2},\quad \zeta=\xi+i\eta.$$

Note that one can construct a canonical matrix for the problem (2.3) and solve the inhomogeneous boundary value problem (2.3) by using the Cauchy type integral. This has been done in our paper [13]. In the present work, we seek for a solution of the inhomogeneous problem (2.3) in a somewhat different way [2].

Multiply the functions $u_1^+(t)$, $u_2^+(t)$ by and $\chi_1^+(t)$, where $\gamma^+(t)$ is defined by (2.23) and $\chi_1^+(t)$ by (3.4).

The matrix $\chi(\zeta)$ defined by (6.1)–(6.10) satisfies the boundary condition (3.5), since the equalities (7.1)–(7.2) are assumed to be fulfilled. This means that the columns of the matrix $\chi(\zeta)$ defined by (6.1)–(6.10) satisfy the boundary condition (3.5).

In order to obtain a sought for solution $\Phi_2(\zeta)$ of the boundary value problem (3.5), we have to take the first column elements of the matrix $\chi(\zeta)$ and construct the vector $\Phi_2(\zeta) = [u_1(\zeta), u_2(\zeta)]$, $\quad \mathrm{Im}(\zeta) \geq 0$.

We have taken the first column elements of the matrix $\chi(\zeta)$ because the ratio $w(\zeta) = u_1(\zeta)/u_2(\zeta)$ provides the general solution of the Schwarz differential equation (3.8), while the ratio $u_1'(\zeta)/u_2'(\zeta)$ does not satisfy Schwarz's equation. This implies that $\omega_2(\zeta) = u_1(\zeta)$, $z_2(\zeta) = u_2(\zeta)$.

The vector $\Phi_1(\zeta) = \chi_1(\zeta)\Phi_2(\zeta)$, where $\chi_1(\zeta)$ is defined by (3.4), satisfies the boundary condition (3.1), and the components of the vector $\Phi_1(\zeta)$ are defined as $\omega_1(\zeta) = \chi_1(\zeta)\omega_2(\zeta)$, $\quad z_1(\zeta) = \chi_1(\zeta)z_2(\zeta)$, $\quad \mathrm{Im}(\zeta) \geq 0$.

The vector $\Phi'(\zeta) = \gamma(\zeta)\Phi_1(\zeta)$, where $\Phi'(\zeta) = d\Phi(\zeta)/d\zeta$, satisfies the boundary condition

$$\Phi'(t) = A_*^{-1}(t)\overline{A_*(t)}\,\overline{\Phi}'(t), \quad -\infty < t < +\infty,$$

where $\gamma(\zeta)$ is defined by (2.23).

Hence, the components of the vector $\Phi'(\zeta)$,

$$\omega'(\zeta) = \gamma(\zeta)\chi_1(\zeta)u_1(\zeta), \quad z'(\zeta) = \gamma(\zeta)\chi_1(\zeta)u_2(\zeta), \quad \mathrm{Im}(\zeta) \geq 0,$$

satisfy the boundary conditions (2.4)–(2.5).

According to [2], we are aware of the behavior of the functions $\omega'(\zeta)$, $z'(\zeta)$ at all singular points $t = e_k$, $k = 1, 2, \ldots, n, n+1$. Therefore the choice of the arguments φ_j, $j = 1, 2, \ldots, n+1$, of the complex numbers $\det A_j(t)$ should be performed with regard for the behavior of the functions $\omega'(\zeta)$, $z'(\zeta)$ at all singular points. In this way we construct uniquely the functions $\omega'(\zeta)$, $z'(\zeta)$. Thus we can write

$$d\omega^+(t) = u_1^+(t)\gamma^+(t)\chi_1^+(t)dt, \quad -\infty < t < +\infty, \tag{9.1}$$

$$dz^+(t) = u_2^+(t)\gamma^+(t)\chi_1^+(t)dt, \quad -\infty < t < +\infty. \tag{9.2}$$

Obviously, the functions (9.1) and (9.2) satisfy the boundary conditions (2.4)–(2.5).

Integrating the equalities (9.1)–(9.2) in the intervals $(-\infty, t)$, (e_j, t), $j = 1, 2, \dots, n$, we obtain

$$\omega^+(t) = \int\limits_{-\infty}^{t} u_1^+(t)\gamma^+(t)\chi_1^+(t)dt + \omega^+(-\infty), \quad -\infty < t < e_1, \tag{9.3}$$

$$z^+(t) = \int\limits_{-\infty}^{t} u_2^+(t)\gamma^+(t)\chi_1^+(t)dt + z^+(-\infty), \quad -\infty < t < e_1, \tag{9.4}$$

$$\omega^+(t)=\int\limits_{e_j}^{t} u_1^+(t)\gamma^+(t)\chi_1^+(t)dt+\omega_j^+(e_j), \ j=1,2,\dots,n, \ e_j<t<e_{j+1}; \tag{9.5}$$

$$z^+(t)=\int\limits_{e_j}^{t} u_2^+(t)\gamma^+(t)\chi_1^+(t)dt+z_j^+(e_j), \ j=1,2,\dots,n, \ e_j<t<e_{j+1}, \tag{9.6}$$

where $\omega^+(-\infty)$, $z^+(-\infty)$, $\omega^+(e_j)$, $z^+(e_j)$ are the right limits of the corresponding functions at the points $-\infty$, e_j, $j = 1, 2, \dots, n$.

It is also evident that the functions $\omega^+(t)$, $z^+(t)$ defined by (9.3)–(9.6) satisfy the boundary conditions (2.1)–(2.2).

In (9.3)–(9.6) we can separate the real and the imaginary parts and obtain the expression for the functions $\varphi(t)$, $\psi(t)$, $x(t)$, $y(t)$.

Moreover, taking $t = e_1$ in (9.3)–(9.4) and $t = e_{j+1}$ in (9.5) and (9.6), we get

$$\omega^+(e_1) = \int\limits_{-\infty}^{e_1} u_1^+(t)\gamma^+(t)\chi^+(t)dt + \omega^+(-\infty), \tag{9.7}$$

$$z^+(e_1) = \int\limits_{-\infty}^{e_1} u_1^+(t)\gamma^+(t)\chi_1^+(t)dt + z^+(-\infty), \tag{9.8}$$

$$\omega^+(e_{j+1}) = \int\limits_{e_j}^{e_{j+1}} u_1^+(t)\gamma^+(t)\chi_1^+(t)dt + \omega^+(e_j), \quad j = 1, 2, \dots, n, \tag{9.9}$$

$$z^+(e_{j+1}) = \int\limits_{e_j}^{e_{j+1}} u_2^+(t)\gamma^+(t)\chi_1^+(t)dt + z^+(e_j), \quad j = 1, 2, \dots, n, \tag{9.10}$$

where $\omega^+(e_{j+1})$, $z^+(e_{j+1})$ are the left limits of the functions $\omega^+(t)$, $z^+(t)$ at the point $t = e_{j+1}$.

In (9.3)–(9.6) the integrands are supposed to be integrable at the left ends of the intervals. If it is not the case, then we can take as the lower limits either the right end or an interior point of the corresponding interval.

For the unknown parameters a_j, c_j appearing in (3.6), we have obtained a system of higher transcendental equations, e.g., the equation (7.24). As to the parameters $t = e_j$ not coinciding with the parameters $t = a_j$ and which functions $\gamma(\zeta)$ and $\chi_1(\zeta)$ depend on, and the parameter Q connected with the discharge of the fluid, we have obtained the system (9.7)–(9.10) for their determination.

Having found all the unknown parameters which the functions $u_1^+(t)$, $u_2^+(t)$, $\gamma^+(t)$, $\chi_1^+(t)$ depend on, by (9.3)–(9.6) we can determine the equations of the unknown parts of the boundary of the domains $s(z)$, $s(\omega)$, $s(w)$ as well as other geometric and mechanical parameters of the fluid flow.

References

1. P. Ya. Polubarinova-Kochina, Application of the theory of linear differential equations to some problems of motion of ground water. (Russian) *Izv. Akad. Nauk SSSR, Ser. Mat.,* No. 5–6, 1939, 579–602.
2. P. Ya. Polubarinova-Kochina, Theory of the motion of ground water. (Russian) *Nauka, Moscow,* 1977.
3. P. Ya. Polubarinova-Kochina, On additional parameters by examples of circular quadrangles. (Russian) *App. Math. Mech.* **55**(1991), No. 2, 1991.
4. P. Ya. Polubarinova-Kochina, On circular polygons in the filtration theory. (Russian) *In: Problems of mathematics and mechanics. Novosibirsk, Nauka*, 1983, 166–177.
5. P. Ya. Polubarinova-Kochina, V. G. Pryazhinskaya and V. N. Emikh, Mathematical methods in irrigation problems. (Russian) *Moscow, Nauka,* 1969.
6. J. Bear, D. Zaslavsky, and S. Irmay, Physical and mathematical foundations of water filtration. (Translated from English) *Mir, Moscow*, 1971.
7. G. Korn and T. Korn, Mathematical hanbook for scientists and engineers. *McGraw-Hill Company, New York-Toronto-London*, 1961.
8. N. I. Muskhelishvili, A course of analytic geometry. (Russian) *GITTL, Moscow–Leningrad*, 1947.
9. A. R. Tsitskishvili, On the construction of analytic functions conformally mapping a half-plane onto circular polygons. (Russian) *Differentsial'nye Uravneniya* **21**(1985), No. 4, 646–656.
10. A. R. Tsitskishvili, On the conformal mapping of a half-plane onto circular polygons. (Russian) *Trudy Tbiliss. Gos. Univ. Ser. Mat. Mekh. Astr.* **185**(1977), 65–89.
11. A. R. Tsitskishvili, On conformal mapping of a half-plane onto circular pentagons with cuts. (Russian) *Differentsial'nye Uravneniya* **12**(1967), No. 1, 2044–2051.
12. A. R. Tsitskishvili, A method of explicit solution of a class of plane problems of the filtration theory. (Russian) *Soobshch. Akad. Nauk Gruzii* **142**(1991), No. 2, 285–288.
13. A. R. Tsitskishvili, Application of the theory of linear differential equations to the solution of some plane problems of the filtration theory. (Russian) *Trudy Tbiliss. Gos. Univ., Ser. Mat. Mekh. Astr.* **259**(1986), 19–20, 295–329.
14. A. R. Tsitskishvili, On the filtration in trapezoidal earthen dams. (Russian) *Trudy Tbiliss. Gos. Univ., Ser. Mat. Mekh. Astr.* **210, 8**(1980), 12–39.

15. A. R. Tsitskishvili, Application of the methods of complex analysis to the solution of a class of two-dimensional problems of the filtration theory. (Russian) *In: Boundary value problems of the underground water filtration.* (*Russian*) *Theses of reports of the Republican Scientific-Technical Seminar. Kazan University Press, Kazan,* 1988, 72–73.
16. A. R. Tsitskishvili, Application of I. A. Lappo-Danilevskiĭ's method to finding functions conformally mapping a half-plane onto circular polygons. (Russian) *Differentsial'nye Uravneniya* **10**(1974), No. 3, 458–469.
17. N. I. Muskhelishvili, Singular Integral Equations. (Russian) *Nauka, Moscow,* 1968.
18. N. P. Vekua, Systems of singular integral equations and some boundary value problems. *Nauka, Moscow,* 1970.
19. E. L. Ince, Ordinary differential equations. (Translated from English) *Kharkov,* 1939.
20. A. Hurvitz and R. Courant, Vorlesungen über Allgemeine Funktionentheorie und Ellyptische Funktionen, Geometrische Funktionentheorie. *Springer-Verlag, Berlin-Göttingen-Heidelberg-New York,* 1968.
21. G. N. Goluzin, Geometrical theory of functions of a complex variable. (Russian) *Nauka, Moscow,* 1966.
22. V. V. Golubev, Lectures in the analytical theory of differential equations. (Russian) *GITTL, Moscow–Leningrad,* 1950.
23. E. A. Coddington, N. Levinson, Theory of ordinary differential equations. *Engl. transl. Moscow,* 1958, p.474.
24. V. Koppenfels and Stallmann, Practice of conformal mappings. *Springer-Verlag, Berlin-Göttingen-Heidelberg*, 1963.
25. E. Kamke, Differentialgleichungen. Lösungsmethoden und Lösungen. I Gewönliche Differentialgleichungen *Leipzig,* 1959.
26. E. T. Whittaker and D. N. Watson, A course of modern analysis. *Cambridge University Press, Cambridge*, 1962.
27. G. Sansone, Equazioni differenziali nel campo reale, Parte Prima. *Bologna,* 1953.
28. G. Bateman and A. Erdelyi. Higher transcendental functions. *McGrow-Hill Book Company, New York-Toronto-London,* 1955.
29. I. A. Aleksandrov, Parametric continuations in the theory of schlicht functions. (Russian) *Nauka, Moscow,* 1976.
30. M. A. Lavrent'ev and B. V. Shabat, Methods of the theory of functions of a complex variable. (Russian) *GIFML, Moscow,* 1958.
31. I. I. Privalov, Introduction to the theory of functions of a complex variable. (Russian) *GITTL, Moscow–Leningrad,* 1948.
32. F. R. Gantmakher, Matrix theory. (Russian) *Nauka, Moscow,* 1967.
33. B. B. Devison, The motion of ground water. (Russian) *In: Some new problems of continuum mechanics, Part 3, Izd. AN SSSR, Moscow-Leningrad,* 1938, 217–356.

Chapter III

Connection Between the Solutions of the Schwarz Nonlinear Differential Equation And Those of the Plane Problems of Filtration

Abstract. Using linearly independent solutions of the Fuchs class linear differential equation which contains a term with the first order derivative of the unknown function, we propose effective methods for solving both the Schwarz nonlinear equation, whose right-hand side is a doubled invariant of the Fuchs class linear differential equation, and the plane problems of filtration with partially unknown boundaries. The modulus of the difference of the characteristic numbers of the Fuchs class linear differential equation for every singular point is equal to the corresponding (divided by π) angle at the vertex of a circular polygon. For the first time it is shown that the coefficients at the poles of second order of the doubled invariant of the Fuchs class linear differential equation and those on the right-hand side of the Schwarz equation coincide completely.

Relying on the property mentioned above, we suggest simpler methods of solving the problems of the theory of stationary motion of incompressible liquid in a porous medium with partially unknown boundaries than those described by us earlier for the solution of the same problems.

1. On the Connection Between Solutions of the Fuchs Class Linear Differential Equation of General Type and the Nonlinear Schwarz Differential Equation

The filtration theory uses analytic function $w(z) = u - iv$, $z = x + iy$, where $w(z)$ the is complex velocity, and u and $(-v)$ are its components satisfying the Cauchy-Riemann conditions [1–6].

Let on the plane $w = u - iv$ be given a simply connected domain $s(w)$ with the boundary $l(w)$ consisting, in a general case, of circular arcs of different radii. Such a domain is called a circular polygon. By A_k, $k = \overline{1,m}$, we denote the angular points of the boundary $l(w)$ and by $\pi\nu_k$, $k = \overline{1,m}$, the interior angles, respectively. In the general case it can be assumed that $-2 \leq \nu_i \leq 2$, [1–31].

We seek for an analytic function $w(\zeta)$ which maps conformally the half-plane $\mathrm{Im}(\zeta) \geq 0$ of the plane $\zeta = t + i\tau$ onto the domain $s(w)$ with the boundary $l(w)$. Denote by $t=a_k$, $k=\overline{1,m}$, the points of the axes $t=a_k$, $k=\overline{1,m}$, of the plane $\zeta=t+i\tau$ which are mapped respectively into the points A_k, $k=\overline{1,m}$, where $-\infty<a_1<a_2<\cdots<a_m<+\infty$. The point $t=\infty$ is assumed to be mapped into a nonangular point of the boundary $l(w)$ of $s(w)$, which may lie between the points A_m and A_1, although one can consider the case in which $t=\infty$ is mapped into an angular point A_k.

Using the linear-fractional transformation, we can map A_mA_1, the arc of the circumference of the boundary $l(w)$ of $s(w)$, onto a straight line or onto a part of a straight line, parallel to or coinciding with the real axis $v=0$.

For the sake of brevity, without restriction of generality, from the very beginning we assume that the side A_mA_1 of $l(w)$ is parallel to or coincides with the axis $v=0$. Therefore the function $w(\zeta)$ can always be extended analytically through the intervals $-\infty<t<a_1$, $a_m<t<+\infty$ to the lower half-plane $\operatorname{Im}(\zeta)<0$. Throughout the paper, it will be assumed that if $\zeta\in\operatorname{Re}\zeta$, then $\zeta=t$.

The unknown function $w(\zeta)$ must satisfy the well-known Schwarz equation [12–17]

$$\{w,\zeta\}\equiv w'''(\zeta)/w'(\zeta)-1,5[w''(\zeta)/w'(\zeta)]^2=R(\zeta), \tag{1.1}$$

$$R(\zeta)=\sum_{k=1}^{m}\{0,5(1-\nu_k^2)(\zeta-a_k)^{-2}+c_k(\zeta-a_k)^{-1}\}, \tag{1.2}$$

where a_k and c_k, $k=\overline{1,m}$, are unknown real parameters to be defined later on.

The extension of the function $R(\zeta)$ in the neighborhood of the point $t=\infty$ in terms of the powers of $1/\zeta$ yields

$$R(\zeta)=\sum_{n=1}^{\infty}N_n\zeta^{-n}.$$

The coefficients N_k, $k=1,2,3$, must satisfy the conditions

$$\begin{gathered}N_1=\sum_{k=1}^{m}c_k=0,\quad N_2=\sum_{k=1}^{m}[a_kc_k+0,5(1-\nu_k^2)]=0,\\ N_3=\sum_{k=1}^{m}[a_k^2c_k+a_k(1-\nu_k^2)]=0,\end{gathered} \tag{1.3}$$

because the point $\zeta=\infty$ is the image of a nonangular point of the boundary $l(w)$ [12–16].

According to the Riemann theorem, three of the parameters $t=a_k$, $k=\overline{1,m}$, can be chosen arbitrarily and fixed. From the system of equations (1.3), the parameters c_1, c_2 and c_3 in the system of equations (1.3) can be expressed in terms of the remaining a_k and c_k. Consequently, the number of unknown parameters a_k and c_k is equal to $2(m-3)$.

By substitution $w'(\zeta) = 1/[u(\zeta)]^2$, the equation (1.1) can be reduced to the linear Fuchs class equation

$$u''(\zeta) + 0,5R(\zeta)u(\zeta) = 0. \tag{1.4}$$

By means of linearly independent particular solutions of (1.4), $u_1(\zeta)$ and $u_2(\zeta)$ with the Wronskian $u_1(\zeta)u_2'(\zeta) - u_2(\zeta)u_1'(\zeta) = 1$, we can construct the general solution of (1.1) as follows:

$$w(\zeta) = [Au_1(\zeta) + Bu_2(\zeta)]/[Cu_1(\zeta) + Du_2(\zeta)], \tag{1.5}$$

where A, B, C, D with $AD - BC = 1$ are the integration constants of the equation (1.1).

The general solution (1.5) of the equation (1.1), along with the $2(m-3)$ essential parameters a_k, c_k, $k = \overline{1,m}$, depends in the general case on three unknown complex parameters A, B, C, D with $AD - BC = 1$, i.e. on six real parameters. Thus the number of unknown parameters is equal to $2m$.

The equation of the boundary $l(w)$ of $s(w)$ can be written as

$$w(\zeta) = [\overline{w(\zeta)}B_0 + iD_0]/[-iA_0\overline{w(\zeta)} + \overline{B}_0], \quad \zeta \in l(w), \tag{1.6}$$

where $w = u - iv$, $\overline{w} = u + iv$, $B_0 = (C_0^* + iB_0^*)/2$, $\overline{B}_0 = (C_0^* - iB_0^*)/2$, A_0, B_0^*, C_0^*, and D_0 are the given real piecewise constant functions which, without restriction of generality, satisfy the condition $B_0\overline{B}_0 - A_0D_0 = 1$.

The coordinates of the centers (u_0, v_0) and the radii of the circumferences (1.6) can be determined as follows:

$$\begin{aligned} u_0 &= -B_0^*/[2A_0], \quad V_0 = -C_0^*/[2A_0], \\ R_0 &= \sqrt{[(B_0^*)^2 + (C_0^*)^2 - 4A_0D_0]/A_0^2}. \end{aligned} \tag{1.7}$$

Suppose that we have constructed linearly independent solutions u_1^* and $u_2^*(\zeta)$ with the Wronskian $u_1^*(\zeta)(u_2^*(\zeta))' - (u_1^*(\zeta))'u_2^*(\zeta) = 1$. Then $w(\zeta) = u_1^*(\zeta)/u_2^*(\zeta)$,

$$u_1^*(\zeta)/u_2^*(\zeta) = [B_0\overline{u_1^*}(\zeta) + iD_0\overline{u_2^*}(\zeta)]/[-iA_0\overline{u_1^*}(\zeta) + \overline{B}_0\overline{u_2^*}(\zeta)]. \tag{1.8}$$

The methods of constructing $w(\zeta)$ in the general case were described in our works [25–31].

The differentiation of (1.8) yields

$$1/[u_2^*(\zeta)]^2 = 1/[-iA_0\overline{u_1^*}(\zeta) + \overline{B}_0\overline{u_2^*}(\zeta)]^2. \tag{1.9}$$

The equalities (1.6)–(1.9) imply that

$$u_1^*(\zeta) = \pm[B_0\overline{u_1^*}(\zeta) + iD_0\overline{u_2^*}(\zeta)], \quad u_2^*(\zeta) = \pm[-iA_0\overline{u_1^*}(\zeta) + \overline{B}_0\overline{u_2^*}(\zeta)]. \tag{1.10}$$

In [24], we have proved the equality (1.10) in somewhat different way. The signs + and − are fixed uniquely by means of the boundary conditions.

Let us consider the Fuchs class second order differential equation [14–16]

$$v''(\zeta) + p(\zeta)v'(\zeta) + q(\zeta)v(\zeta) = 0, \tag{1.11}$$

where

$$p(\zeta) = \sum_{j=1}^{m} [1 - (\alpha_{1j} + \alpha_{2i})](\zeta - a_i)^{-1},$$
$$q(\zeta) = \sum_{j=1}^{m} [\alpha_{1j}\alpha_{2i}(\zeta - a_j)^{-2} + c_j^*(\zeta - a_j)^{-1}]. \tag{1.12}$$

For the points $t = a_j$, $j = \overline{1, m}$, $t = \infty$ to be regular singular points, it is necessary and sufficient that $p(\zeta)$ and $q(\zeta)$ have the form (1.12) and the parameters c_j^*, $j = \overline{1, m}$, satisfy the condition [11–20]

$$M_1 = \sum_{k=1}^{m} c_k^* = 0. \tag{1.13}$$

Suppose that the parameters a_j, α_{kj}, c_j^*, $k = 1, 2$, $j = \overline{1, m}$, are real and $t = a_j$, $j = \overline{1, m}$, are the same as in (1.2). Using the linearly independent particular solutions (1.1) $v_1(\zeta)$ and $v_2(\zeta)$, we construct the general solution of the Schwarz equation

$$w(\zeta) = [A_1 w_1(\zeta) + B_1]/[C_1 w_1(\zeta) + D_1], \tag{1.14}$$

where $w_1(\zeta) = v_1(\zeta)/v_2(\zeta)$ is a particular solution of the Schwarz equation with the right-hand side equal to

$$\{w, \zeta\} = 2q(\zeta) - p'(\zeta) - 0,5[p(\zeta)]^2, \tag{1.15}$$

and A_1, B_1, C_1, D_1, $A_1 D_1 - B_1 C_1 \neq 0$ are the integration constants of (1.14).

The Wronskian for (1.11) has the form

$$v_{1j}(\zeta)v'_{2j}(\zeta) - v'_{1j}(\zeta)v_{2j}(\zeta) = c_{*j} \prod_{j=1}^{m} (\zeta - a_j)^{\alpha_{1j} + \alpha_{2j} - 1}. \tag{1.16}$$

In [14, p. 300] it is stated that to reduce the right-hand side of (1.15) to the function $R(\zeta)$ appearing in (1.2), we have to choose two functions $p(\zeta)$ and $q(\zeta)$ which make the problem indeterminate. In [14], the author considers the linear second order equation of general type. But if one takes (1.11), where α_{1j}, α_{2j}, $j = \overline{1, (m+1)}$, satisfy the conditions

$$\alpha_{1j} - \alpha_{2j} = \nu_i, \quad j = \overline{1, m}, \quad \alpha_{1(m+1)} - \alpha_{2(m+1)} = 1, \quad t = a_{m+1} = \infty,$$
$$\alpha_{1(m+1)} = 3, \quad \alpha_{2(m+1)} = 2, \quad \sum_{k=1}^{m} [1 - (\alpha_{1j} + \alpha_{2i})] = 6, \tag{1.17}$$

then the right-hand side of (1.15) is, as it can be directly verified, represented in the form

$$\{w, \zeta\} = 2q(\zeta) - p'(\zeta) - 0,5[p(\zeta)]^2 =$$

$$= \sum_{j=1}^{m} \{0,5[1-(\alpha_{1i}-\alpha_{2i})^2](\zeta-a_j)^{-2} + c_j^{**}(\zeta-a_j)^{-1}\}, \tag{1.18}$$

where

$$c_j^{**} = 2c_j^* - \beta_j \sum_{k=1, k\neq j}^{m} \beta_k (a_j - a_k)^{-1}, \quad \beta_k = 1-(\alpha_{1k}+\alpha_{2k}), \quad k=\overline{1,m}. \tag{1.19}$$

Since $\alpha_{1j} - \alpha_{2j} = \nu_i$, $j = \overline{1,m}$, the coefficients at $(\zeta - a_j)^{-2}$ in (1.2) and (1.18) coincide.

The expansion of the function $2q(\zeta) - p'(\zeta) - 0,5[p(\zeta)]^2$ in the neighborhood of the point $\zeta = \infty$ into the series with respect to the powers $1/\zeta$ results in

$$2q(\zeta) - p'(\zeta) - 0,5[p(\zeta)]^2 = \sum_{k=1}^{m} M_k^* \zeta^{-k}. \tag{1.20}$$

The point $\zeta = \infty$ is not a branching point of (1.11), therefore the conditions

$$\begin{gathered} M_1^* \equiv \sum_{j=1}^{m} c_j^{**} = 0, \quad M_2^* = \sum_{k=1}^{m} [a_k c_k^{**} + 0,5(1-\nu_k^2)] = 0, \\ M_3^* = \sum_{k=1}^{m} [a_k^2 c_k^{**} + a_k(1-\nu_k^2)] = 0 \end{gathered} \tag{1.21}$$

must be fulfilled.

The condition $M_1^* = 0$ coincides with (1.13). Below we will see that the last two equations of (1.21) can be obtained in somewhat different, natural way.

As is known, an equation of the type (1.11) can be reduced to that of the type (1.4). The expression $q(\zeta) - 0,5p(\zeta) - 0,25[p(\zeta)]^2$ is, in a certain sense, an invariant of (1.4) [23, p. 243]. Indeed, using the substitution $v(\zeta) = \exp[-0,5\int p(\zeta)d\zeta]v_0(\zeta)$ [23], we reduce the equation (1.11) to the type

$$v_0''(\zeta) + (q(\zeta) - 0,25p^2(\zeta) - 0,5p'(\zeta))v_0(\zeta) = 0. \tag{1.22}$$

If the characteristics α_{1j}, α_{2j}, $j = \overline{1,m}$, of the equation (1.11) satisfy the conditions $\alpha_{1j} + \alpha_{2j} = 1$, $j = \overline{1,m}$, then $p'(\zeta) = 0$, $p(\zeta) = 0$ and hence $R(\zeta) = 2q(\zeta)$, $2c_j^* = c_j$, $j = \overline{1,m}$.

The parameters α_{1j} and α_{2j} in the case of the equation (1.1) are defined by the equalities $\alpha_{1j} = 0,5(1+\nu_j)$, $\alpha_{2j} = 0,5(1-\nu_j)$, $\alpha_{1j} + \alpha_{2j} = 1$, $\alpha_{1j} - \alpha_{2j} = \nu_j$, $j = \overline{1,m}$.

In (1.6) there take place indeterminate constants c_{*j}, $j = \overline{1,m}$, which can be defined by the equality (1.16).

Indeed, if we divide both sides of the equality (1.16), by $(\zeta - a_j)^{\alpha_{1j}+\alpha_{2j}-1}$ and then pass to the limit $\zeta \to a_j$, we will get a system of equations for determination of c_{*j}, $j = \overline{1,m}$.

Note here that the equalities (1.10) can be generalized even in the case where $\alpha_{1j} + \alpha_{2j} \neq 1$.

2. Solution of Plane with Partially Unknown Boundaries Problems of Filtration

Consider some plane problems of the theory of stationary motion of incompressible liquid in a porous medium subjected to the Darcy law. The porous medium is assumed to be undeformable, isotropic and homogeneous [1–7].

The plane of the liquid motion coincides with the plane of the complex variable $z = x+iy$. In the domain $s(z)$ with the boundary $l(z)$ we seek for a complex potential $w(z) = \varphi(x,y) + i\psi(x,y)$, where $\varphi(x,y)$ and $\psi(x,y)$ are, respectively, the velocity potential and the stream function which satisfies the boundary conditions given below. The functions $\varphi(x,y)$ and $\psi(x,y)$ are connected by means of the Cauchy-Riemann conditions. If the analytic function $\omega(z)$ is found, then due to the dependencies [1–7]

$$\begin{gathered}\varphi(x,y) = -k(p/\gamma + y) + c, \quad w(z) = u - iv, \\ u = \frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x},\end{gathered} \tag{2.1}$$

where p is the hydrodynamic pressure, γ is the specific weight of the liquid, u and v are the vector components of filtration velocity, $\omega'(z) \equiv w(z)$ is the complex velocity, k is the coefficient of filtration, and c is an arbitrary constant, all the characteristics of the filtration stream can be found, namely: filtration velocity, pressure head, pressure, liquid discharge for filtration and unknown parts of the boundary $l(z)$ of $s(z)$ [1–7; 24–31]. Below we shall consider the reduced complex potential $\omega(z)$, the complex potential divided by the coefficient of filtration. Next we assume that the boundary $l(z)$ of $s(z)$ is a simple, piecewise analytic contour consisting of a finite number of unknown depression curves, segments of straight lines, half-lines and straight lines. The domains $s(z)$, $\omega(z)$ and $w(z) = \omega'(z)$ may be bounded or unbounded. In particular, if the boundary $l(z)$ has no depression curves, then the domain $s(z)$ turns into a linear polygon.

In the domain $s(z)$ we have to find an analytic function $\omega(z) = \varphi(x,y) + i\psi(x,y)$ which must satisfy the boundary conditions [1–7]

$$a_{k1}\varphi(x,y) + a_{k2}\psi(x,y) + a_{k3}x + a_{k4}y = f_k, \quad k = 1,2, \quad (x,y) \in l(z), \tag{2.2}$$

where a_{kj}, f_k, $j = \overline{1,4}$, are the given piecewise constant real functions.

Before we proceed to the solution of the basic problem of filtration, we can determine the boundary $l(w)$ of the domains $s(w)$ and also a part of the boundary $l(\omega)$ of $s(\omega)$ [1–7].

Using the functions $\omega(z)$ and $\omega'(z) = d\omega(z)/dz$, the domain $s(z)$ with the boundary $l(z)$ is mapped conformally respectively onto the domain $s(\omega)$ and $s(w)$ with the boundaries $l(\omega)$ and $l(w)$, where the domain $s(w)$ is a circular polygon with the boundary $l(w)$ consisting of a finite number of circular arcs, segments of straight lines, half-lines and straight lines.

If we take arbitrarily any part of the boundary $l(z)$ of $s(z)$ and differentiate (2.2) along that part of the boundary $l(z)$ with respect to the parameter s, where s is the arc length of the curve, we get

$$(a_{11}u - a_{12}v + a_{13})\cos(x,s) + (a_{11}v + a_{12}u + a_{14})\cos(y,s) = 0, \quad (2.3)$$

$$(a_{21}u - a_{22}v + a_{23})\cos(x,s) + (a_{21}v + a_{22}u + a_{24})\cos(y,s) = 0, \quad (2.4)$$

where $dx/ds = \cos(x,s)$ and $dy/ds = \cos(y,s)$.

For the system (2.3) and (2.4) to have a nontrivial solution with respect to dx/ds and dy/ds, it is necessary and sufficient that the determinant of the system at the given part of the boundary be equal to zero,

$$A_{11}(u^2 + v^2) + A_{12}u + A_{13}v + A_{14} = 0. \quad (2.5)$$

The coefficients a_{kj}, $k = 1, 2$, $j = \overline{1, 4}$, are given by (2.2), and therefore the coefficients A_{11}, A_{12}, A_{13}, and A_{14} are fixed.

The equation (2.5) can be written in a complex form

$$w = \left[B\overline{w} + i2A_{14}\right]\left[-2iA_{11}\overline{w} + \overline{B}\right]^{-1}, \quad (2.6)$$

where $w = u - iv$, $\overline{w} = u + iv$, $B = A_{13} + iA_{12}$, $\overline{B} = A_{13} - iA_{12}$,

$$A_{11} = a_{11}a_{22} - a_{21}a_{12}, \quad A_{12} = a_{11}a_{24} - a_{21}a_{14} + a_{13}a_{22} - a_{23}a_{12},$$
$$A_{13} = a_{14}a_{22} - a_{24}a_{12} + a_{13}a_{21} - a_{23}a_{11}, \quad A_{14} = a_{13}a_{24} - a_{23}a_{14}. \quad (2.7)$$

The coordinates (u_*, v_*) of the center and the radius R_* of the circumference (2.5) for the chosen by us part of the boundary $l(w)$ are defined as follows:

$$u_* = -A_{12}/[2A_{11}], \quad v_* = -A_{13}/[2A_{11}],$$
$$R_* = \frac{1}{2}\sqrt{[A_{12}/A_{11}]^2 + [A_{13}/A_{11}]^2 - 4A_{14}/A_{11}}. \quad (2.8)$$

We can require the condition $B\overline{B} - 4A_{11}A_{14} \neq 0$, and not the condition $B\overline{B} - 4A_{11}A_{14} = 1$ because the parameters a_{kj}, $k = 1, 2$, $j = \overline{1, 4}$, are fixed by the condition (2.2).

To solve the problems of filtration, one usually introduces the plane $\zeta = t + i\tau$ and maps conformally the half-plane $\mathrm{Im}(\zeta) > 0$ onto the domains $s(z)$, $s(\omega)$ and $s(w)$. We denote the conformally mapping functions respectively by $z(\zeta)$, $\omega(\zeta)$ and $w(\zeta) = \omega'(\zeta)/z'(\zeta)$, where $d\omega(\zeta)/d\zeta = \omega'(\zeta)$

and $dz(\zeta)/d\zeta = z'(\zeta)$. B_k, $k=\overline{1,n}$, denote angular points of the boundary $l(z)$, $l(\omega)$ and $l(w)$ of the domains $s(z)$, $s(\omega)$ and $s(w)$ which will be met at least on one of the above-mentioned boundaries $l(z)$, $l(\omega)$ and $l(w)$, as a result of a circuit in the positive direction. By $t=e_k$, $k=\overline{1,n}$, we denote the points of the t-axis of the plane ζ which are mapped, respectively, into the points B_k, $k=\overline{1,n}$, where $-\infty < e_1 < e_2 < \cdots < e_n < +\infty$. The point $t=e_{n+1}=\infty$ is mapped into the nonangular point which lies on some part of the boundary B_nB_1.

The boundary values of the functions $z(\zeta)$, $\omega(\zeta)$ and $w(\zeta)$, as $\zeta \to t$, $\zeta \in \operatorname{Im}(\zeta) > 0$ will be denoted by $z(t)=x(t)+iy(t)$, $\omega(t)=\varphi(t)+i\psi(t)$, $w(t)=u(t)-iv(t)$, while the complex conjugates to the functions $z(t)$, $\omega(t)$ and $w(t)$ will be denoted by $\overline{z(t)}$, $\overline{\omega(t)}$, and $\overline{w(t)}$.

Introduce the vectors $\Phi(\zeta)=[\omega(\zeta),z(\zeta)]$, $\overline{\Phi(\zeta)}=[\overline{\omega(t)},\overline{z(t)}]$, $\Phi'(\zeta)=[\omega'(\zeta),z'(\zeta)]$, $\overline{\Phi'(\zeta)}=[\overline{\omega'(\zeta)},\overline{z'(\zeta)}]$, $f(t)=[f_1(t),f_2(t)]$. Then the boundary conditions (2.2) can be written as follows:

$$(a_{k2}+ia_{k1})\omega(t)+(a_{k4}+ia_{k3})z(t)=(a_{k2}-ia_{k1})\overline{\omega(t)}+$$
$$+(a_{k4}-ia_{k3})\overline{z(t)}+2if_k(t),\quad -\infty<t<+\infty,\quad k=1,2. \tag{2.9}$$

The condition (2.9) can by means of the vector $\Phi(z)$ be rewritten as

$$\Phi(t)=g(t)\overline{\Phi(t)}+2iG^{-1}f(t),\quad -\infty<t<+\infty, \tag{2.10}$$

where $g(t)=G^{-1}(t)\overline{G(t)}$ is a piecewise constant nonsingular second order matrix with the discontinuity points $t=e_k$, $k=\overline{1,n}$. $G^{-1}(t)$ is the inverse to $G(t)$ matrix and $\overline{G(t)}$ is the complex-conjugate to $G(t)$ matrix.

Below, instead of $a_{kj}(t)$, $k=1,2$, $j=\overline{1,4}$ we will write a_{kj}, $k=1,2$, $j=\overline{1,4}$.

The matrices $G(t)$ and $G^{-1}(t)$ are defined by the formulas

$$G(t)=\begin{pmatrix} a_{12}+ia_{11}, & a_{14}+ia_{13} \\ a_{22}+ia_{21}, & a_{24}+ia_{23}\end{pmatrix} \tag{2.11}$$

and

$$G^{-1}(t)=\frac{1}{\det G(t)}\begin{pmatrix} a_{24}+ia_{23}, & -(a_{14}+ia_{13}) \\ -(a_{22}+ia_{21}), & a_{12}+ia_{11}\end{pmatrix}. \tag{2.12}$$

The matrix $g(t)$ in the interval (a_j,a_{j+1}) is defined as

$$g_j(t)=G_j^{-1}\overline{G}_j=\frac{1}{\det G_j(t)}\begin{pmatrix} A_{11}^{*j}, & iA_{12}^{*j} \\ iA_{21}^{*j}, & \overline{A}_{11}^{*j}\end{pmatrix},\quad a_j<t<a_{j+1}, \tag{2.13}$$

but for $j = n-1$ we have

$$\begin{aligned} A_{11}^{*(n-1)} &= (-1)(A_{13}^{n-1} + iA_{12}^{n-1}), \\ A_{12}^{*(n-1)} &= (-2)A_{14}^{(n-1)},\ A_{21}^{*(n-1)} = 2A_{11}^{(n-1)}, \\ A_{11}^{*(n-1)} &= a_{24}a_{12} + a_{23}a_{11} - a_{14}a_{22} - a_{13}a_{21} + \\ &\quad + i(a_{23}a_{12} - a_{24}a_{11} + a_{21}a_{14} - a_{13}a_{22}). \end{aligned} \tag{2.14}$$

The function $\overline{A}_{11}^{*(n-1)}$ is the complex-conjugate to $A_{11}^{*(n-1)}$.

Differentiation of (2.10) yields

$$\Phi'(t) = g(t)\overline{\Phi}'(t), \quad -\infty < t < +\infty. \tag{2.15}$$

It can be easily verified that the equality $\overline{g(t)} = [g(t)]^{-1} = \overline{G}^{-1}G$ holds, where $[g(t)]^{-1}$ is the matrix, inverse to $g(t)$, and $\overline{g}(t)$ is the matrix, complex-conjugate to $g(t)$.

For the point $t = e_j$ we compose the characteristic equation

$$\det(g_{j+1}^{-1}(e_i + 0)g_j(e_j - 0) - \lambda E) = 0, \tag{2.16}$$

where $g_{j+1}^{-1}(e_j + 0)g_j(e_j - 0)$ is a matrix, E is the unit matrix, λ is the parameter, and $g_j(e_j+0)$, $g_{j-1}(e_j-0)$ are the limiting values of matrices $g_j(t)$, $g_{j-1}(y)$ at the point $t = e_j$ from the right and from the left, respectively; $g_j^{-1}(e_j + 0)$ is the inverse to $g_j(e_j + 0)$ matrix.

If we denote by λ_{kn} the characteristic numbers of the matrix $g_{(n-1)}(t)$, then the equalities

$$\begin{gathered} \lambda_{1n} + \lambda_{2n} = [A_{11}^{*(n-1)} + \overline{A}_{11}^{*(n-1)}]/[2\det G_{n-1}], \\ \lambda_{1n}\cdot\lambda_{2n} = \det\overline{G}_{n-1}/\det G_{n-1}, \\ |\lambda_{1n}||\lambda_{2n}| = 1, \quad |\det g(t)| = 1, \quad \overline{\lambda}_{1n}\cdot\overline{\lambda}_{2n} = 1/[\lambda_{1n}\cdot\lambda_{2n}], \\ 1/\lambda_{1n} + 1/\lambda_{2n} = \overline{\lambda}_{1n} + \overline{\lambda}_{2n}, \quad \lambda_{1n} + \lambda_{2n} = \lambda_{1n}\lambda_{2n}(\overline{\lambda}_{1n} + \overline{\lambda}_{2n}) \end{gathered}$$

hold [1-31].

Let us introduce the characteristic numbers $\alpha_{kn} = \frac{1}{2\pi i}\ln\lambda_{kn}$, $k = 1, 2$. Then $\alpha_{1n} + \alpha_{2n} = \alpha_{0j}$, where $\alpha_{0j} = \frac{1}{2\pi i}\arg\det(\overline{G}_j/G_j)$,

$$\alpha_{1n} - \alpha_{2n} = \frac{1}{2\pi i}\ln(\lambda_{1n}/\lambda_{2n}) = \nu_n, \tag{2.17}$$

where $\pi\nu_n$ is the interior angle of the contour $l(w)$ of $s(w)$ at the point A_n.

The roots λ_{kn}, $k = 1, 2$, for the point $t = e_n$ are calculated by the formula [1–7]

$$\begin{gathered} \lambda_{kn} = [A_{11}^{*(n-1)} + \overline{A}_{11}^{*n} \pm \\ \pm i\sqrt{4\det G_n \det\overline{G}_n - (A_{11}^{*n} + \overline{A}_{11}^{*n})^2}]/[2\det G_n],\ k = 1, 2. \end{gathered} \tag{2.18}$$

For the points $t = e_j$, $j = 1, 2, \dots, n-1$, we have

$$g_{j+1}^{-1}(a_j+0)g_j(e_j-0) = \overline{G}_{j+1}^{-1}G_{j+1}G_j^{-1}\overline{G}_j,$$

$$g_{j+1}^{-1}(a_j+0)g_j(e_j-0) =$$

$$= \frac{1}{\det \overline{G}_{j+1}} \cdot \frac{1}{\det G_j} \begin{pmatrix} \overline{A}_{11}^{*(j+1)}, & -iA_{12}^{*(j+1)} \\ -iA_{21}^{*(j+1)}, & A_{11}^{*(j+1)} \end{pmatrix} \begin{pmatrix} \overline{A}_{11}^{*j)}, & iA_{12}^{*j)} \\ iA_{21}^{*j}, & \overline{A}_{11}^{*j} \end{pmatrix}, \quad (2.19)$$

$$\lambda_{1j} + \lambda_{2j} = [\overline{A}_{11}^{*(j+1)}A_{11}^{*j} + A_{12}^{*(j+1)}A_{21}^{*j} +$$
$$+ A_{21}^{*(j+1)}A_{12}^{*j} + A_{11}^{*(j+1)}\overline{A}_{11}^{*j}]/[\det \overline{G}_{j+1} \det G_j], \quad (2.20)$$

$$\lambda_{1j}\lambda_{2j} = \det G_{j+1} \det \overline{G}_j / [\det \overline{G}_{j+1} \det G_j]. \quad (2.21)$$

Using (2.19), (2.20) and (2.21), we can calculate

$$\lambda_{1j}/\lambda_{2j}, \quad \alpha_{1j},\ \alpha_{2j}, \quad \alpha_{1j} + \alpha_{2j} = \alpha_{0j}^*, \quad \alpha_{1j} - \alpha_{2j} = \nu_j, \quad (2.22)$$
$$\alpha_{0j}^* = \frac{1}{\pi}[\alpha_{0(j+1)} - \alpha_{0j}], \quad \det G_j = R_0 \exp(i\alpha_{0j}).$$

The characteristic numbers α_{kj}, $k = 1, 2$, $j = \overline{1,(n+1)}$, must satisfy the Fuchs condition [1–31]

$$\sum_{j=1}^{n+1}[1 - (\alpha_{1j} + \alpha_{2j})] = 2,$$
$$\alpha_{1(n+1)} = 3, \quad \alpha_{2(n+1)} = 2, \quad t_\infty = a_{n+1} = \infty. \quad (2.23)$$

The equality $\alpha_{1j} + \alpha_{2j} = 1$, $j = \overline{1,n}$, under the condition (2.5) may fail to be fulfilled, and hence we are unable to apply the equation (1.4) and solve the equation (2.15). As it will be seen below, to solve (2.15) completely it suffices to use the linearly independent solutions (1.11).

Of all singular angular points of the boundaries $l(z)$ and $l(\omega)$, we select such angular points to which on the boundary $l(w)$ of $s(w)$ there correspond regular nonangular points. Such angular points on the boundaries $l(z)$ and $l(\omega)$ are usually called removable singular points [1–7]. For the sake of simplicity, we assume that the number of removable singular points is equal to two. Denote these points by $t = e_k$ and $t = e_{k+j}$. The angles corresponding to such points on the contours $l(z)$ and $l(\omega)$ are equal to $\pi/2$. To remove those singular points from the boundary conditions (2.15), we introduce the new unknown vector $\Phi_1(\zeta)$ by the formula

$$\Phi'(\zeta) = \Phi_1(\zeta)\sqrt{\frac{(\zeta - e_{k-1})(\zeta - e_{k+j-1})}{(\zeta - e_k)(\zeta - e_{k+j})}},$$
$$\sqrt{\frac{(\zeta - e_{k-1})(\zeta - e_{k+i-1})}{(\zeta - e_k)(\zeta - e_{k+j})}} > 0, \quad \zeta > e_{k+j}. \quad (2.24)$$

When passing from the vector $\Phi(\zeta)$ to $\Phi_1(\zeta)$, the matrix $g(t)$ in the interval $(e_{k-1}, e_k), (e_{k+j-1}, e_{k+j})$ is multiplied by (-1).

We enumerate the remaining singular points along the t-axis as $t = a_k$, $k = \overline{1, m}$. To these points there correspond the points A_k, $k = \overline{1, m}$, on the contour $l(w)$. In what follows, the notation for the matrices $g(t) = G^{-1}\overline{G}$ will remain unchanged, but all the changes occurring while introducing $\Phi_1(\zeta)$, will be taken into account.

If one or several elements in the matrix $g(t)$ are equal to zero, and moreover, $\det g(t) \neq 0$, then the problem (2.10) is solved completely by means of the Cauchy type integral [1–31]. Besides the above-mentioned one, we come across the cases where all the elements in the matrix $g(t)$ are different from zero and then the problem (2.10) is solved by elementary means [16, 26].

The boundary condition with respect to $\Phi_1(\zeta)$ can be written as

$$\Phi_1(t) = g(t)\overline{\Phi}_1(t), \quad -\infty < t < +\infty. \tag{2.25}$$

To solve the problem (2.25), we first find all the roots λ_{kj}, $k = 1, 2$, $j = \overline{1, m+1}$, from (2.16) and then, taking into account (2.23), we find α_{ki}, $k = 1, 2$, $j = \overline{1, m+1}$ [1,7]. Having found the above-mentioned quantities, we substitute α_{kj}, $k = 1, 2$, $j = \overline{1, m}$ into (1.11).

All the equations and formulas (1.11)–(1.16) remain valid and will be used later on for solving (2.10), (2.15) and (2.25).

3. The Fuchs Class Equation in the Form of a System

The equation (1.11) in the neighborhood of every singular point $t = a_k$, $k = \overline{1, m+1}$, and in the neighborhood of any regular point, where $p(\zeta)$ and $q(\zeta)$ are analytic, has two linearly independent local solutions which are constructed by means of infinite series whose coefficients are defined in the well-known manner. These series converge respectively in the circles with centers at the points for which these series have been constructed, and the convergence radii of the series are bounded by the distance from the centers of the given circles to the nearest to the centers singular points.

We denote the local linearly independent solutions of the equation (1.11) for singular points $\zeta = a_k$, $k = 1, 2, \dots, m+1$, by $v_{kj}(\zeta)$, $j = \overline{1, (m+1)}$, and for $t = a_j^* = (a_j + a_{j+1})/2$, $j = 1, 2, \dots, m-1$, by $\sigma_{kj}(\zeta)$, $k = 1, 2$, $j = 1, 2, \dots, m-1$.

Suppose

$$u_1(\zeta) = pu_{1j}(\zeta) + qu_{2j}(\zeta), \quad u_2(\zeta) = ru_{1j}(\zeta) + su_{2j}(\zeta), \tag{3.1}$$

where p, q, r, s are the integration constants of (1.15).

The equation (1.11) can be written in the form of the system

$$\chi_1'(\zeta) = \chi_1(\zeta)\mathcal{P}(\zeta), \tag{3.2}$$

$$\chi_1(\zeta)=\begin{pmatrix}u_1(\zeta), & u_1'(\zeta)\\ u_2(\zeta), & u_2'(\zeta)\end{pmatrix},\mathcal{P}(\zeta)=\begin{pmatrix}0, & -q(\zeta)\\ 1, & -p(\zeta)\end{pmatrix}, \tag{3.3}$$

where $u_1(\zeta)$ and $u_2(\zeta)$ are linearly independent solutions of (1.11).

A solution of the boundary value problem (2.25) will be sought by means of the matrix $\chi_1(\zeta)$. It is known that if the matrix $\chi_1(\zeta)$ is a solution of (3.2), then the matrix $T\chi_1(\zeta)$ is also the solution of (3.2), where

$$T=\begin{pmatrix}p, & q\\ r, & s\end{pmatrix}, \quad \det T\neq 0. \tag{3.4}$$

If we construct the local linearly independent solutions $u_{kj}(\zeta)$ and $\sigma_{kj}(\zeta)$ of (1.11), for the points $\zeta=a_j$, $j=\overline{1,m+1}$, $\zeta=a_j^*=(a_j+a_{j+1})/2$, $j=\overline{1,m-1}$, respectively, then the local fundamental matrices for (3.2) will have the form

$$\Theta_j(\zeta)=\begin{pmatrix}u_{1j}(\zeta), & u_{1j}'(\zeta)\\ u_{2j}(\zeta), & u_{2j}'(\zeta)\end{pmatrix}, \quad j=\overline{1,m+1}, \tag{3.5}$$

$$\sigma_j(\zeta)=\begin{pmatrix}\sigma_{1j}(\zeta), & \sigma_{1j}'(\zeta)\\ \sigma_{2j}(\zeta), & \sigma_{2j}'(\zeta)\end{pmatrix}, \quad j=\overline{1,m-1}. \tag{3.6}$$

Suppose that the inequality $|a_m|>|a_1|$ holds. Then at the point $a_m^*=-|a_m|$ we construct the local series $\sigma_{*k}(\zeta)$, $k=1,2$, and the corresponding local matrix $\sigma_{*j}(\zeta)$. The convergence radii of these series are bounded by the distance from the point $t=a_m$ to the singular point $t=a_1$, and if $|a_1|>|a_m|$, then we construct at the point $a_1^*=|a_1|$ the local series $\sigma_{*k}(\zeta)$, $k=1,2$, and the matrix $\sigma^*(\zeta)$. The convergence radius of these series will be bounded by the distance from the point a_1^* to the point $t=a_m$.

It becomes evident that there exists a finite number of circles with centers $\zeta=a_j$, $j=\overline{1,m+1}$, $\zeta=a_j^*=(a_j+a_{j+1})/2$, $j=\overline{1,m-1}$, $\zeta=a_m^*$ (or $\zeta=a_1^*$) which cover completely the x-axis, $-\infty<t<+\infty$. Note that the circle with the center $\zeta=\infty$ is assumed to be the exterior of the circle $|\zeta|<r_0$, where r_0 is equal to the largest (in absolute value) of the numbers a_1 and a_m .

The equation (1.11) in the neighborhood of $\zeta=a_j$ can be written as

$$(\zeta-a_j)^2v''(\zeta)+(\zeta-a_j)p_j(\zeta)v'(\zeta)+q_j(\zeta)v(\zeta)=0, \tag{3.7}$$

where

$$\begin{aligned}p_j(\zeta)&=p_{0j}+\sum_{n=1}^{\infty}p_{nj}(\zeta-a_j)^n,\\ p_{nj}&=(-1)^{n-1}\sum_{k=1,k\neq j}^{m}[1-\alpha_{1k}-\alpha_{2k}](a_j-a_k)^n,\end{aligned} \tag{3.8}$$

$$p_{0j} = 1 - \alpha_{1j} - \alpha_{2j},$$

$$q_j(\zeta) = \alpha_{1j}\alpha_{2j} + c_j^*(\zeta - a_j) + \sum_{n=2}^{\infty} q_{nj}(\zeta - a_j)^n, \tag{3.9}$$

$$q_{nj} = (-1)^{n-2} \sum_{k=1, k\neq j}^{m} [\alpha_{1k}\alpha_{2k}(n-1) + c_k^*(a_j - a_k)](a_j - a_k)^{-n}, \tag{3.10}$$

$n = 2, 3, \ldots$

$$q_{0j} = \alpha_{1j}\alpha_{2j}, \quad q_{1j} = c_j^*, \quad j = \overline{1, m}. \tag{3.11}$$

The local solutions of (3.7) for the point $t = a_j$, $j = \overline{1, m}$, will be sought in the form

$$u_j(\zeta) = (\zeta - a_j)^{\alpha_j} \widetilde{u}_j(\zeta), \quad \widetilde{u}_j(\zeta) = 1 + \sum_{n=1}^{\infty} \gamma_{nj}(\zeta - a_j)^n. \tag{3.12}$$

For definition of the coefficients γ_{nj}, $n = \overline{1, \infty}$, $j = \overline{1, m}$, we have the following recursion formulas:

$$f_{0j}(\alpha_j) = \alpha_j(\alpha_j - 1) + p_{0j}\alpha_j + q_{0j} = 0, \tag{3.13}$$

$$\gamma_{1j} f_0(\alpha_j + 1) + f_1(\gamma_j) = 0, \tag{3.14}$$

$$\gamma_{2j} f_0(\alpha_{j+2}) + \gamma_{1j} f_1(\alpha_j + 1) + f_2(\alpha_j) = 0, \tag{3.15}$$

. . . .

$$\gamma_{nj} f_0(\alpha_j + n) + \gamma_{(n-1)j} f_1(\alpha_j + n - 1) +$$
$$+ \gamma_{(n-2)j} f_2(\alpha_j + n - 2) + \cdots + \gamma_{1j} f_{(n-1)}(\alpha_j + 1) + f_n(\alpha_j) = 0, \tag{3.16}$$

where

$$f_n(\alpha_j) = \alpha_j p_{nj} + q_{nj}. \tag{3.17}$$

The defining equation (3.13) for every point $t = a_j$, $j = \overline{1, m}$, has two roots, α_{1j} and α_{2j}. If the difference $\alpha_{1j} - \alpha_{2j}$ is not an integer, then using the formulas (3.14)–(3.16), we can construct for every point $t = a_j$ two linearly independent solutions

$$u_{kj}(\zeta) = (\zeta - a_j)^{\alpha_{kj}} \widetilde{u}_{kj}(\zeta), \quad \widetilde{u}_{kj} = 1 + \sum_{n=1}^{\infty} \gamma_{nj}^k (\zeta - a_j)^n, \quad k = 1, 2. \tag{3.18}$$

But if the difference $\alpha_{1j} - \alpha_{2j}$ is an integer, then $u_{1j}(\zeta)$, $j = \overline{1, m}$, can be constructed by the formulas (3.14)–(3.16), while if $u_{2j}(\zeta)$ involves a logarithmic term, $u_{2j}(\zeta)$ can be constructed with the help of the Frobenius method [15, 27-31].

Let us pass now to the construction of $u_{2j}(\zeta)$ when the difference $\alpha_{1j} - \alpha_{2j} = 2$ and $u_{2j}(\zeta)$ does not involve a logarithmic term. For such a point $t = a_j$, on the contour $l(w)$ there is a cut (circular or linear) with the angle 2π. P. Ya. Polubarinova-Kochina has proved [2] that $u_{2j}(\zeta)$ does not contain a logarithmic term. She also obtained the equation connecting the parameters a_j, c_j^* [1-7, 25-31]. To construct $u_{2j}(\zeta)$, we will act as follows [25-31].

For the point $t = a_k$, the equality (3.15) fails to be fulfilled because

$$f_{0j}(\alpha_j + 2) = 0 \tag{3.19}$$

as $\alpha_j \to \alpha_{2j}$.

In order for the equality (3.15) to take place as $\alpha_j \to \alpha_{2j}$, it is necessary and sufficient that the condition

$$\gamma_{1j} f_j(\alpha_j + 1) + f_2(\alpha_j) = 0, \quad \alpha_j \to \alpha_{2j} \tag{3.20}$$

be fulfilled.

After certain transformations, the equation (3.20) takes the form

$$q_{2j} + q_{1j}^2 + q_{1j} p_{1j} = 0. \tag{3.21}$$

Note that for the cut end $t = a_j$ with the angle 2π the equality $dw(\zeta)/d\zeta = 0$ holds for $t = a_j$, where $w(\zeta)$ is the general solution of (1.1) or (1.18).

To construct $u_{2j}(\zeta)$ for the cut end, it suffices to calculate $\gamma_{2j}^2(\alpha_{2j})$ uniquely; the remaining coefficients $\gamma_{nj}^2(\alpha_{2j})$, $n = 1, 3, 4, 5, \dots$, can be calculated by the formula (3.16). Under the conditions (3.19) and (3.20) the equation (3.15) is fulfilled.

To define $\gamma_{2j}^2(\alpha_{2j})$ and, consequently, $u_{2j}(\zeta)$ uniquely, we suppose that $\alpha_j \neq \alpha_{2j}$. Then (1.5) implies that

$$\gamma_{2j}(\alpha_j) = -[\gamma_{1j}(\alpha_j) f_{1j}(\alpha_j + 1) + f_{2j}(\alpha_j)]/f_{0j}(\alpha_j + 2). \tag{3.22}$$

The numerator and denominator on the right-hand side of (3.22) vanish as $\alpha_j \to \alpha_{2j}$, and hence there is an indeterminacy. Developing this indeterminacy by the L'Hospital rule, we obtain

$$\gamma_{2j}^2 = -0,5[p_{1j}(p_{1j} + 2q_{1j}) + p_{2j}]. \tag{3.23}$$

Thus γ_{2j}^2, and hence $u_{2j}(\zeta)$, are defined uniquely.

Let us proceed now to the determination of local solutions in the neighborhood of the point $\zeta = a_{m+1} = \infty$.

We Represent $p(\zeta)$ and $q(\zeta)$ in the neighborhood of $\zeta = \infty$ as follows:

$$p(\zeta) = \zeta^{-1} \sum_{n=0}^{\infty} p_{n\infty} \zeta^{-n}, \quad q(\zeta) = \zeta^{-2} \sum_{n=0}^{\infty} q_{n\infty} \zeta^{-n}, \tag{3.24}$$

where

$$p_{n\infty} = \sum_{k=1}^{m} [1 - (\alpha_{1k} + \alpha_{2k})] a_k^n, \quad p_{0\infty} = 6, \tag{3.25}$$

$$q_{n\infty} = \sum_{k=1}^{m} [\alpha_{1k}\alpha_{2k}(n+1) + c_k^* a_k] a_k^n, \tag{3.26}$$

$$q_{0\infty} = \sum_{k=1}^{m} [\alpha_{1k}\alpha_{2k} + c_k^* a_k], \tag{3.27}$$

$$q_{1\infty} = \sum_{k=1}^{m} [2\alpha_{1k}\alpha_{2k} a_k + c_k^* a_k^2]. \tag{3.28}$$

The local solutions in the neighborhood of the point $t = \infty$ will be sought in the form

$$u_\infty(\zeta) = \zeta^{-\alpha_\infty} + \sum_{n=1}^{\infty} \gamma_{n\infty}\zeta^{-\alpha_\infty - n}. \tag{3.29}$$

For definition of $\gamma_{n\infty}$, $n = \overline{1,\infty}$, we have the formulas

$$f_{0\infty}(\alpha_\infty) = \alpha_\infty(\alpha_\infty + 1) - p_{0\infty}\alpha_\infty + q_{0\infty} = 0, \tag{3.30}$$

$$\gamma_{1\infty} f_{0\infty}(\alpha_\infty + 1) - p_{1\infty} + q_{1\infty} = 0, \tag{3.31}$$

$$\gamma_{2\infty} f_{0\infty}(\alpha_\infty + 2) + \gamma_{1\infty} f_{1\infty}(\alpha_\infty + 1) - p_{2\infty}\alpha_\infty + q_{2\infty} = 0, \tag{3.32}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{aligned} &\gamma_{n\infty} f_{0\infty}(\alpha_\infty + n) + \gamma_{(n-1)\infty} f_{1\infty}(\alpha_\infty + n - 1) + \\ &\quad + \gamma_{(n-2)\infty} f_{2\infty}(\alpha_\infty + n - 2) + \cdots + \gamma_{1\infty} f_{(n-1)}(\alpha_\infty + 1) - \\ &\quad - p_{n\infty}\alpha_\infty + q_{n\infty} = 0, \end{aligned} \tag{3.33}$$

where

$$f_{k\infty} = q_{k\infty} - (\alpha_\infty + k)p_{k\infty}. \tag{3.34}$$

Owing to the fact that $t = \infty$ is the image of the nonangular point, the equation (3.30) must have the roots $\alpha_{1\infty} = 3$ and $\alpha_{2\infty} = 2$, and hence the free term $q_{0\infty}$ must satisfy the condition

$$q_{0\infty} = \sum_{k=1}^{m} [\alpha_{1k}\alpha_{2k} + a_k c_k^*] = 6. \tag{3.35}$$

Since $\alpha_{1\infty} - \alpha_{2\infty} = 1$, the equality (3.31) fails to be fulfilled, therefore the formulas (3.31)–(3.33) allow one to determine only $\gamma'_{n\infty}$, $n = \overline{1,\infty}$, and hence the solution $u_{1\infty}(\zeta)$. For the equality (3.31) to take place for $\alpha_\infty = \alpha_{2\infty}$, it is necessary and sufficient that the condition

$$q_{1\infty} - p_{1\infty}\alpha_{2\infty} = 0 \tag{3.36}$$

be fulfilled.

To define $\gamma_{1\infty}^2$, we act as follows: from (3.31) for $\alpha_\infty \neq \alpha_{2\infty}$ we define $\gamma_{1\infty}$ and obtain

$$\gamma_{1\infty} = [p_{1\infty} - q_{1\infty}]/f_{0\infty}(\alpha_\infty + 1). \tag{3.37}$$

Since the numerator and the denominator in (3.37) vanish as $\alpha_\infty \to \alpha_{2\infty}$, we can develop the indeterminacy in the well-known manner and get

$$\gamma_{1\infty}^2 = p_{1\infty}. \tag{3.38}$$

Next we define $\gamma^2_{n\infty}$, $n=\overline{2,\infty}$, by the formulas (3.32)–(3.33). Thus we have obtained the solution $u_{2\infty}(\zeta)$.

Finally, we have

$$u_{k\infty}(\zeta)=\zeta^{-\alpha_{k\infty}}+\sum_{n=1}^{\infty}\gamma^n_{n\infty}\zeta^{-\alpha_{2\infty}-n},\quad k=1,2. \tag{3.39}$$

The equations (1.21) coincide respectively with the equations (1.13), (3.35) and (3.36).

4. Local Representations of the Matrices $\chi_j(\zeta)$, $j=\overline{1,m+1}$

Of each set of branches of the functions $\exp[\alpha_{kj}\ln(t-a_j)]$ appearing in the local solutions $u_{kj}(\zeta)$, we choose one as follows:

$$\exp[\alpha_{kj}\ln(t-a_j)]>0,\quad t>a_j,$$
$$[\exp[\alpha_{kj}\ln(t-a_j)]]^{\pm}=\exp[\pm i\alpha_{kj}]\exp[\alpha_{kj}\ln(a_j-t)],\quad t<a_j,$$
$$[\exp[-\alpha_{k\infty}\ln(-t)]]^{\pm}>0,\quad -\infty<t<a_j;$$
$$[\exp[-\alpha_{k\infty}\ln t]]^{\pm}=\exp[\pm i\pi(-\alpha_{k\infty})]\exp[-\alpha_{k\infty}\ln t],\quad a_m<t<\infty.$$

Along with (3.5) and (3.6), we introduce the matrices

$$\Theta^*_j(t)=\begin{pmatrix}u^*_{1j}(t), & u'^*_{1j}(t)\\ u^*_{2j}(t), & u'^*_{2j}(t)\end{pmatrix},\quad a_{j-1}<t<a_j, \tag{4.1}$$

where

$$\begin{gathered}u^*_{kj}(t)=(a_j-t)^{\alpha_{kj}}\widetilde{u}_{kj}(t),\\ u'_{kj}(t)=-(a_j-t)^{\alpha_{kj}-1}\widetilde{u}'^*_{kj}(t),\quad u'_{kj}(t)=du_{kj(t)/dt},\\ \widetilde{u}'^*_{kj}(t)\equiv\alpha_{kj}+\sum_{n=1}^{\infty}\gamma^k_{nj}(\alpha_{kj}+n)(t-a_j)^n.\end{gathered} \tag{4.2}$$

Between the matrices $\Theta_j(t)$ and $\Theta^*_j(t)$ there is the connection

$$\Theta^{\pm}_j(t)=\theta^{\pm}_j\Theta^*_j(t),\quad a_{j-1}<t<a_j, \tag{4.3}$$
$$\Theta^{\pm}_{\infty}(t)=\theta^{\pm}_{\infty}(t)\Theta^*_{\infty}(t),\quad a_m<t<+\infty, \tag{4.4}$$

where the matrices $\theta^{\pm}_j$ are defined by the formula

$$\theta^{\pm}=\begin{pmatrix}\exp(\pm i\pi\alpha_{1j}), & 0\\ 0, & \exp(\pm i\pi\alpha_{2j})\end{pmatrix} \tag{4.5}$$

for $\alpha_{1j}-\alpha_{2j}\neq n$, while for $n=0,1,2$ they are defined by the equality

$$\theta^{\pm}_j=e^{\pm\pi\alpha_{2j}}\begin{pmatrix}1, & 0\\ \mp\pi i, & 1\end{pmatrix},\ n=0,2;\quad \theta^{\pm}_j=e^{\pm i\pi\alpha_{2i}}\begin{pmatrix}-1, & 0\\ \pm\pi i, & -1\end{pmatrix},\ n=1. \tag{4.6}$$

For the cut end $w = A_j$, the matrices $\theta_j^{\pm}$ are defined in the following manner. If the eigenvalues are of the type $\alpha_{1j} = 3/2$, $\alpha_{2j} = -1/2$, then $\theta_j^{\pm} = \mp iE$, where E is the unit matrix, but if $\alpha_{1i} = 2$, $\alpha_{2j} = 0$, then $\theta_j^{\pm} = E$.

The elements of the matrix $\Theta_j^*(t)$ involving logarithmic terms are defined by the formulas

$$u_{2j}^*(t) = (a_j - t)^{\alpha_{2i}}[(t - a_j)^n \widetilde{u}_{1j}(t)\ln(a_j - t) + \widetilde{u}_{2i}^2(t)], \tag{4.7}$$

$$u_{2i}'^*(t) = -(a_j - t)^{\alpha_{2j}-1}\{[(a_j - t)^n e^{i\pi n}\widetilde{u}_{2j}^1 \ln(a_j - t) + \widetilde{u}_{1j}(t)] + \\ + \widetilde{u}_{2j}^2(t)\}, \quad n = 0, 1, 2. \tag{4.8}$$

5. The Fundamental Matrix

Let us construct the matrix $\chi(\zeta)$. The domains of convergence of the matrices $\Theta_i(t)$ and $\sigma_j(t)$ have always a part in common in which one can write the equalities

$$\Theta_j^*(t) = T_j^*\sigma_j(t), \quad \sigma_j(t) = T_{0j}\Theta_{j-1}(t), \quad T_{j-1} = T_j^* T_{0j}, \quad a_{j-1} < t < a_j, \tag{5.1}$$

$$\Theta_1^*(t) = T_{-m}\sigma_{-m}(t), \quad \sigma_{-m}(t) = T_{-\infty}\Theta_\infty(t), \quad -\infty < t < a_1 \tag{5.2}$$

$$\Theta_\infty^*(t) = T_\infty \Theta_m(t), \quad a_m < t < +\infty. \tag{5.3}$$

Here T_j^*, T_{0j}, T_{-m}, T_{j-1}, $T_{-\infty}$, T_∞ are constant real matrices defined by the equalities (5.1)–(5.3). In these equalities we can fix t arbitrarily in the domain where the two local matrices, appearing in the above-mentioned equalities, converge.

Define the matrix $\chi_1(\zeta)$ along the t-axis of the plane ζ as

$$\chi_1^{\pm}(t) = T\Theta_m^{\pm}(t), \quad \Theta_m^+(t) = \Theta_m^-(t), \quad a_m < t < +\infty, \tag{5.4}$$

$$\chi_1^{\pm}(t) = T\theta_m^{\pm}\Theta_m^*(t), \quad a_{m-1} < t < a_m; \tag{5.5}$$

$$\chi_1^{\pm}(t) = T\theta_m^{\pm}T_{m-1}\Theta_{m-1}^{\pm}(t), \quad T_{m-1} = T_m^* T_{0m}, \quad a_{m-1} < t < a_m; \tag{5.6}$$

$$\chi_1^{\pm}(t) = T\theta_m^{\pm}T_{m-1}\theta_{m-1}^{\pm}\Theta_{m-1}^*(t), \quad a_{m-2} < t < a_{m-1}; \tag{5.7}$$

. . . .

$$\chi_1^{\pm}(t) = T\theta_m^{\pm}T_{m-1}\ldots T_1\theta_1^{\pm}\Theta_1^*(t), \quad -\infty < t < a_1, \tag{5.8}$$

$$\chi_1^{\pm}(t) = T\theta_m^{\pm}T_{m-1}\ldots\theta_1^{\pm}T_{-\infty}\Theta_\infty(t), \quad -\infty < t < a_1, \tag{5.9}$$

$$\chi_1^{\pm}(t) = TT_\infty\Theta_\infty^{\pm}(t), \quad a_m < t < +\infty, \tag{5.10}$$

where the matrix T is defined by the formula (3.4).

The upper signs ($\pm$) in the matrices (5.4)–(5.10) denote the limiting values of the matrix $\chi(\zeta)$ respectively in the upper (when $\zeta \in \mathrm{Im}(\zeta) > 0$, $\zeta \to t$) and in the lower (when $\zeta \in \mathrm{Im}(\zeta) < 0$, $\zeta \to t$) half-planes. The limiting values of $\chi^+(t)$ and of $\chi^-(t)$ are connected as follows: $\chi^-(t) = \overline{\chi^+(t)}$, where $\overline{\chi^+(t)}$ is the complex conjugate of the matrix $\chi^+(t)$.

6. Solution of the Boundary Value Problem (2.25)

A straightforward checking shows that the matrices (5.4)–(5.10) satisfy the equation (3.2). Therefore, by appropriate choice of the parameters a_j, c_j, $j = \overline{1,m}$, p, q, r, s, the same matrices must satisfy the condition (2.25). Indeed, we start our proof with the interval $(a_m, +\infty)$. We have

$$\begin{gathered} T\Theta_m^+(t) = g_m(t)\overline{T}\Theta_m^-(t), \quad g_m(t) = E, \\ \Theta_m^+(t) = \Theta_m^-(t), \quad T = \overline{T}, \quad a_m < t < +\infty. \end{gathered} \tag{6.1}$$

For the interval (a_{m-1}, a_m) in the neighborhood of $A = a_m$ we obtain the equality

$$T\theta_m^+\Theta_m^*(t) = g_{m-1}T\theta_m^-\Theta_m^*(t), \quad a_{m-1} < t < a_m. \tag{6.2}$$

The expressions (6.1) and (6.2) result in the matrix equations

$$(\theta_m^+)^2 = T^{-1}G_{m-1}^{-1}\overline{G}_{m-1}T, \tag{6.3}$$

from which one can see that the matrices $(\theta_m^+)^2$ and $G_{m-1}^{-1}\overline{G}_{m-1}$ are similar.

The matrix equation (6.2) can be rewritten in the form

$$T\begin{pmatrix} \widetilde{\lambda}_{1(m)}, & 0 \\ 0, & \widetilde{\lambda}_{2(m)} \end{pmatrix} = \begin{pmatrix} A_{11}^{*(m-1)}, & iA_{12}^{*(m-1)} \\ iA_{21}^{*(m-1)}, & \overline{A}_{11}^{*(m-1)} \end{pmatrix} T, \tag{6.4}$$

$$\lambda_{km} = \widetilde{\lambda}_{km}/\det G_{m-1}, \quad k = 1, 2, \tag{6.5}$$

which in its turn results in the system consisting of two equations

$$r/p = \Big\{\sqrt{\det G_{m-1}\det\overline{G}_{m-1} - (\operatorname{Re} A_{11}^{*(m-1)})^2} - \operatorname{Im} A_{11}^{*(m-1)}\Big\}\Big/A_{12}^{*(m-1)}, \tag{6.6}$$

and

$$s/q = \Big\{\operatorname{Im} A_{11}^{*(m-1)} - \sqrt{\det G_{m-1}\det\overline{G}_{m-1} - (\operatorname{Re} A_{11}^{*(m-1)})^2}\Big\}\Big/A_{21}^{*(m-1)}. \tag{6.7}$$

Analogously to the matrix equation (6.3), we find the matrix equations successively for the points $\zeta = a_j$, $j = m-1, m-2, \dots, 2, 1$. We have

$$T\theta_m^+T_{m-1}\theta_{m-1}^+ = g_{m-2}(t)T\theta_m^-T_{m-1}\theta_{m-1}^-, \tag{6.8}$$

$$T\theta_m^+T_{m-1}\theta_{m-1}^+T_{m-2}\theta_{m-2}^+ = g_{m-3}T\theta_m^-T_{m-1}\theta_{m-1}^-T_{m-2}\theta_{m-2}^-, \tag{6.9}$$

. . . .

$$\begin{aligned} &T\theta_m^+T_{m-1}\theta_{m-1}^+T_{m-2}\theta_{m-2}^+\dots T_1\theta_1^+ = \\ &= T\theta_m^-T_{m-1}\theta_{m-1}^-T_{m-2}\theta_{m-2}^-\dots T_1\theta_1^-. \end{aligned} \tag{6.10}$$

Similarly to the system of equations (6.6) and (6.7), from the matrix equations (6.8)–(6.10) we get two equations for every singular point.

The matrix equation (6.3) can be written as

$$p\cdot \exp(i\pi\alpha_{1m}) = \\ = A_{11}^{*(m-1)}p \cdot \exp(-i\pi\alpha_{1m}) + iA_{12}^{*(m-1)}r\cdot\exp(-i\pi\alpha_{1m}), \tag{6.11}$$

$$r\cdot \exp(i\pi\alpha_{1m}) = \\ = iA_{21}^{*(m-1)}p\cdot\exp(-i\pi\alpha_{1m}) + \overline{A}_{11}^{*(m-1)}r\cdot\exp(-i\pi\alpha_{1m}), \tag{6.12}$$

$$q\cdot \exp(i\pi\alpha_{2m}) = \\ = A_{11}^{*(m-1)}q\cdot\exp(-i\pi\alpha_{2m}) + iA12^{*(m-1)}s\cdot\exp(-i\pi\alpha_{2m}), \tag{6.13}$$

$$s\cdot \exp(i\pi\alpha_{2m}) = \\ = iA_{21}^{*(m-1)}q\cdot\exp(-i\pi\alpha_{2m}) + \overline{A}_{11}^{*(m-1)}s\cdot\exp(-i\pi\alpha_{2m}). \tag{6.14}$$

Dividing the corresponding parts of the equations (6.11) and (6.12), (6.13) and (6.14), one can see that the ratios p/r, q/s in the interval (a_{m-1}, a_m) satisfy the boundary condition (2.25),

$$\frac{p}{r} = \frac{iA_{11}^{*(m-1)}p/r + A_{12}^{*(m-1)}}{iA_{21}^{*(m-1)}p/r + i\overline{A}_{11}^{*(m-1)}}, \quad \frac{q}{s} = \frac{A_{11}^{*(m-1)}q/s + iA_{12}^{*(m-1)}}{iA_{21}^{*(m-1)}q/s + \overline{A}_{11}^{*(m-1)}}. \tag{6.15}$$

The coordinates of the points $w = A_m$, $w = A'_m$ also satisfy the same condition and, consequently,

$$p/r = A_m, \quad q/s = A'_m, \tag{6.16}$$

where A'_m is the second point of intersection of the two neighboring circumferences.

Remind that by A_k, A'_k, $k = 1, 2, \dots, m$, we have denoted the complex coordinates of the angular points of the circular polygon $s(w)$ at which two neighboring circumferences may intersect; note that the point A'_k lies more often outside of the contour $l(w)$.

On the plane w, if the origin coincides with the point $w = A_m$, then $A_m = 0$ and $A'_m = \infty$. Consequently, $p = 0$ and $s = 0$. It should be noted that for the interval (a_{m-1}, a_m), if $\nu_m \neq 0$, one can always suppose that

$$G_{m-1} = \begin{pmatrix} A_{11}^{*(m-1)}, & 0 \\ 0, & \overline{A}_{11}^{*(m-1)} \end{pmatrix}. \tag{6.17}$$

Consider the matrix equation (6.8),

$$T_{*(m-1)}\theta_{m-1}^{+} = g_{m-2}\overline{T}_{*(m-1)}\theta_{m-1}^{-}, \quad T_{*(m-1)} = T\theta_m^{+}T_{m-1}. \tag{6.18}$$

From (6.18) we get the system of equations

$$p_{*(m-1)}/r_{*(m-1)} = A_{m-1}, \quad q_{*(m-1)}/s_{*(m-1)} = A'_{m-1}, \tag{6.19}$$

where $p_{*(m-1)}$, $q_{*(m-1)}$, $r_{*(m-1)}$ and $s_{*(m-1)}$ are the elements of the matrix $T_{*(m-1)}$. Taking into account (6.18), the equalities (6.19) can be rewritten as follows:

$$\frac{p_*p_{m-1}+q_*r_{m-1}}{r_*p_{m-1}+s_*r_{m-1}}=A_{m-1},\quad \frac{p_*q_{m-1}+q_*s_{m-1}}{r_*p_{m-1}+s_*s_{m-1}}=A'_{m-1}, \tag{6.20}$$

where p_*, q_*, r_* and s_* are the elements of the matrix $T_*=T\theta^+_m$.

The equalities (6.20) with regard for (6.19) can in their turn be rewritten as

$$\begin{aligned}\frac{r_*p_{m-1}A_m+s_*r_{m-1}A'_m}{r_*p_{m-1}+s_*r_{m-1}}&=A_{m-1},\\ \frac{r_*q_{m-1}A_m+s_*s_{m-1}A'_m}{r_*q_{m-1}+s_*s_{m-1}}&=A'_{m-1}.\end{aligned} \tag{6.21}$$

After simplification, the equations (6.21) take the form

$$r_*p_{m-1}(A_m-A_{m-1})+s_*r_{m-1}(A'_m-A_{m-1})=0, \tag{6.22}$$

$$r_*q_{m-1}(A_m-A'_{m-1})+s_*s_{m-1}(A'_m-A'_{m-1})=0. \tag{6.23}$$

The condition of compatibility of (6.22) and (6.23) with respect to r_* and s_* has the form

$$\frac{p_{m-1}s_{m-1}}{r_{m-1}q_{m-1}}=\frac{A'_m-A_{m-1}}{A_m-A_{m-1}}\cdot\frac{A_m-A'_{m-1}}{A'_m-A'_{m-1}}. \tag{6.24}$$

From the matrix equation (6.9) we obtain the system of equations

$$\begin{aligned}\frac{p_{*(m-1)}p_{m-2}+q_{*(m-1)}r_{m-2}}{r_{*(m-1)}p_{m-2}+s_{*(m-1)}r_{m-2}}&=A_{m-2},\\ \frac{p_{*(m-1)}q_{m-2}+q^*_{*(m-1)}s_{m-2}}{r_{*(m-1)}q_{m-2}+s_{*(m-1)}s_{m-2}}&=A'_{m-2},\end{aligned} \tag{6.25}$$

where $p_{*(m-1)}$, $q_{*(m-1)}$, $r_{*(m-1)}$, $S_{*(m-1)}$ are the elements of the matrix

$$T_{*(m-1)}=T\theta^+_mT_{m-1}\theta^+_{m-1}. \tag{6.26}$$

After certain transformations the above system takes the form

$$r_{*(m-1)}p_{m-2}(A_{m-1}-A_{m-2})+s_{*(m-1)}r_{m-2}(A'_{m-1}-A_{m-2})=0, \tag{6.27}$$

$$r_{*(m-1)}q_{m-2}(A_{m-1}-A'_{m-2})+s_{*(m-1)}s_{m-2}(A'_{m-1}-A'_{m-2})=0. \tag{6.28}$$

The equations (6.27) and (6.28) imply

$$\frac{p_{m-2}s_{m-2}}{r_{m-2}q_{m-2}}=\frac{A'_{m-1}-A_{m-2}}{A_{m-1}-A_{m-2}}\cdot\frac{A_{m-1}-A'_{m-2}}{A'_{m-1}-A'_{m-2}}, \tag{6.29}$$

The remaining matrix equations can be investigated analogously [25-31].

The equations (6.24) and (6.29) are nothing but the invariant cross-ratios of four points of the same circumference at which the given circumference intersects with the two neighboring ones.

From (6.3)–(6.10) we can get all the needed equations with respect to a_k, c_k, $k = \overline{1, m}$, and to the integrations constants p, q, r and s.

For every point $t = a_j$ we have obtained a system of two equations which are homogeneous with respect to the elements of the matrices T_k, $k = \overline{1, m}$; their conditions of compatibility, for e.g., the points $t = a_m$ and a_{m-1}, have the form (6.24) and (6.29). The above-mentioned systems of equations have been obtained under the assumption that $\alpha_{1j} - \alpha_{2j} \neq n$, $n = 0, 1, 2$.

Consider briefly the case where $\alpha_{1j} - \alpha_{2j} = n$, $n = 0, 1, 2$. According to the representations (5.4)–(5.10), the unknown matrices $\chi^+(t)$ and $\chi^-(t)$ in the interval (a_{j-1}, a_j) must satisfy the boundary condition

$$\begin{aligned} &T\theta_m^+ T_{m-1}\theta_{m-1}^+ T_{m-2}\theta_{m-2}^+ \dots T_j\theta_j^+ = \\ &= g_{j-1} T\theta_m^- T_{m1}\theta_{m-1}^- T_{m-2}\theta_{m-2}^- \dots T_j\theta_j^-, \end{aligned} \tag{6.30}$$

where

$$\theta_j^+ = e^{i\pi\alpha_{2j}} \begin{pmatrix} 1, & 0 \\ \pm\pi i, & 1 \end{pmatrix}, \quad \theta_j^- = \overline{\theta}_j^+, \quad n = 0, 2;$$

$$\theta_j^+ = e^{i\pi\alpha_{2j}} \begin{pmatrix} -1, & 0 \\ -\pi i, & 1 \end{pmatrix}, \quad n = 1, \quad \theta_j^- = \overline{\theta}_j^+.$$

It can immediately be verified that (6.30) leads to a usual system of two equations with respect to p_j, q_j, r_j, s_j, but the condition of their compatibility does not provide now the relations analogous to (6.24) and (6.29).

As is mentioned above, matrix equations similar to (6.1)–(6.10) can be obtained for all points $\zeta = a_k$, with the exclusion of the points $\zeta = a_j$ to which there correspond the cut ends of the boundary $l(w)$ of the circular polygon $w = A_j$ for which $\nu_j = 2$. For such points we have either the condition (3.20) or (3.21). This allows one to obtain one equation for each point, the second equation being obtained after determination of $l(z)$, $l(\omega)$ and $l(w)$.

From the matrix representations we first define $u_1^+(t)$ and $u_2^+(t)$ and then compose the ratio $w^+(t) = u_1^+(t)/u_2^+(t)$. According to (5.4)–(5.10), the function $w^+(t)$ for the interval (a_j, a_{j+1}) can be represented as

$$w^+(t) = [A_j^* u_{1j}^+(t) + B_j^* u_{2j}^*(t)]/[C_j^* u_{1j}^+ + D_j^* u_{2j}^+(t)], \tag{6.31}$$

where A_j^*, B_j^*, C_j^*, D_j^* are defined by (5.4)–(5.10).

Calculating the limit as $\zeta \to a_j$ by means of (6.31), we obtain the equation

$$A_j = B_j^*/D_j^*. \tag{6.32}$$

The corresponding equations for the other points $t = a_k$, $k = \overline{1, m+1}$, can be obtained analogously.

Finally, for every point $t = a_j$ we obtain two real, homogeneous with respect to p_j, q_j, r_j and s_j equations, for example, (6.6) and (6.7). From the condition of compatibility of homogeneous equations for $\nu_j \neq 0,1,2$, we obtain invariant cross-ratios for four points of one circumference, for example, (6.24) and (6.29). In the case where $\nu_j = 0,1,2$, the conditions of compatibility of two equations provide certain equations which, however are not anharmonic.

From each system of two equations we can take one equation and, in addition, one more equation of compatibility, i.e. we take two equations for each point $\zeta = a_j$. The number of equations is equal to $2m$ and the number of unknown parameters a_k, c_k^* $k=\overline{1,m}$, p, q, r, s with $ps-rq\neq 0$ is $2m-3$. Consequently, the number of equations will be greater by three than that number of unknown parameters. It should be noted here that from the very beginning we have supposed that the linear fractional transformation over the domain $s(w)$ was performed with a view to have the equation $G_m = E$ (E is the unit matrix) on one part of the boundary $l(w)$. Thus the parameters p, q, r, and s turned out to be real and their number equals to three, since $ps - rq \neq 0$. The above-described method of constructing the functions $w(\zeta)$, $\omega(\zeta)$ and $z(\zeta)$ and the system of equations with respect to a_j, c_j^*, $j = \overline{1,m}$, is assumed to be much more convenient than some other methods. One can give up transformation of the domain $s(w)$. In this case the parameters p, q, r and s will be complex and the number of the unknown parameters will be equal to $2m$. The appearance of three additional equations can be explained just as in the case of linear polygons.

Having constructed the system of equations for determining a_k, c_k^*, p, q, r and s, $k=\overline{1,m}$, we have first to establish the intervals of variation of the parameters c_k^*, $k=\overline{1,m}$, then to solve the system with respect to a_k, c_k^*, $k=\overline{1,m}$ and finally, to define p, q, r and s.

Remind that p_j, q_j, r_j and s_j, $j=\overline{1,m}$, depend implicitly, through the coefficients of the generalized hypergeometric series, on the parameters a_k, c_k^*, $k=\overline{1,m}$. The intervals of variation of the parameters can be established according to [27].

As is known, the series $u_{kj}(\zeta)$, $j=\overline{1,m+1}$, $k=1,2$, converge, respectively, in the neighborhood of the points $\zeta=a_j$, $j=\overline{1,m+1}$, $t=a_{m+1}=\infty$ and the series $\sigma_{kj}(\zeta)$ in the neighborhood of the points $a_j^* = (a_j+a_{j+1})/2$. The convergence radii of these series are bounded by the distance from the given point $t=a_j$ (or from the point a_j^*) to the nearest points $\zeta = a_{j-1}, a_{j+1}$.

The series $u_{kj}(\zeta)$, $k=1,2$, $j=\overline{1,m}$, are whole functions of the parameters c_j^*, $j=\overline{1,m}$, but with respect to ζ these series converge slowly, which makes numerical calculations difficult. As n increases, the coefficients γ_{nj}^k sometimes rapidly increase, although their multipliers $(\zeta-a_j)^n$ on the contrary rapidly decrease. Electronic computers fail to multiply γ_{nj}^k by $(t-a_j)^n$ despite the fact that the series converge. To eliminate this drawback, we have suggested to write the same series in the form of rapidly and uniformly convergent functional series [28-31].

Let us consider the structure of the recurrence formulas (3.15)–(3.16), (3.31)–(3.33). The sum of the first lower indices in the expressions $\gamma_{(k-n)j} \cdot f_{nj}(\alpha_j + k - n)$ is always equal to k, i.e., to the exponent of $(t-a_j)^k$. Instead of the series (3.18) let us consider the function series of the type

$$u_{kj}(t) = (t-a_j)^{\alpha_{kj}} \widetilde{u}_{kj}(t-a_j), \quad \widetilde{u}_{kj}(t) = \sum_{n=0}^{\infty} \gamma_{nj}^{k}(t-a_j), \quad \gamma_{0j}^{k} = 1, \tag{6.33}$$

where γ_{nj}^{k} is defined according to (3.15)–(3.16), in terms of $\gamma_{1j}, \gamma_{2j}, \dots, \gamma_{(n-1)j}$, and the latter in their turn are defined in terms of $f_{kj}(\alpha_j)$, where

$$f_{kj}(t-a_j, \alpha_j) = \alpha_j p_{kj}(t-a_j) + q_{kj}(t-a_j) \tag{6.34}$$

$$p_{nj}(t-a_j) = (-1)^{n-1} \sum_{k=1, j\neq k}^{k} [1 - \alpha_{1k} - \alpha_{2k}] \Big(\frac{t-a_j}{a_j - a_k}\Big)^n, \tag{6.35}$$

$n = 1, 2, \dots,$

$$q_{nj}(t-a_j) = (-1)^{n-2} \sum_{k=1, k\neq j} [\alpha_{1k}\alpha_{2k}(n-1) + c_k^*(a_j - a_k)] \Big(\frac{t-a_j}{a_j - a_k}\Big)^n, \tag{6.36}$$

$n = 2, 3, \dots,$

$$\Big|\frac{t-a_j}{a_j - a_k}\Big| < 1, \quad k \neq j \tag{6.37}$$

$$p_{n\infty}(t) = \sum_{k=1}^{m} [1 - \alpha_{1k}\alpha_{2k}](a_k/t)^n, \tag{6.38}$$

$$q_{n\infty}(t) = \sum_{k=1}^{\infty} [\alpha_{1k}\alpha_{2k}(n+1) + c_k^* a_k](a_k/t)^n, \tag{6.39}$$

$n = 0, 1, 2, \dots,$

$$|a_k/t| < 1. \tag{6.40}$$

The formulas (6.34)–(6.40) show that the fundamental series (6.33) and the series

$$u_{k\infty}(t) = \zeta^{-\alpha k\infty}\Big[1 + \sum_{n=1}^{\infty} \gamma_{n\infty}^{k}(t)\Big] \tag{6.41}$$

converge in the domain $|\zeta - a_j|$ more rapidly than the series (3.18).

The matrices $\chi^{\pm}(t)$ defined by the formulas (5.1)–(5.10) satisfy the boundary condition (2.25).

7. Definition of the Functions $\omega(\zeta)$ and $z(\zeta)$

Along the real t-axis, the function $w^+(t)$ is defined by the equality

$$w^+(t) = u_1^+(t)/u_2^+(t), \quad -\infty < t < +\infty, \tag{7.1}$$

where $u_1^+(t)$ and $u_2^+(t)$, being the linear independent solutions of (1.11), are defined by the formulas (5.1)–(5.10).

Knowing $w(\zeta)$ along the entire real t-axis of the plane, we can find $w(\zeta)$ for $\mathrm{Im}(\zeta) > 0$ for all $t = e_k$, $k = \overline{1, n+1}$, with the help of the well-known formula given in [16].

Note that using the matrix $\chi(\zeta)$ defined by the formulas (5.4)–(5.10), we can construct a canonical matrix for the corresponding homogeneous problem (2.10) with regard for all singular points $t = e_k$, $k = \overline{1, n+1}$, after which it becomes possible to solve the nonhomogeneous boundary value problem (2.10) by means of the Cauchy type integral. This has been done by us in [27]. In the present paper we find the solution of (2.10) in a more simple way than that described in [27] and [29]. We rely here on the linear independent solutions (1.11) and on the general solution of (1.18).

Let us multiply the functions $u_1^+(t)$ and $u_2^+(t)$ by

$$\chi_0(\zeta) = \sqrt{\frac{(\zeta - e_{k-1})(\zeta - e_{k+j-1})}{(\zeta - e_k)(\zeta - e_{k+j})}}.$$

The matrix $\chi_1(\zeta)$ defined by the formulas (5.4)–(5.10) satisfies the boundary condition (2.25), as far as we take for granted that the equalities (6.1)–(6.32) are fulfilled. This means that the columns of the matrix $\chi_1(\zeta)$ defined by the formulas (5.4)–(5.10) satisfy the boundary condition (2.25). To obtain the solution $\Phi_1(\zeta)$, we have to take the elements of the first column of the matrix $\chi_1(\zeta)$ which are defined by the formulas (6.1)–(6.32) and then to compose the vector $\Phi_1(\zeta) = [u_1(\zeta), u_2(\zeta)]$, $\mathrm{Im}(\zeta) \geq 0$.

We have taken the elements of the first column of the matrix $\chi_1(\zeta)$ because the relation $w(\zeta) = u_1(\zeta)/u_2(\zeta)$ gives the general solution of the Schwarz equation with the right-hand side (1.18), while the ratio $u_1'(\zeta)/u_2'(\zeta)$ does not satisfy the equation (1.18).

The vector $\Phi'(\zeta) = \Phi_1(\zeta)\chi_0(\zeta)$, where $\chi_0(\zeta) = \sqrt{\frac{(\zeta-e_{k-1})(\zeta-e_{k+j-1})}{(\zeta-e_k)(\zeta-e_{k+j})}}$, will be a solution of the problem (2.15). Consequently, the elements of the vector $\Phi'(\zeta)$, $\omega'(\zeta) = u_1^+(\zeta)\chi_0(\zeta)$ and $z'(\zeta) = u_2^+(\zeta)\chi_0(\zeta)$, satisfy both the boundary conditions (2.15) and the conditions at the singular points $t = e_k$, $k = \overline{1, n+1}$.

Now we can write the following equalities:

$$d\omega(t) = u_1^+(t)\chi_0^+(t)dt, \quad -\infty < t < +\infty, \tag{7.2}$$

$$dz(t) = u_2^+(t)\chi_0^+(t)dt, \quad -\infty < t < +\infty. \tag{7.3}$$

Integrating the equalities (7.2) and (7.3) in the intervals $(-\infty, t)$, (e_j, t), $j = 1, 2, \dots, n$, we obtain

$$\omega^+(t) = \int\limits_{-\infty}^{t} u_1^+(t)\chi_0^+(t)dt + \omega^+(-\infty), \quad -\infty < t < e_1, \tag{7.4}$$

$$z^+(t) = \int\limits_{-\infty}^{t} u_2^+(t)\chi_0^+(t)dt + z^+(-\infty), \quad -\infty < t < e_1, \tag{7.5}$$

$$\omega^+(t) = \int\limits_{e_j}^{t} u_1^+(t)\chi_0^+(t)dt + \omega_j^+(e_j), \quad j = \overline{1, n+1}, \quad e_j < t < e_{j+1}, \tag{7.6}$$

$$z^+(t) = \int\limits_{e_j}^{t} u_2^+(t)\chi_0^+(t)dt + z_j^+(e_j), \quad j = \overline{1, n+1}, \quad e_j < t < e_{j+1}, \tag{7.7}$$

where $\omega^+(-\infty)$, $z^+(-\infty)$, $\omega^+(e_j)$, $z^+(e_j)$ are the limiting values of the corresponding functions $\omega^+(t)$, $z^+(t)$ from the right at the points $-\infty$, e_j, $j = \overline{1, n+1}$.

Obviously, the functions $\omega^+(t)$, $z^+(t)$ defined by the formulas (7.4)–(7.7) satisfy the boundary conditions (2.10).

In the formulas (7.4)–(7.7) we can separate the real and imaginary parts and get expressions for the functions $\varphi(t)$, $\psi(t)$, $\chi(t)$ and $y(t)$.

Passing in the formulas (7.4) and (7.5) to the limit as $t \to e_j$ from the left, we arrive at

$$\omega^+(e_1) = \int\limits_{-\infty}^{e_1} u_1^+(t)\chi_0^+(t)dt + \omega^+(-\infty), \tag{7.8}$$

$$z^+(e_1) = \int\limits_{-\infty}^{e_1} u_2^+(t)\chi_0^+(t)dt + z^+(-\infty), \tag{7.9}$$

$$\omega^+(e_{j+1}) = \int\limits_{e_j}^{e_{j+1}} u_1^+(t)\chi_0^+(t)dt + \omega^+(e_j), \quad j = \overline{1, n+1}, \tag{7.10}$$

$$z + (e_{j+1}) = \int\limits_{e_j}^{e_{j+1}} u_2^+(t)\chi_0^+(t)dt + z^+(e_j), \quad j = \overline{1, n+1}, \tag{7.11}$$

where $\omega^+(e_{j+1})$, $z^+(e_{j+1})$, are the limiting values of the functions $\omega^+(t)$, $z^+(t)$ from the left at the point $t = e_{j+1}$.

In the formulas (7.4)–(7.11) it is assumed that the integrands at the points $t=-\infty$, $t=e_j$, $j=\overline{1,n}$ are integrable. In case the integrands are nonintegrable at some point $t=e_j$ of e_1, $e_2, e_2, \dots, e_{n+1}$, we take the integrals from the other end of the interval, where they are integrable. But if the above-mentioned functions are nonintegrable at both ends of the interval, then we take any interior point of the interval and from that point (as the lower limit) from which the integral is taken.

For determination of the parameters a_j and c_j, $j=\overline{1,m}$, we have obtained a system of higher transcendent equations, e.g., the equations (6.6)–(6.32); as for the parameters $t=e_j$, $j=\overline{1,n}$, which do not coincide with the parameters $t=a_j$ and the function $\chi_0(\zeta)$ depends on, and also as for the parameter Q which is connected with the liquid discharge, for their determination we have obtained the system of equations (7.8)–(7.11).

Having found all the unknown parameters on which the functions $\omega(\zeta)$, $z(\zeta)$, and $w(\zeta)$ depend, by the formulas (7.6)–(7.7) we can find the equations for the unknown parts of the boundaries of the domains $s(z)$, $s(\omega)$ and $s(w)$, as well as for the other geometric and mechanical parameters of the liquid flow [30, 31].

8. Another Method of Solving the System (6.3)–(6.10) with Respect to p_j/r_j, s_j/q_j

Of the system (6.3)–(6.10), we consider the matrix equations for two neighboring points $t=a_j$ and $t=a_{j-1}$. We have

$$A^+_{j+1}=g_jA^-_{j+1},\quad A^+_{j+1}T_j\theta^+_j=g_{j-1}A^-_{j+1}T_j\theta^-_j, \tag{8.1}$$

where

$$A^\pm_{j+1}=T\theta^\pm_mT_{m-1}\theta^\pm_{m-1}\dots T_{j+1}\theta^\pm_{j+1}. \tag{8.2}$$

Excluding A^+_{j+1} from the system (8.1), we obtain the equation with respect to T_j:

$$T_j(\theta^+_j)^2=B^jT_j,\quad B^j=\begin{pmatrix}B^j_{11}, & B^j_{12}\\ B^j_{21}, & B^J_{22}\end{pmatrix}=(A^-_{j+1})^{-1}g_j^{-1}g_{j-1}A^-_{j+1}. \tag{8.3}$$

When solving (8.3), we consider the following cases: (1) the difference $\alpha_{1j}-\alpha_{2j}$ is not an integer; (2) the difference $\alpha_{1j}-\alpha_{2j}$ is an integer.

1. The solution of (8.3) with respect to the elements of the matrix T_j has the form

$$p_j/r_j=B^j_{12}[\lambda_{1j}-B^j_{11}]^{-1},\quad p_j/r_j=(B^j_{21})^{-1}[\lambda_{1j}-B^j_{22}] \tag{8.4}$$

$$s_j/q_j=(B^j_{12})^{-1}[\lambda_{2j}-B^j_{11}]^{-1},\quad s_j/q_j=B^j_{21}[\lambda_{2j}-B^j_{22}]^{-1}. \tag{8.5}$$

We take one equation from each of (8.4) and (8.5) because the second equations coincide with the first ones owing to the fact that

$$\det B^j = \lambda_{1j}\lambda_{2j}, \quad B^j_{11} + B^j_{22} = \lambda_{1j} + \lambda_{2j}. \tag{8.6}$$

Consequently, the solution of (8.3) for one point is given in the form of two scalar equations with respect to the parameters a_j, c_j, $j = \overline{1, m}$. Recall (5.1) and (5.2) in which it is seen that the parameters p_j, q_j, r_j, s_j depend implicitly on a_j, c_j, $j = \overline{1, m}$.

The solution of the matrix equation

$$T\theta^+_m = g_{m-1}T\theta^-_m, \quad g_{m-1} = \begin{pmatrix} g^{m-1}_{11}, & g^{m-1}_{12} \\ g^{m-1}_{21}, & g^{m-1}_{22} \end{pmatrix} \tag{8.7}$$

have the form

$$p/r = g^{m-1}_{12}[\lambda_{1m} - g^{m-1}_{11}]^{-1}, \quad s/q = B^{m-1}_{12}[\lambda_{2m} - g^{m-1}_{11}], \tag{8.8}$$

where p/r and s/q are the integration constants of the Schwarz differential equation (1.15).

We can immediately verify that the solutions (8.4) and (8.5) are real, hence the equation

$$p_j s_j/(r_j q_j) = [\lambda_{2j} - B^j_{11}][\lambda_{1j} - B^j_{11}]^{-1}, \tag{8.9}$$

is real as well. This equation is connected with the invariant cross-ratio of four intersection points of one circumference with two neighboring circumferences (see, e.g., (6.29) or (6.24)).

2(a). The difference $\lambda_{1j} - \lambda_{2j} = n$, $n = 0, 2$. In this case the equation (8.3) takes the form

$$\lambda_{2j}T_j \begin{pmatrix} 1, & 0 \\ 2\pi i, & 1 \end{pmatrix} = B^j T_j. \tag{8.10}$$

The solution (8.10) has the form

$$\lambda_{2j}(p_j + 2\pi i q_j) = B^j_{11}p_j + B^j_{12}r_j, \quad \lambda_{2j}[r_j + 2\pi i s_j) = B^j_{21}p_j + B^j_{22}r_j, \tag{8.11}$$

$$s_j/q_j = [\lambda_{2j} - B^j_{11}](B^j_{12})^{-1}, \quad s_j/q_j = B^j_{2j}[\lambda_{2j} - B^j_{22}]^{-1}. \tag{8.12}$$

We take one equation from each of (8.11) and (8.12) because the second equations coincide with the first ones. Indeed, this is obvious for (8.12), while for (8.11) it is necessary to indicate the way of proving. First we define q_j/s_j from (8.12) and substitute the obtained value into the first of the equations (8.11), then we divide by s_j the left and right sides of both equations (8.11) and obtain

$$\frac{p_j}{s_j}(\lambda_{2j} - B^j_{11}) - \frac{r_j}{s_j}B^j_{12} = -2\pi i\lambda_{1j}B^j_{12}[\lambda_{1j} - B^j_{11}], \tag{8.13}$$

$$\frac{p_j}{s_j}(-B^j_{21}) + (\lambda_{2j} - B^j_{22})\frac{r_j}{s_j} = 2\pi i\lambda_{2j}. \tag{8.14}$$

These equations coincide because the coefficients (including free terms) are proportional.

2(b). If $\alpha_{1j} - \alpha_{2j} = 1$, then the equation (8.3) takes the form

$$\lambda_{2j} T_j \begin{pmatrix} 1, & 0 \\ -2\pi i, & 1 \end{pmatrix} = B^j T_j. \tag{8.15}$$

In this case (8.12) remains invariable, and proportionality of the coefficients (including free terms) is not violated if in the systems (8.13) and (8.14) we replace $2\pi i$ by $-2\pi i$.

Defining the elements p_j, q_j, r_j and s_j from (5.1) as depending on a_j, c_j, $j = \overline{1, m}$, and substituting them in (8.4), (8.5), (8.9), (8.11) and (8.12), we obtain equations with respect to a_j, c_j, $j = \overline{1, m}$.

References

1. P. Ya. Polubarinova-Kochina, Application of the theory of linear differential equations to some problems of underground water motion. (Russian) *Izv. Acad. Nauk SSSR, Ser. Mat.* No. 5–6(1939), 579–608.
2. P. Ya. Polubarinova-Kochina, The theory of underground water motion. 2nd ed. (Russian) *Moscow, Nauka*, 1977.
3. P. Ya. Polubarinova-Kochina, On additional parameters after examples of circular quadrangles. (Russian) *Prikl. Mat. Mekh.* **55**(1991), No. 2, 222–227; translation in *J. Appl. Math. Mech.* **55**(1991), No. 2, 176–180 (1992).
4. P. Ya. Polubarinova-Kochina, Circular polygons in filtration theory. (Russian) *Problems of mathematics and mechanics,* 166–177, *"Nauka" Sibirsk. Otdel., Novosibirsk,* 1983.
5. P. Ya. Polubarinova-Kochina, Analytic theory of linear differential equations in the theory of filtration. Mathematics and problems of water handling facilities. *Collection of scientific papers,* 19–36. *Naukova Dumka, Kiev,* 1986,
6. P. Ya. Polubarinova-Kochina, V. G. Prjazhinskaya, and V. N. Emikh, Mathematical methods in irrigation. (Russian) *Moscow, Nauka*, 1969.
7. Ya. Bear, D. Zaslavskii, and S. Irmey, Physical and mathematical foundations of water filtration. (Translated from English) *Mir, Moscow*, 1971.
8. G. Korn and T. Korn, Mathematical handbook for scientists and engineers. *McGraw-Hill Company, New York-Toronto-London*, 1961.
9. N. I. Muskhelishvili, Singular integral equations. 3rd ed. (Russian) *Nauka, Moscow*, 1968.
10. N. P. Vekua, Systems of singular integral equations and certain boundary value problems. 2nd ed. (Russian) *Nauka, Moscow*, 1970.
11. E. L. Ince, Ordinary differential equations. (Translation from English) *Gos. Nauchn.-Tekhn. Izd. Ukrainy, Kharkov,* 1939.
12. A. Hurwitz and R. Courant, Theory of functions. (Translation from German) *Nauka, Moscow*, 1968.
13. G. N. Goluzin, Geometrical theory of functions of a complex variable. 2nd ed. (Russian) *Nauka, Moscow*, 1966.
14. V. V. Golubev, Lectures in analytical theory of differential equations. 2nd ed. (Russian) *Gostekhizdat, Moscow-Leningrad*, 1950.

15. E. A. Codington and N. Levinson, Theory of ordinary differential equations. *McGraw-Hill Book Company, Inc., New York-Toronto-London,* 1955.
16. V. Koppenfels and F. Stallmann, Practice of conformal mappings. German translation. *Springer-Verlag, Berlin-Göttingen-Heidelberg*, 1963.
17. E. Kamke, Referenbook in ordinary differential equations. 2nd ed. (Translation from German) *Gosizdat, Moscow,* 1961.
18. E. T. Whittaker and D. N. Watson, A course of modern analysis. *Cambridge University Press, Cambridge*, 1962.
19. G. Sansone, Ordinary differential equations, I. (Translation from English) *Izd. Inostr. Lit., Moscow*, 1953.
20. G. Bateman and A. Erdelyi, Higher transcendental functions. *McGrow-Hill Book Company, New York-Toronto-London,* 1955.
21. I. A. Alexandrov, Parametric continuations in the theory of schlicht functions. (Russian) *Nauka, Moscow*, 1976.
22. M. A. Lavrent'ev and B. V. Shabat, Methods of the theory of functions of a complex variable. (Russian) *Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow,* 1958.
23. V. V. Stepanov, Course of differential equations. 5th edition. (Russian) *Gosudarstv. Izdat. Tekhniko-Teoretich. Lit., Moscow-Leningrad,* 1950.
24. A. P. Tsitskishvili, Conformal mapping of a half-plane on circular polygons. (Russian) *Trudy Tbiliss. Univ.* **185**(1977), 65–89.
25. A. P. Tsitskishvili, On the conformal mapping of a half-plane onto circular polygons with a cut. (Russian) *Differentsial'nye Uravneniya* **12**(1976), No. 1, 2044–2051.
26. A. P. Tsitskishvili, A method of explicit solution of a class of plane problems of the filtration theory. (Russian) *Soobshch. Acad. Nauk Gruzii* **142**(1991), No. 2, 285–288.
27. A. P. Tsitskishvili, Application of the theory of linear differential equations to the solution of certain plane problems of filtration theory. (Russian) *Trudy Tbiliss. Univ. Mat. Mekh. Astronom.,* No. 19-20 (1986), 295–329.
28. A. Tsitskishvili, Solution of the Schwarz differential equations. *Mem. Differential Equations Math. Phys.* **11**(1997), 129–156.
29. A. Tsitskishvili, Solution of some plane filtration problems with partially unknown boundaries. *Mem. Differential Equations Math. Phys.* **15**(1998), 109–138.
30. A. Tsitskishvili, Solution of a problem of the theory of filtration through a plane earth dam (coffer-dam) when water depth in a downstream can be neglected. *Proc A. Razmadze Math. Inst.* **126**(2001), 75–96.
31. A. Tsitskishvili, On the motion of underground waters towards a slope of an earth structure. *Proc A. Razmadze Math. Inst.* **128**(2002), 121–146.

Chapter IV

Exact Solution of Spatial with Partially Unknown Boundaries Axisymmetric Problems of The Filtration Theory

Abstract. We suggest a general method of solution of spatial axisymmetric problems of steady liquid motion in a porous medium with partially unknown boundaries. The liquid motion of ground waters in a porous medium is subjected to the Darcy law. The porous medium is undeformable, isotropic and homogeneous. The velocity potential $\varphi(z,\rho)$ and the flow function $\psi(z,\rho)$ are mutually connected and separately satisfy different equations of elliptic type, where z is the coordinate of the axis of symmetry, and ρ is the distance to that axis.

To the domain $S(\sigma)$ of the liquid motion on the plane of complex velocity there corresponds a circular polygon. The mapping $\omega = \varphi + i\psi$ belongs to the class of quasi-conformal mappings. Using the functions $\omega_0(\zeta) = \varphi_0(\xi,\eta) + i\psi_0(\xi,\eta)$ and $\sigma(\zeta) = z(\xi,\eta) + i\rho(\xi,\eta)$ we map conformally the half-plane $\mathrm{Im}\,(\zeta) > 0$ onto the domains $S(\sigma)$, $S(\omega_0)$ and $S(\omega_0'(\zeta)/\sigma'(\zeta))$. These functions satisfy all the boundary conditions, while the functions $\varphi_1(\xi,\eta) = \varphi(\xi,\eta) - \varphi_0(\xi,\eta)$, $\psi_1(\xi,\eta) = \psi(\xi,\eta) - \psi_0(\xi,\eta)$ satisfy the system of differential equations and also the zero boundary conditions. The solution of these equations is reduced to a system of Fredholm integral equations of second kind which are solved uniquely by rapidly converging series.

1. Liquid Motion with Axial Symmetry

In this paper we suggest an effective algorithm allowing one to construct solutions of spatial with partially unknown boundaries axisymmetric problems of filtration.

Let us consider some spatial axisymmetric problems (with partially unknown boundaries) of the theory of steady motion of incompressible liquid in a porous medium obeying the Darcy law. The porous medium is assumed to be undeformable, isotropic and homogeneous ([1]–[39]).

The liquid motion is said to be axisymmetric if all velocity vectors lie in half-planes passing through some line which is called the symmetry axis. The picture of the liquid flow is the same for all such planes. The field of velocities of an axisymmetric liquid motion is completely described by the plane field taken from any of such half-planes. The symmetry axis is assumed to be the z-axis which is directed vertically downwards. The distance to the oz-axis is denoted by $\rho = \sqrt{x^2+y^2}$, v_z and v_ρ denote the coordinates of the vector of velocity $\vec{v}(v_z, v_\rho)$ which is connected with the velocity potential as follows: $\vec{v}(v_z, v_\rho) = \operatorname{grad}\varphi(z,\rho)$ ([1]–[39]).

Of an infinite set of half-planes we select arbitrarily the one passing through the symmetry axis on which the moving liquid occupies a certain simply connected domain $S(\sigma)$, where $\sigma = z + i\rho$. Some part of its boundary is unknown and should be defined.

The lines of intersection of the surface and the planes passing through the oz-axis of rotation are called meridians, and the lines of intersection with the planes perpendicular to the oz-axis are called parallels.

1. The Notion of a Stream Function for an Axisymmetric Flow. Let us cite once again the definition of axisymmetric flow, analogous to that we presented above. The flow is called axisymmetric if the stream planes passing through the given axis, and every such plane have the same picture of distribution of flow lines ([1]–[6]). oz is assumed to be the symmetry axis of the cylindrical system of coordinates ρ, θ, z. Then it follows from the definition that the component of velocity, when the liquid flow is potential, has the form $v_\theta = 0$. Then the equation of continuity takes the form

$$\frac{\partial(\rho v_z)}{\partial z} + \frac{\partial(\rho v_\rho)}{\partial \rho} = 0. \tag{1.1}$$

The differential equation of any stream line for axisymmetric flow, $v_\rho\, dz - v_z\, d\rho = 0$, multiplied by ρ, is a full differential of some stream function $\psi(\rho, z)$, $d\psi = \rho v_\rho\, dz - \rho v_z\, d\rho$. Thus $v_z = \frac{1}{\rho}\frac{\partial\psi}{\partial\rho}$, $v_\rho = -\frac{1}{\rho}\frac{\partial\psi}{\partial z}$. On the other hand, $v_z = \frac{\partial\varphi}{\partial z}$, $v_\rho = \frac{\partial\varphi}{\partial\rho}$, and hence

$$v_z = \frac{\partial\varphi}{\partial z} = +\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}, \quad v_\rho = \frac{\partial\varphi}{\partial\rho} = -\frac{1}{\rho}\frac{\partial\psi}{\partial z}. \tag{1.2}$$

If the liquid flow is irrotational, i.e. potential, $\frac{\partial v_z}{\partial\rho} = \frac{\partial v_\rho}{\partial z}$, then the stream function should satisfy the equation

$$\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial z}\right) + \frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\right) = 0. \tag{1.3}$$

Recall that $\varphi(z,\rho)$ is a harmonic function of the cylindrical system of coordinates. Unlike the plane case, the stream function $\psi(z,\rho)$ is not harmonic and it follows from (1.2) that

$$\frac{\partial\varphi}{\partial z}\frac{\partial\psi}{\partial z} + \frac{\partial\varphi}{\partial\rho}\frac{\partial\psi}{\partial\rho} = 0. \tag{1.4}$$

The system (1.1), (1.3) can be rewritten as

$$\Delta\varphi(z,\rho)+\frac{1}{\rho}\frac{\partial\varphi}{\partial\rho}=0, \tag{1.5}$$

$$\Delta\psi(z,\rho)-\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}=0, \tag{1.6}$$

where Δ is the Laplace operator. We rewrite the system (1.5), (1.6) as follows:

$$\frac{\partial^2\varphi}{\partial z^2}+4\alpha\frac{\partial^2\varphi}{\partial\alpha^2}+4\frac{\partial\varphi}{\partial\alpha}=0, \tag{1.7}$$

$$\frac{\partial^2\psi}{\partial z^2}+4\alpha\frac{\partial^2\psi}{\partial\alpha^2}=0, \tag{1.8}$$

where $\alpha=\rho^2$.

It can be seen from (1.7) and (1.8) that for $\alpha=\rho^2\neq 0$ the system is elliptic. Hence $\varphi(z,\rho)=\text{const}$ and $\psi(z,\rho)=\text{const}$ are orthogonal. However, the mapping $f(z+i\rho)=\varphi(z,\rho)+i\psi(z,\rho)$ is not conformal. The mappings under consideration constitute a class of quasi-conformal mappings. The system (1.2) is elliptic only in the domains not adjoining the axis of rotation. The system degenerates on that axis and quasi-conformity violates.

When the point $z+i\rho$ approaches the axis of rotation, the ratio of half-axes of these ellipses infinitely increases. Such violation of quasi-conformity is a geometric criterion of degeneration of a system on the axis of rotation. In the domains whose closure does not intersect the axis of rotation, the mappings $f=\varphi+i\psi$, satisfying the system (1.2), are quasi-conformal, possessing, owing to the system (1.2) the principal properties of quasi-conformal mappings ([1]–[39]).

A linear elliptic equation is said to be degenerated if in some part of its domain of definition the quadratic form is defined nonpositively.

It can be seen from (1.7) and (1.8) that the given system for $\alpha=\rho^2\neq 0$ is elliptic.

Along the oz-axis, as $\alpha\to 0$, we have

$$\frac{\partial^2\varphi}{\partial z^2}+4\frac{\partial\varphi}{\partial\alpha}=0, \tag{1.9}$$

$$\frac{\partial^2\psi}{\partial z^2}=0. \tag{1.10}$$

Along the oz-axis of symmetry we have

$$\begin{gathered}\lim_{\rho\to 0}\frac{\partial\varphi}{\partial\rho}=0,\quad \lim_{\rho\to 0}\frac{\partial\psi}{\partial\rho}=0,\quad \lim_{\rho\to 0}\frac{\partial\psi}{\partial z}=0,\\ \lim_{\rho\to 0}\frac{1}{\rho}\frac{\partial\varphi}{\partial\rho}=\frac{\partial^2\varphi}{\partial\rho^2},\quad \lim_{\rho\to 0}\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}=\frac{\partial^2\psi}{\partial\rho^2}.\end{gathered} \tag{1.11}$$

We map the half-plane $\mathrm{Im}(\zeta) > 0$ (or $\mathrm{Im}(\zeta) < 0$) of the complex plane $\zeta = \xi + i\eta$ conformally onto the domains $S(\sigma)$,

$$\sigma(\zeta) = z(\xi, \eta) + i\rho(\xi, \eta). \tag{1.12}$$

The system (1.2) takes on the plane $\xi + i\eta$ the form

$$\frac{\partial\varphi}{\partial\xi} = \frac{1}{\rho(\xi, \eta)} \frac{\partial\psi}{\partial\eta}, \tag{1.13}$$

$$\frac{\partial\varphi}{\partial\eta} = -\frac{1}{\rho(\xi, \eta)} \frac{\partial\psi}{\partial\xi}, \tag{1.14}$$

that is,

$$\frac{\partial}{\partial\xi}\left[\rho(\xi, \eta) \frac{\partial\varphi(\xi, \eta)}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[\rho(\xi, \eta) \frac{\partial\varphi(\xi, \eta)}{\partial\eta}\right] = 0, \tag{1.15}$$

$$\frac{\partial}{\partial\xi}\left[\frac{1}{\rho(\xi, \eta)} \frac{\partial\psi(\xi, \eta)}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[\frac{1}{\rho(\xi, \eta)} \frac{\partial\psi(\xi, \eta)}{\partial\eta}\right] = 0. \tag{1.16}$$

From (1.13) and (1.14) follows the condition (1.4).

The boundary conditions have the following forms.

(1) On the free (depression) surface:

$$\varphi(z, \rho) - kz = \text{const}, \tag{1.17}$$

$$\psi(z, \rho) = \text{const}, \tag{1.18}$$

where $k = \text{const}$ is the filtration coefficient;

(2) along the boundary of water basins:

$$\varphi(z, \rho) = \text{const}, \tag{1.19}$$

$$a_1 z + b_1 \rho + c_1 = 0, \quad a_1, b_1, c_1 = \text{const}; \tag{1.20}$$

(3) along the leaking intervals:

$$\varphi(z, \rho) - kz = \text{const}, \tag{1.21}$$

$$a_2 z + b_2 \rho + c_2 = 0, \quad a_2, b_2, c_2 = \text{const}; \tag{1.22}$$

(4) along the symmetry axis, when a segment of the oz-axis of symmetry coincides with a segment of the boundary of $S(\sigma)$:

$$\rho = 0, \tag{1.23}$$

$$\psi(z, \rho) = 0, \tag{1.24}$$

but if the symmetry axis does not coincide with some part of the boundary of the flow domain $S(\sigma)$, then

$$\rho \neq 0, \quad \rho = \text{const}, \quad const \neq 0, \tag{1.25}$$

$$\psi(z, \rho) = \text{const}, \quad const \neq 0; \tag{1.26}$$

(5) along impermeable boundaries:

$$\psi(z, \rho) = \text{const}, \tag{1.27}$$

$$a_3 z + b_3 \rho + c_3 = 0, \quad a_3, b_3, c_3 = \text{const}; \tag{1.28}$$

(6) along the impermeable boundary, the velocity vector is directed along that boundary.

(7) the velocity vector is perpendicular to the boundary of water basins.

(8) along the free surface (depression curve) we have

$$v_z^2 + v_\rho^2 - k v_z = 0. \tag{1.29}$$

As is stated in our work [31] on the plane of complex velocity we have circular polygons of particular types. But this class of problems is much more wider. There are axisymmetric spatial problems with partially unknown boundaries when the boundary of the domain does not involve the symmetry axis, but as is mentioned above, there are problems when the boundary of the domain involves the axis of symmetry or its parts.

For circular polygons, in particular, for linear polygons, we are able to solve plane problems of filtration with partially unknown boundaries. The statement and solution of the corresponding plane problems with partially unknown boundaries of filtration can be found in [26]–[39].

Suppose we have solved the plane problem, i.e. constructed analytic functions by which the half-plane $\mathrm{Im}(\zeta) > 0$ (or $\mathrm{Im}(\zeta) < 0$) of the plane $\zeta = \xi + i\eta$ is mapped conformally onto a circular polygon.

For general discussion we assume that there is a circular polygon with number of vertices m. To find such an analytic function, we have to solve a nonlinear third order Schwarz differential equation whose solution is reduced to that of a differential Fuchs class equation. The Schwarz equation, and hence the corresponding Fuchs class equation, involves $2(m-3)$ essential unknown parameters. After integration of the Schwarz equation there appear six additional parameters of integration. To find these parameters, we write a system of $2(m-3)$ higher transcendent equations and a system consisting of six equations. The boundary conditions for the problem of filtration contain additional unknown parameters. Further, using the solutions of the plane problems, we construct the solutions $\varphi(\xi, \eta)$ and $\psi(\xi, \eta)$ for the systems (1.13)–(1.16) of differential equations of spatial axisymmetric problems. They allow one to construct the functions which map quasi-conformally the half-plane $\mathrm{Im}(\zeta) \geq 0$ onto the domain of the complex potential and onto the domains of the complex velocity, i.e., onto $S(\omega_0)$ and $S(\omega_0'(\zeta)/\sigma'(\zeta))$.

For three analytic functions

$$\sigma(\zeta) = z(\xi, \eta) + i\rho(\xi, \eta), \quad \omega_0(\zeta) = \varphi_0(\xi, \eta) + i\psi_0(\xi, \eta),$$
$$w_0(\zeta) = \omega_0'(\zeta)/\sigma'(\zeta)$$

we introduce the notation

$$\begin{gathered}\Delta z(\zeta,\eta)=0,\ \ \Delta\rho(\zeta,\eta)=0,\ \ \Delta\varphi_0(\zeta,\eta)=0,\\ \Delta\psi_0(\zeta,\eta)=0,\ \ \operatorname{Im}(\zeta)\geq 0,\end{gathered}\tag{1.30}$$

which map conformally the half-plane $\operatorname{Im}(\zeta)\geq 0$ onto the domain $S(\sigma)$ of liquid motion, the domains of the complex potential $S(\omega_0)$ and the domains of the complex velocity $S(\omega_0'(\zeta)/\sigma'(\zeta))$.

Below, for the half-plane we will need the Dirichlet problem. Suppose that on the real axis there is a function $u(\xi)$ bounded by a finite number of points of discontinuity. To find a value at the point $\zeta=\xi+i\eta$ of the harmonic in the upper half-plane function, we have to use the Poisson integral

$$u(\xi)=\frac{1}{\pi}\int\limits_{-\infty}^{+\infty}u(t)\,\frac{\eta}{(t-\xi)^2+\eta^2}\,dt,\tag{1.31}$$

where $\zeta=\xi+i\eta$.

2. Solution of the System (1.13), (1.14)

We rewrite the system (1.13), (1.14) as follows:

$$\Delta\varphi(\xi,\eta)+a(\xi,\eta)\,\frac{\partial\varphi}{\partial\xi}+b(\xi,\eta)\,\frac{\partial\varphi}{\partial\eta}=0,\tag{2.1}$$

$$\Delta\psi(\xi,\eta)-a(\xi,\eta)\,\frac{\partial\psi}{\partial\xi}-b(\xi,\eta)\,\frac{\partial\psi}{\partial\eta}=0,\tag{2.2}$$

where

$$a(\xi,\eta)=\frac{1}{\rho(\xi,\eta)}\,\frac{\partial\rho}{\partial\xi}\,,\quad b(\xi,\eta)=\frac{1}{\rho(\xi,\eta)}\,\frac{\partial\rho}{\partial\eta}\,,\quad \Delta=\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\,.\tag{2.3}$$

Below we will pass to the consideration of the problem of solvability of the system of differential equations (2.1), (2.2) with respect to the functions $\varphi(\xi,\eta)$ and $\psi(\xi,\eta)$ which should satisfy both the compatibility conditions (1.13) and (1.14) and the mixed boundary conditions (1.17)–(1.28) on the known and unknown parts of the boundary. First of all, we replace $\varphi(\xi,\eta)$ and $\psi(\xi,\eta)$ by $\varphi_0(\xi,\eta)+\varphi_1(\xi,\eta)$ and $\psi_0(\xi,\eta)+\psi_1(\xi,\eta)$, where $\varphi_0(\xi,\eta)$ and $\psi_0(\xi,\eta)$ are conjugate, harmonic in the domain $\operatorname{Im}(\zeta)>0$ functions satisfying the boundary conditions. This transformation makes it possible for the unknown functions $\varphi_1(\xi,\eta)$ and $\psi_1(\xi,\eta)$ to satisfy the zero boundary conditions. Note that the system of equations (2.1) and (2.2) will alter hereat.

As is said above, a solution of the system (2.1) and (2.2) will be sought with regard for (1.13) and (1.14) in the form

$$\varphi(\xi,\eta)=\varphi_0(\xi,\eta)+\varphi_1(\xi,\eta),\tag{2.4}$$

$$\psi(\xi,\eta) = \psi_0(\xi,\eta) + \psi_1(\xi,\eta), \tag{2.5}$$

where $\varphi_0(\xi,\eta)$, $\psi_0(\xi,\eta)$ are the conjugate harmonic functions,

$$\Delta\varphi_0(\xi,\eta) = 0, \quad \Delta\psi_0(\xi,\eta) = 0, \tag{2.6}$$

which satisfy the Cauchy–Riemann conditions

$$\frac{\partial\varphi_0}{\partial\xi} = \frac{\partial\psi_0}{\partial\eta}\,, \quad \frac{\partial\varphi_0}{\partial\eta} = -\frac{\partial\psi_0}{\partial\zeta} \tag{2.7}$$

and also all the boundary conditions.

By means of the functions $\omega_0(\xi) = \varphi_0(\xi,\eta) + i\psi_0(\xi,\eta)$, $\sigma(\zeta) = z(\xi,\eta) + i\rho(\xi,\eta)$, the half-plane $\mathrm{Im}(\zeta) > 0$ (or $\mathrm{Im}(\zeta) < 0$) of the plane $\zeta = \xi + i\eta$ is, as is said above, mapped conformally onto the domains $S(\omega)$, $S(\sigma)$, $S(w)$, where $w(\zeta) = \omega'(\zeta)/\sigma'(\zeta)$. The functions $z(\xi,\eta)$ and $\rho(\xi,\eta)$ should satisfy the conditions

$$\Delta z(\xi,\eta) = 0, \quad \Delta\rho(\xi,\eta) = 0, \tag{2.8}$$

$$\frac{\partial z}{\partial\xi} = \frac{\partial\rho}{\partial\eta}\,, \quad \frac{\partial z}{\partial\eta} = -\frac{\partial\rho}{\partial\xi}\,. \tag{2.9}$$

The system (2.1), (2.2) can be written with respect to $\varphi_1(\xi,\eta)$, $\psi_1(\xi,\eta)$ as follows:

$$\begin{aligned} &\Delta\varphi_1(\xi,\eta) + a(\xi,\eta)\,\frac{\partial\varphi_1(\xi,\eta)}{\partial\xi} + b(\xi,\eta)\,\frac{\partial\varphi_1(\xi,\eta)}{\partial\eta} = \\ &\quad = -\Big[\Delta\varphi_0(\xi,\eta) + a(\xi,\eta)\,\frac{\partial\varphi_0}{\partial\xi} + b(\xi,\eta)\,\frac{\partial\varphi_0}{\partial\eta}\Big], \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\Delta\psi_1(\xi,\eta) - a(\xi,\eta)\,\frac{\partial\psi_1(\xi,\eta)}{\partial\xi} - b(\xi,\eta)\,\frac{\partial\psi_1(\xi,\eta)}{\partial\eta} = \\ &\quad = -\Big[\Delta\psi_0(\xi,\eta) - a(\xi,\eta)\,\frac{\partial\psi_0}{\partial\xi} - b(\xi,\eta)\,\frac{\partial\psi_0}{\partial\eta}\Big]. \end{aligned} \tag{2.11}$$

To simplify our investigation and solution of the system (2.10), (2.11), we have deliberately left in the right-hand sides of (2.10) and (2.11) the terms $\Delta\varphi_0(\xi,\eta)$ and $\Delta\psi_0(\xi,\eta)$ which are, according to (2.6), equal to zero.

Transforming the unknown functions $\varphi_1(\xi,\eta)$, $\psi_1(\xi,\eta)$, $\varphi_0(\xi,\eta)$, $\psi_0(\xi,\eta)$ as

$$\varphi_1(\xi,\eta) = \rho^{-1/2}(\xi,\eta)\varphi_2(\xi,\eta), \quad \psi_1(\xi,\eta) = \rho^{1/2}(\xi,\eta)\psi_2(\xi,\eta), \tag{2.12}$$

$$\varphi_0(\xi,\eta) = \rho^{-1/2}(\xi,\eta)\varphi_2^*(\xi,\eta), \quad \psi_0(\xi,\eta) = \rho^{1/2}(\xi,\eta)\psi_2^*(\xi,\eta), \tag{2.13}$$

we obtain

$$\begin{aligned} \frac{\partial\varphi_1}{\partial\xi} &= -\frac{1}{2}\,\rho^{-3/2}\,\frac{\partial\rho}{\partial\xi}\,\varphi_2(\xi,\eta) + \rho^{-1/2}\,\frac{\partial\varphi_2}{\partial\xi}\,, \\ \frac{\partial\varphi_1}{\partial\eta} &= -\frac{1}{2}\,\rho^{-3/2}\,\frac{\partial\rho}{\partial\eta}\,\varphi_2(\xi,\eta) + \rho^{-1/2}\,\frac{\partial\varphi_2}{\partial\eta}\,, \end{aligned} \tag{2.14}$$

$$\frac{\partial^2\varphi_1}{\partial\xi^2} = \frac{3}{4}\,\rho^{-5/2}\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 - \frac{1}{2}\,\rho^{-3/2}\,\frac{\partial^2\rho}{\partial\xi^2}\,\varphi_2 -$$

$$-\,\frac{1}{2}\,\rho^{-3/2}\,\frac{\partial\rho}{\partial\xi}\,\frac{\partial\varphi_2}{\partial\xi} - \frac{1}{2}\,\rho^{-3/2}\,\frac{\partial\rho}{\partial\xi}\,\frac{\partial\varphi_2}{\partial\xi} + \rho^{-1/2}\,\frac{\partial^{\varphi}_2}{\partial\xi^2}\,, \tag{2.15}$$

$$\frac{\partial^2\varphi_1}{\partial\eta^2} = \frac{3}{4}\,\rho^{-5/2}\Big(\frac{\partial\rho}{\partial\eta}\Big)^2\varphi_2 - \frac{1}{2}\,\rho^{-3/2}\,\frac{\partial^2\rho}{\partial\eta^2}\,\varphi_2 -$$

$$-\,\frac{1}{2}\,\rho^{-3/2}\,\frac{\partial\rho}{\partial\eta}\,\frac{\partial\varphi_2}{\partial\eta} - \frac{1}{2}\,\rho^{-3/2}\,\frac{\partial\rho}{\partial\eta}\,\frac{\partial\varphi_2}{\partial\eta} + \rho^{-1/2}\,\frac{\partial^{\varphi}_2}{\partial\eta^2}\,, \tag{2.16}$$

$$\Delta\varphi_1 = \frac{3}{4}\,\rho^{-5/2}\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\varphi_2 -$$

$$-\,\rho^{-3/2}\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\varphi_2}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\varphi_2}{\partial\eta}\Big) + \rho^{-1/2}\Delta\varphi_2, \tag{2.17}$$

$$\Delta\varphi_1 + \frac{1}{\rho}\,\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\varphi_2}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\varphi_2}{\partial\eta}\Big) = \frac{3}{4}\,\rho^{-5/2}\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\varphi_2 -$$

$$-\,\rho^{-3/2}\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\varphi_2}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\varphi_2}{\partial\eta}\Big) + \rho^{-1/2}\Delta\varphi_2 +$$

$$+\,\frac{1}{\rho}\,\frac{\partial\rho}{\partial\xi}\,\Big(-\,\frac{1}{2}\,\rho^{-3/2}\,\frac{\partial\rho}{\partial\xi}\,\varphi_2 + \rho^{-1/2}\,\frac{\partial\varphi_2}{\partial\xi}\Big) +$$

$$+\,\frac{1}{\rho}\,\frac{\partial\rho}{\partial\eta}\,\Big(-\,\frac{1}{2}\,\rho^{-3/2}\,\frac{\partial\rho}{\partial\eta}\,\varphi_2 + \rho^{-1/2}\,\frac{\partial\varphi_2}{\partial\eta}\Big), \tag{2.18}$$

$$\Delta\varphi_1 + \frac{1}{\rho}\,\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\varphi_1}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\varphi_1}{\partial\eta}\Big) =$$

$$= \rho^{-1/2}\Big\{\Delta\varphi_2 + \frac{1}{4}\,\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\varphi_2\Big\}, \tag{2.19}$$

$$\Delta\psi_1 - \frac{1}{\rho}\,\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\psi_1}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_1}{\partial\eta}\Big) = -\Big[\Delta\psi_0 - \frac{1}{\rho}\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\psi_0}{\partial\eta} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_0}{\partial\eta}\Big)\Big], \tag{2.20}$$

$$\frac{\partial\psi_1}{\partial\xi} = \frac{1}{2}\,\rho^{-1/2}\,\frac{\partial\rho}{\partial\xi}\,\psi_2 + \rho^{1/2}\,\frac{\partial\psi_2}{\partial\xi}\,, \quad \frac{\partial\psi_1}{\partial\eta} =$$

$$= \frac{1}{2}\,\rho^{-1/2}\,\frac{\partial\rho}{\partial\eta}\,\psi_2 + \rho^{1/2}\,\frac{\partial\psi_2}{\partial\eta}\,, \tag{2.21}$$

$$\frac{\partial^2\psi_1}{\partial\xi^2} = -\frac{1}{4}\,\rho^{-3/2}\Big(\frac{\partial\rho}{\partial\xi}\Big)^2\psi_2 + \frac{1}{2}\rho^{-1/2}\,\frac{\partial^2\psi_2}{\partial\xi^2} + \frac{1}{2}\,\rho^{-1/2}\frac{\partial\rho}{\partial\xi}\,\frac{\partial\psi_2}{\partial\xi} +$$

$$+\,\frac{1}{2}\,\rho^{-1/2}\frac{\partial\rho}{\partial\xi}\,\frac{\partial\psi_2}{\partial\xi} + \rho^{1/2}\frac{\partial^2\psi_2}{\partial\xi^2}\,, \tag{2.22}$$

$$\frac{\partial^2\psi_1}{\partial\eta^2} = -\frac{1}{4}\,\rho^{-3/2}\Big(\frac{\partial\rho}{\partial\eta}\Big)^2\psi_2 + \frac{1}{2}\rho^{-1/2}\,\frac{\partial^2\psi_2}{\partial\eta^2}\,\psi_2 + \frac{1}{2}\,\rho^{-1/2}\frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_2}{\partial\eta} +$$

$$+\,\frac{1}{2}\,\rho^{-1/2}\frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_2}{\partial\eta} + \rho^{1/2}\frac{\partial^2\psi_2}{\partial\eta^2}\,, \tag{2.23}$$

$$\Delta\psi_1 = -\frac{1}{4}\,\rho^{-3/2}\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\psi_2 + {}$$
$$+\,\rho^{-1/2}\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\psi_2}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_2}{\partial\eta}\Big) + \rho^{-1/2}\Delta\psi_2, \tag{2.24}$$

$$\Delta\psi_1 - \frac{1}{\rho}\,\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\psi_1}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_1}{\partial\eta}\Big) = -\frac{1}{4}\,\rho^{-3/2}\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\psi_2 + {}$$
$$+\,\rho^{-1/2}\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\psi_2}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_2}{\partial\eta}\Big) + \rho^{1/2}\Delta\psi_2, \tag{2.25}$$

$$\Delta\psi_1 - \frac{1}{\rho}\,\frac{\partial\rho}{\partial\xi}\,\Big(\frac{1}{2}\,\rho^{-1/2}\,\frac{\partial\rho}{\partial\xi}\,\psi_2 + \rho^{1/2}\,\frac{\partial\psi_2}{\partial\xi}\Big) - {}$$
$$-\,\frac{1}{\rho}\,\frac{\partial\rho}{\partial\eta}\,\Big(\frac{1}{2}\,\rho^{-1/2}\,\frac{\partial\rho}{\partial\eta}\,\psi_2 + \rho^{1/2}\,\frac{\partial\psi_2}{\partial\eta}\Big) = {}$$
$$= -\frac{1}{4}\,\rho^{-3/2}\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\psi_2 + {}$$
$$+\,\rho^{-1/2}\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\psi_2}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_2}{\partial\eta}\Big) + \rho^{1/2}\Delta\psi_2, \tag{2.25_1}$$

$$\Delta\psi_1 = \rho^{1/2}\Big\{\Delta\psi_2 - \frac{3}{4}\,\frac{1}{\rho^2}\,\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\psi_2\Big\}. \tag{2.26}$$

Taking into account (2.13), we represent the functions $\varphi_0(\xi,\eta)$ and $\psi_0(\xi,\eta)$ analogously to (2.19) and (2.26) with respect to $\varphi_2^*(\xi,\eta)$, $\psi_2^*(\xi,\eta)$ and obtain

$$\Delta\varphi_0 + \frac{1}{\rho}\,\Big(\frac{\partial\rho}{\partial\xi}\,\frac{\partial\varphi_0}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\varphi_0}{\partial\eta}\Big) = {}$$
$$= \rho^{-1/2}\Big\{\Delta\varphi_2^* + \frac{1}{4}\,\rho^{-2}\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\varphi_2^*\Big\}, \tag{2.27}$$
$$\Delta\psi_0 - \frac{1}{\rho}\,\Big(\frac{\partial\rho}{\partial\psi}\,\frac{\partial\psi_0}{\partial\xi} + \frac{\partial\rho}{\partial\eta}\,\frac{\partial\psi_0}{\partial\eta}\Big) = {}$$
$$= \rho^{1/2}\Big\{\Delta\psi_2^* - \frac{3}{4}\,\rho^{-2}\Big[\Big(\frac{\partial\rho}{\partial\xi}\Big)^2 + \Big(\frac{\partial\rho}{\partial\eta}\Big)^2\Big]\psi_2^*\Big\}, \tag{2.28}$$

where $\Delta\varphi_0 = 0$, $\Delta\psi_0 = 0$.

Bearing in mind (2.19), (2.27), (2.26) and (2.28), we represent the system (2.10) and (2.11) as follows:

$$\rho^{-1/2}\big[\Delta\varphi_2 + \lambda\rho_1\varphi_2\big] = -\rho^{-1/2}\Big[\Delta\varphi_2^* + \frac{1}{4}\,\rho_1\varphi_2^*\Big], \tag{2.29}$$
$$\rho^{1/2}\big[\Delta\psi_2 - \mu\rho_1\psi_2\big] = -\rho^{1/2}\Big[\Delta\psi_2^* - \frac{3}{4}\,\rho_1\psi_2^*\Big], \tag{2.30}$$

where $\rho_1 = \frac{1}{\rho^2}\,\big[\big(\frac{\partial\rho}{\partial\xi}\big)^2 + \big(\frac{\partial\rho}{\partial\eta}\big)^2\big]$, $\lambda = \frac{1}{4}$, $\mu = \frac{3}{4}$.

The equalities (2.29) and (2.30) can be rewritten in the form

$$\Delta\varphi_2 + \frac{1}{4}\rho_1\varphi_2 = -\Big[\Delta\varphi_2^* + \frac{1}{4}\,\rho_1\varphi_2^*\Big], \tag{2.31}$$

$$\Delta\psi_2 - \frac{3}{4}\rho_1\psi_2 = -\rho^{-1/2}\Big[\Delta\psi_2^* - \frac{3}{4}\,\rho_1\psi_2^*\Big]. \tag{2.32}$$

Assuming that $\varphi_2^*(\xi,\eta)$ and $\psi_2^*(\xi,\eta)$ are the known functions, we rewrite the equations (2.31), (2.32) as

$$\Delta(\varphi_2 + \varphi_2^*) = -\frac{1}{4}\,\rho_1(\varphi_2 + \varphi_2^*) \equiv f_1^*(\xi,\eta), \tag{2.33}$$

$$\Delta(\psi_2 + \psi_2^*) = \frac{3}{4}\,\rho_1(\psi_2 + \psi_2^*) \equiv f_2^*(\xi,\eta). \tag{2.34}$$

Consider the Poisson equation

$$\Delta u(\xi,\eta) = f_1^*(\xi,\eta), \ \ (\xi_1,\eta_1) \in \mathrm{Im}(\zeta) > 0. \tag{2.35}$$

Define the function $u(\xi,\eta)$ by the formula

$$u(\xi,\eta) = -\frac{1}{2\pi}\iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)f_1(x,y)\,dx\,dy, \tag{2.36}$$

where

$$G(\xi,\eta;x,y) = \frac{1}{4\pi}\,\ln\frac{(\xi-x)^2+(\eta+y)^2}{(\xi-x)^2+(\eta-y)^2} \tag{2.37}$$

is the Green's function of the Dirichlet problem for the harmonic in $\mathrm{Im}(\zeta) > 0$ function, while the function $f_1^*(\xi,\eta)$ is bounded and has continuous first derivatives bounded in $\mathrm{Im}(\zeta) > 0$, $U(\xi,\eta)$ is a regular solution of the Poisson equation (2.35). It is proved that (2.36) satisfies the boundary condition [4]

$$\lim_{(\xi,\eta)\to(\xi_0,\eta_0)} u(\xi,\eta) = 0, \ \ (\xi,\eta) \in \mathrm{Im}(\zeta) > 0, \ \ (\xi_0,\eta_0) \in \mathrm{Im}(\zeta_0). \tag{2.38}$$

Using (2.35) and (2.36) with respect to (2.33) and (2.34), we obtain

$$\varphi_2(\xi,\eta) = -\varphi_2^*(\xi,\eta) + \frac{1}{2\pi}\iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)f_1^*(x,y)\,dx\,dy, \tag{2.39}$$

$$\psi_2(\xi,\eta) = -\psi_2^*(\xi,\eta) + \frac{1}{2\pi}\iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)f_2^*(x,y)\,dx\,dy. \tag{2.40}$$

The equalities (2.39) and (2.40) can be written as follows:

$$\varphi_2(\xi,\eta) = -\rho^{1/2}\varphi_0(\xi,\eta) +$$

$$+\frac{1}{8\pi}\iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)\rho_1(x,y)\big[\varphi_2(x,y)+\rho^{1/2}\varphi_0(x,y)\big]\,dx\,dy, \tag{2.41}$$

$$\psi_2(\xi,\eta) = -\psi^*(\xi,\eta) + \\ + \frac{3}{8\pi} \iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)\rho_1(x,y)\big[\psi_2(x,y) + \psi_2^*(x,y)\big]\,dx\,dy. \tag{2.42}$$

We rewrite the equations (2.40) and (2.42) in the form

$$\varphi_2(\xi,\eta) = f_3(\xi,\eta) + \frac{1}{8\pi} \iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)\rho_1(x,y)\varphi_2(x,y)\,dx\,dy, \tag{2.43}$$

$$\psi_2(\xi,\eta) = f_4(\xi,\eta) + \frac{3}{8\pi} \iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)\rho_1(x,y)\psi_2(x,y)\,dx\,dy, \tag{2.44}$$

where

$$f_3(\xi,\eta) = -\,\rho^{1/2}\varphi_0(\xi,\eta) + \\ + \frac{1}{8\pi} \iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)\rho_1(x,y)\rho^{1/2}\varphi_0(x,y)\,dx\,dy, \tag{2.45}$$

$$f_4(\xi,\eta) = -\,\psi^*(\xi,\eta) + \\ + \frac{3}{8\pi} \iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)\rho_1(x,y)\psi_2^*(x,y)\,dx\,dy. \tag{2.46}$$

Thus we have obtained the second kind Fredholm's integral equations (2.43) and (2.44) with respect to $\varphi_2(\xi,\eta)$ and $\psi_2(\xi,\eta)$. The problems (2.33) and (2.34) are, respectively, equivalent to the integral equations (2.43) and (2.44) which will be solved by using the exact methods.

Solutions of the integral equations (2.43) and (2.44) will be sought by the method of successive approximations in the form of the series

$$\varphi_2(\xi,\eta) = \sum_{n=0}^{\infty} \lambda^n \varphi_{2(n)}(\xi,\eta), \tag{2.47}$$

$$\psi_2(\xi,\eta) = \sum_{n=0}^{\infty} \mu^n \psi_{2(n)}(\xi,\eta), \tag{2.48}$$

where $\lambda = \frac{1}{8\pi}$, $\mu = \frac{3}{8\pi}$.

Substituting the series (2.47) and (2.48) respectively into the integral equations and equating the coefficients with the same powers of the parameters λ and μ, we obtain

$$\varphi_{2(0)}(\xi,\eta) = f_3(\xi,\eta), \tag{2.49}$$

. . . .

$$\varphi_{2(n)}(\xi,\eta) = \iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)\rho_1(x,y)\varphi_{2(n-1)}(x,y)\,dx\,dy, \qquad (2.50)$$

. . . .

$$\psi_{2(0)}(\xi,\eta) = f_4(\xi,\eta), \qquad (2.51)$$

. . . .

$$\psi_{2(n)}(\xi,\eta) = \iint\limits_{\mathrm{Im}(\zeta)>0} G(\xi,\eta;x,y)\rho_1(x,y)\psi_{2(n-1)}(x,y)\,dx\,dy, \qquad (2.52)$$

. . . .

$$n = 1,2,3,\dots .$$

The parameters $\lambda = \frac{1}{8\pi}$ and $\mu = \frac{3}{8\pi}$ of the integral equations (2.43) and (2.44) are small enough; this ensures the convergence of the series (2.47) and (2.48). Recall here that as initial approximations, as usual, are taken the free terms $f_3(\xi,\eta)$ and $f_4(\xi,\eta)$.

Basing on (2.47)–(2.52), we can construct general formulas which allow one to express any approximations through the free terms by means of iterated kernels.

Assuming that the series (2.47) and (2.48) are already constructed, we can multiply them respectively by $\rho^{-1/2}$ and $\rho^{1/2}$. We obtain

$$\varphi(\xi,\eta) = \varphi_0(\xi,\eta) + \varphi_1(\xi,\eta), \qquad (2.53)$$

$$\psi(\xi,\eta) = \psi_0(\xi,\eta) + \psi_1(\xi,\eta). \qquad (2.54)$$

Recall that the boundary conditions along the oz-axis of symmetry, when some parts of oz coincide with the boundary $S(\sigma)$, have in the coordinates (z,ρ) the form

$$\rho \to 0, \quad \alpha = \rho^2,$$

$$\frac{\partial\varphi}{\partial\rho} = \frac{\partial\varphi}{\partial\alpha}\frac{\partial\alpha}{\partial\rho} = \frac{\partial\varphi}{\partial\alpha}\cdot 2\rho \to 0; \quad \frac{\partial\psi}{\partial\rho} = \frac{\partial\psi}{\partial\alpha}\frac{\partial\alpha}{\partial\rho} = \frac{\partial\psi}{\partial\alpha}\cdot 2\rho \to 0. \qquad (2.55)$$

In the coordinates (ξ,η), the boundary conditions along the oz-axis have the form

$$\rho(\xi,\eta) = 0, \quad \frac{\partial\rho}{\partial\xi} = 0, \quad \frac{\partial\rho}{\partial\eta} = 0;$$

$$\Big(\frac{\partial\varphi}{\partial\xi}\Big)_{\rho\to 0} = \Big(\frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial\zeta} + \frac{\partial\varphi}{\partial\alpha}\frac{\partial\alpha}{\partial\rho}\frac{\partial\rho}{\partial\xi}\Big)_{\rho\to 0} = \Big(\frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial\xi}\Big)_{\rho\to 0}, \qquad (2.56)$$

$$\Big|\frac{\partial\varphi}{\partial\alpha}\frac{\partial\alpha}{\partial\rho}\frac{\partial\rho}{\partial\xi}\Big|_{\rho\to 0} \to 0,$$

$$\left(\frac{\partial\varphi}{\partial\rho}\right)_{\rho\to 0} = \left(\frac{\partial\varphi}{\partial\alpha}\,\frac{\partial\alpha}{\partial\rho}\right)_{\rho\to 0} \to 0,$$
$$\left(\frac{\partial\varphi}{\partial\eta}\right) = \left(\frac{\partial\varphi}{\partial z}\,\frac{\partial z}{\partial\eta} + \frac{\partial\varphi}{\partial\alpha}\,\frac{\partial\alpha}{\partial\rho}\,\frac{\partial\rho}{\partial\eta}\right)_{\rho\to 0} = \left(\frac{\partial\varphi}{\partial z}\,\frac{\partial z}{\partial\eta}\right)_{\rho\to\infty} \quad \frac{\partial\rho}{\partial\eta} \to 0. \tag{2.57}$$

Suppose that the oz-axis of symmetry (or its parts) does not coincide with the boundary of the filtration domain $S(\sigma)$,

$$\rho = \text{const} \neq 0, \quad \alpha = \rho^2,$$
$$\frac{\partial\varphi}{\partial\rho} = \frac{\partial\varphi}{\partial\alpha}\,\frac{\partial\alpha}{\partial\rho} = \frac{\partial\varphi}{\partial\alpha}\cdot 2\rho = \frac{\partial\varphi}{\partial\alpha}\cdot 2\,\text{const}, \quad \rho\to\text{const}\,. \tag{2.58}$$
$$\left(\frac{\partial\psi}{\partial\alpha}\,\frac{\partial\alpha}{\partial\rho}\right)_{\rho\to\text{const}} = \left(\frac{\partial\psi}{\partial\alpha}\right)\cdot 2\,\text{const}, \quad (\psi)_{\rho=\text{const}} = \text{const},$$

$$\left(\frac{\partial\psi}{\partial\xi}\right)_{\rho\to\text{const}} \to 0, \quad \left(\frac{\partial\psi}{\partial\eta}\right)_{\rho\to\text{const}} \to 0,$$
$$\left(\frac{\partial\varphi}{\partial\eta}\right)_{\rho=\text{const}} = \left(\frac{\partial\varphi}{\partial z}\,\frac{\partial z}{\partial\eta} + \frac{\partial\varphi}{\partial\alpha}\,\frac{\partial\alpha}{\partial\rho}\,\frac{\partial\rho}{\partial\eta}\right)_{\rho=\text{const}} = \left(\frac{\partial\varphi}{\partial z}\,\frac{\partial z}{\partial\eta}\right)_{\rho=\text{const}}, \tag{2.59}$$
$$\frac{\partial\rho}{\partial\eta} = 0.$$

Below we will present another way of solution of the system (2.10) and (2.11).

1^0. Green's function belongs to the class of fundamental solutions of the Laplace equation. It is determined that as a harmonic function of a pair of points $(P;Q)$, is symmetric with respect to P and Q, equals to zero on the boundary and is analytic at all points P of the domain D_i, except of the points $P = Q$ at which it has logarithmic singularity, i.e. at the point P of the neighborhood of Q, the relation

$$G(P;Q) = \frac{1}{2\pi}\,\ln r(P;Q) + g(P;Q) \tag{2.60}$$

is fulfilled, where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$ is the distance between the points P and Q. Moreover, Green's function, as a function of P, should have everywhere inside of D_i, except the point of Q, continuous derivatives up to the second order and satisfy the Laplace equation, while on the boundary it should satisfy the limiting condition. Next, $G(P;Q)$, as a function of P, must have singularity at the point Q corresponding to the initial charge (or to the mass) concentrated at the point Q. Green's function of the Laplace operator for the plane simply connected domain under the limiting condition $U_\ell = 0$ is tightly connected with the function which transforms conformally the above-mentioned domain onto the circle $|W| \leq 1$.

$G(P;Q)$ is a harmonic in the domain D_i function of the coordinates x and y ([4], [17], [33]–[36]).

If d is the diameter of the domain D_i, then the inequality

$$0 \le G(P;Q) \le \ln\left(\frac{d}{r}\right) \tag{2.61}$$

is valid. Green's function for the circle of radius $R = 1$ has the form

$$G(P;Q) = \frac{1}{2\pi}\ln\left(\frac{\rho r_1}{r}\right), \tag{2.62}$$

where $\rho = \sqrt{\xi^2 + \eta^2}$ is the distance of the point $Q(\xi, \eta)$ from the center of the circle. r_1 is defined as follows: $r_1 = \sqrt{(x - \xi/\rho^2)^2 + (y - \eta/\rho^2)^2}$.

2^0. Consider the inhomogeneous equation

$$\Delta U(x,y) = -\varphi(x,y). \tag{2.63}$$

We seek for a solution of (2.63), continuous up to the contour of the domain and satisfying the limiting equation $U\big|_\ell = 0$. There may be only one such solution ([35]).

The unknown solution has the form

$$U(x,y) = \iint\limits_{D_i} G(x,y;\xi,\eta)\varphi(\xi,\eta)\, d\xi\, d\eta, \tag{2.64}$$

that is,

$$U(x,y) = \frac{1}{2\pi}\iint\limits_{D_i} \varphi(\xi,\eta)\ln\frac{1}{r}\, d\xi\, d\eta + \\ + \iint\limits_{D_i} g(x,y;\xi,\eta)\varphi(\xi,\eta)\ln\frac{1}{r}\, d\xi\, d\eta, \tag{2.65}$$

otherwise.

The first summand of (2.65) has inside of D_i continuous derivatives up to the second order, and its Laplace operator is equal to $[-\varphi(\xi,\eta)]$. It is proved that the second summand of (2.65) can be differentiated under the integral sign with respect to the coordinates (x, y) of the point $P(x, y)$ as many times as desired. This implies that this summand is a function harmonic inside of D_i, because $g(P;Q)$ is a harmonic function of the point $P(x, y)$. $g(P;Q)$ is a harmonic function of the point Q with limiting values $\left(\frac{1}{2\pi}\ln r\right)$, where $r = \sqrt{(x - \xi)^2 + (y - \eta)^2}$. It is assumed that $P(x, y)$ is inside of D_i. The formula (2.65) provides us with the solution of the equation (2.63) satisfying the condition $U\big|_\ell = 0$. Recall that there exists a generalized solution of (2.63).

3^0. The linear-fractional conformal mapping of the half-plane $\mathrm{Im}(\zeta) > 0$ onto the circle $|W| < 1$ has the form

$$W = \frac{1 + i\zeta}{i + zt}, \quad \zeta = \xi + i\eta, \quad w = u + iv. \tag{2.66}$$

It follows from (2.66) that

$$u = \frac{2\xi}{\xi^2 + (1+\eta)^2}\,, \quad v = \frac{\xi^2 + \eta^2 - 1}{\xi^2 + (1+\eta)^2}\,. \tag{2.67}$$

On the other hand, from (2.66) we have

$$\zeta = \frac{i+w}{1+iw}\,, \quad \xi = \frac{2u}{u^2 + (1-v)^2}\,, \quad \eta = \frac{1 - v^2 - u^2}{u^2 + (1-v)^2}\,. \tag{2.68}$$

4^0. Harmonic and analytic functions of a complex variable.

Let

$$w = f(z) = u(x,y) + iv(x,y), \quad z = x + iy \tag{2.69}$$

be some function of the complex variable $z = x + iy$; u and v are the real functions of the variables x and y. The Cauchy–Riemann conditions $u_x = v_y$, $u_y = -v_x$ are necessary and sufficient for the function to be analytic. It follows from these conditions that $\Delta U = 0$, $\Delta V = 0$, where Δ is the Laplace operator.

Consider the transformation

$$x = x(u,v), \quad y = y(u,v), \tag{2.70}$$

$$u = u(x,y), \quad v = v(x,y), \tag{2.71}$$

where $u(x,y)$ and $v(x,y)$ are the conjugate harmonic functions. Then the above transformation is equivalent to (2.69).

By virtue of the Cauchy–Riemann conditions, the relations [36]

$$u_x^2 + u_y^2 = u_x^2 + v_x^2 = v_y^2 + v_x^2 = |f'(z)|^2, \quad u_x v_x + u_y v_y = 0 \tag{2.72}$$

should be satisfied for the functions u and v.

Let us find out how the Laplace operator varies under that transformation. We obtain

$$U = U\big[x(u,v), y(u,v)\big] = \widetilde{U}(u,v), \tag{2.73}$$

$$U_{xx} + U_{yy} = (\widetilde{U}_{uu} + \widetilde{V}_{uu})|f'(z)|^2, \tag{2.74}$$

whence it follows that as a result of the transformation $w = f(z) = u + iv$, the function $U(x,y)$, harmonic in the domain G, transforms into the function $\widetilde{U} = \widetilde{U}(u,v)$, harmonic in the domain G', and if only if $|f'(z)|^2 \neq 0$.

Consider the equations

$$\Delta\varphi + \frac{1}{4}\,\rho_1\varphi = 0, \tag{2.75}$$

$$\Delta\psi - \frac{3}{4}\,\rho_1\psi = 0, \tag{2.76}$$

where $\rho_1 = \frac{1}{\rho^2}\left[\left(\frac{\partial\rho}{\partial x}\right)^2 + \left(\frac{\partial\rho}{\partial y}\right)^2\right]$. The transformations

$$\varphi = \varphi\big[x(u,v), y(u,v)\big] = \widetilde{\varphi}(u,v), \tag{2.77}$$

$$\psi = \psi\big[x(u,v), y(u,v)\big] = \widetilde{\psi}(u,v) \tag{2.78}$$

result in the equalities

$$\varphi_{xx} + \varphi_{yy} + \frac{1}{4}\,\frac{1}{\rho^2(x,y)}\Big[\Big(\frac{\partial\rho}{\partial x}\Big)^2 + \Big(\frac{\partial\rho}{\partial y}\Big)^2\Big]\varphi(x,y) = \Big\{\widetilde{\varphi}_{uu} + \widetilde{\varphi}_{vv} +$$

$$+\frac{1}{4}\,\frac{1}{\widetilde{\rho}(u,v)}\Big[\Big(\frac{\partial\widetilde{\rho}}{\partial u}\Big)^2 + \Big(\frac{\partial\widetilde{\rho}}{\partial v}\Big)^2\Big]\widetilde{\varphi}(u,v)\Big\}|f'(z)|^2 = 0, \quad |f'(z)|^2 \neq 0, \tag{2.79}$$

$$\psi_{xx} + \psi_{yy} - \frac{3}{4}\,\frac{1}{\rho^2(x,y)}\Big\{\Big(\frac{\partial\rho}{\partial x}\Big)^2 + \Big(\frac{\partial\rho}{\partial y}\Big)^2\Big\}\psi(x,y) =$$

$$= \Big\{\widetilde{\psi}_{uu} + \widetilde{\psi}_{vv} - \frac{3}{4}\,\frac{1}{\widetilde{\rho}(u,v)}\Big[\Big(\frac{\partial\widetilde{\rho}}{\partial u}\Big)^2 + \Big(\frac{\partial\widetilde{\rho}}{\partial v}\Big)^2\Big]\widetilde{\varphi}(u,v)\Big\}|f'(z)|^2 = 0, \tag{2.80}$$

since $|f'(z)| \neq 0$, and hence from (2.79) and (2.80) we have

$$\widetilde{\varphi}_{uu} + \widetilde{\varphi}_{vv} + \frac{1}{4}\,\frac{1}{\widetilde{\rho}^2}\Big[\Big(\frac{\partial\widetilde{\rho}}{\partial u}\Big)^2 + \Big(\frac{\partial\widetilde{\rho}}{\partial v}\Big)^2\Big]\widetilde{\varphi} = 0, \tag{2.81}$$

$$\widetilde{\psi}_{uu} + \widetilde{\psi}_{vv} - \frac{3}{4}\,\frac{1}{\widetilde{\rho}^2}\Big[\Big(\frac{\partial\widetilde{\rho}}{\partial u}\Big)^2 + \Big(\frac{\partial\widetilde{\rho}}{\partial v}\Big)^2\Big]\widetilde{\psi} = 0. \tag{2.82}$$

It follows from the above-said that using the transformations (2.68), (2.71) and Green's function (2.62), we can reduce the problem (2.31) (or (2.32)) to the solution of Fredholm's integral equation of second kind, where the given functions, the kernel and the right-hand side are defined in the domain of the unit circle. In this case, for the convergence of Neumann's series we can indicate a simpler condition. In particular, if the kernel is bounded, then for the convergence of Neumann's series there exist more plausible condition.

In hydrodynamics, there exists the method of sources and channels. This method has been for the first time applied by Rankin to the spatial problem of body streamline. The method consists in the replacement of the body streamline by such a system of sources and channels that the body surface is one of the stream surfaces; note that the algebraic sum of abundance sources should be equal to zero. The choice of a system of sources and channels by means of a preassigned surface formed of a body streamline is of great mathematical difficulty ([1-3], [7]).

Below we will present an algorithm for finding the functions $\varphi_0(\xi,\eta)$, $\psi_0(\xi,\eta)$, $z(\xi,\eta)$ and $\rho(\xi,\eta)$. Recall that the plane of liquid motion coincides with that of the complex variable $\sigma = z + i\rho$, $i = -\sqrt{-1}$.

In the domain $S(\sigma)$ with the boundary $\ell(\sigma)$ we seek for a complex potential (i.e., a potential divided by the filtration coefficient) $\omega(\sigma) = \varphi_0(z,\rho) + i\psi_0(z,\rho)$. The velocity potential $\varphi_0(z,\rho)$ and the stream function $\psi_0(z,\rho)$ satisfy the Cauchy–Riemann conditions and the boundary conditions

$$a_{kj}\varphi_0(z,\rho) + a_{k2}\psi_0(z,\rho) + ak_3 z + a_{k4}\rho = f_k, \quad k = 1,2, \quad (z,\rho) \in \ell(\sigma), \tag{2.83}$$

where a_{kj}, f_k, $k=1,2$, $j=1,\dots,n$, are the given piecewise constant real functions; f_k, $k=1,2$, depend on an unknown parameter Q of the filtrated liquid discharge.

Using (2.60), we can find a part of the boundary $\ell(\omega_0)$ of $S(\omega_0)$ and the boundary $\ell(w_0)$ of the domain of complex velocity $S(w_0)$, where $w_0(z) = d\omega_0/d\sigma = \frac{\omega_0'(\zeta)}{\sigma'(\zeta)}$, excluding some coordinates of those vertices of circular polygons which are connected with cut ends. By means of the functions $\omega_0(\sigma)$ and $w(\sigma)$, we map conformally the domain $\ell(\sigma)$ with the boundary $\ell(\sigma)$ onto the domains $S(\omega_0)$ and $S(w)$. The domain $S(w)$ is a circular polygon.

Angular points of the boundaries $\ell(\sigma)$, $\ell(\omega_0)$ and $\ell(w)$ which can be encountered at least at one of them when passing in the positive direction we denote by A_k, $k=1,\dots,n$.

The half-plane $\mathrm{Im}(\zeta)>0$ of the plane $\zeta=\xi+i\eta$ is mapped conformally onto the domains $S(\sigma)$, $S(\omega_0)$ and $S(w_0)$. We denote the corresponding mapping functions by $\sigma(\zeta)$, $\omega_0(\zeta)$, $w(\zeta_0)=\omega_0'(\zeta)/\sigma'(\zeta)$, $d\omega_0(\zeta)/d\zeta=\omega_0'(\zeta)$, $d\sigma(\zeta)/d\zeta=\sigma'(\zeta)$. To the angular points A_k, $k=1,\dots,n$, there correspond the points $\zeta=e_k$, $k=1,\dots,n$, along the axis t with $-\infty<e_1<e_2<\cdots<e_{n-1}<e_n<+\infty$, and $\xi=e_{n+1}=0$ is mapped into the nonangular point A_∞ of the boundary $\ell(\sigma)$ which is located between the points A_n and A_1.

3. Construction of the Functions $d\omega_0(\zeta)/d\sigma(\zeta)$, $\omega_0(\zeta)$ and $\sigma(\zeta)$

By $\sigma(\xi)=z(\xi)+i\rho(\xi)$, $\omega_0(\xi)=\varphi_0(\xi)+i\psi_0(\xi)$, $w_0(\xi)=u_0(\xi)+iv_0(\xi)$ we denote the boundary values of the functions $\sigma(\zeta)$, $\omega_0(\zeta)$ and $w_0(\zeta)$ as $\zeta\to\xi$, $\zeta\in\mathrm{Im}(\zeta)>0$. By $\overline{\sigma}(\xi)$, $\overline{\omega}_0(\xi)$ and $\overline{w}_0(\xi)$ we denote the complex conjugate functions corresponding to the functions $\sigma(\xi)$, $\omega_0(\xi)$ and $w_0(\xi)$.

Introduce the vectors $\Phi_0(\xi)=[\omega_0(\xi),\sigma(\xi)]$, $\overline{\Phi}_0(\xi)=[\overline{\omega}_0(\xi),\overline{\sigma}(\xi)]$, $\Phi_0'(\xi)=[\omega_0'(\xi),\sigma'(\xi)]$, $\overline{\Phi}_0{}'(\xi)=[\overline{\omega}_0{}'(\xi),\overline{\sigma}{}'(\xi)]$, $f(\xi)=[f_1(\xi),f_2(\xi)]$. Then the boundary conditions ([26]–[31])

$$\Phi_0(\xi)=g(\xi)\overline{\Phi}_0(\xi)+i\cdot 2G^{-1}f(\xi),\ \ -\infty<\xi<+\infty, \tag{3.1}$$

where $G^{-1}(\xi)\overline{G}(\xi)=g(\xi)$ is a piecewise constant nonsingular second rank matrix with the points of discontinuity $\xi=e_k$, $k=1,\dots,n$; $G^{-1}(\xi)$ and $\overline{G}(\xi)$ are, respectively, the inverse and complex conjugate matrices to the matrix $G(\xi)$, and the vector $f(\xi)$ is defined by means of (2.83).

Differentiating (3.1) along the boundary ξ, we obtain

$$\Phi_0'(\xi)=g(\xi)\Phi_0'(\xi),\ \ -\infty<\xi<+\infty. \tag{3.2}$$

It can be verified that the equality $\overline{g}(\xi)=\overline{G}^{\,-1}(\xi)G(\xi)$ holds. For the points $\xi=e_j$, $j=1,\dots,n$, let us consider the characteristic equation ([1]–[31])

$$\det\left[g_{j+1}^{-1}(e_j+0)g_j(e_j-0)-\lambda E\right] \tag{3.3}$$

with respect to the parameter λ, where E is the unit matrix, $g_j(\xi)$, $e_j < \xi < e_{j+1}$, $g_{j+1}^{-1}(e_j+0)$, $g_j(e_j-0)$ are the limiting values of the matrices $g_{j+1}^{-1}(\xi)$, $g_i(\xi)$ at the point $\xi = e_j$ from the right and from the left, respectively.

By means of the roots λ_{kj} of the equation (3.3) we define uniquely the numbers $\alpha_{kj} = (2\pi i)^{-1} \ln \lambda_{kj}$, $k = 1, 2$; $j = 1 \ldots, n$ ([1]–[30]).

Suppose that among the points A_k, $k = 1 \ldots, n$, of the boundaries $\ell(\sigma)$ and $\ell(\omega_0)$ there exist removable points to which on the boundary $\ell(w_0)$ of $S(w_0)$ there correspond regular nonangular points ([26]–[30]).

For the sake of simplicity we assume that the removable singular point coincides with the point $\xi = e_j$ to which on the boundaries $\ell(\sigma)$ and $\ell(\omega_0)$ there correspond the angles $\pi/2$, while on the boundary $\ell(w_0)$ the angle π. To remove this point from the homogeneous boundary conditions (3.2), we introduce a new unknown vector $\Phi_1(\xi)$ ([26]–[30])

$$\Phi_0'(\xi) = \chi_1(\xi)\Phi_1(\xi), \tag{3.4}$$

where

$$\chi_1(\xi = \sqrt{\frac{\xi - e_{j-1}}{\xi - e_j}}\,. \tag{3.5}$$

After the passage from the vector $\Phi'(\xi)$ to $\Phi_1(\xi)$, we multiply the matrix $g_i(\xi)$ by (-1).

The boundary conditions with respect to $\Phi_1(\xi)$ take the form

$$\Phi_1(\xi) = g^*(\xi)\overline{\Phi}_1(\xi), \tag{3.6}$$

where

$$g^*(\xi) = [\chi_1(\xi)]^{-1} g(\xi)[\overline{\chi}_1(\xi)]. \tag{3.7}$$

On the contour $\ell(w_0)$ we renumerate singular points and denote them by a_j, $j = 1, \ldots, m$. We denote the characteristic points defined uniquely and those corresponding to the points $t = a_j$ again by α_{kj}, $k = 1, 2$, $j = 1, \ldots, m$. They satisfy the Fuchs condition.

Now we write the Fuchs class equation ([1]–[39])

$$u''(\xi) + p(\xi)u'(\zeta) + g(\xi)u(\xi) = 0, \tag{3.8}$$

where

$$p(\xi) = \sum_{j=1}^{m} (1 - \alpha_{1j} - \alpha_{2j})(\xi - a_j)^{-1}, \tag{3.9}$$

$$q(\xi) = \sum_{j=1}^{m} \big[\alpha_{1j}\alpha_{2j}(\xi - a_j)^{-2} + c_j(\xi - a_j)^{-1}\big], \tag{3.10}$$

where c_j are unknown real accessory parameters satisfying the conditions

$$N_1 = \sum_{j=1}^{m} c_j = 0. \tag{3.11}$$

By means of matrices, we write the equation (3.8) in the form of the system ([26]–[39)]

$$\chi'(\xi) = \chi(\xi)\Phi(\xi), \tag{3.12}$$

$$\chi(\xi) = \begin{pmatrix} u_1(\xi) & u_1'(\xi) \\ u_2(\xi) & u_2'(\xi) \end{pmatrix}, \quad \Phi(\xi) = \begin{pmatrix} 0 & -g(\xi) \\ 1 & -p(\xi) \end{pmatrix}. \tag{3.13}$$

Further, using linearly independent solutions $u_1(\xi)$ and $u_2(\xi)$ of the equation (3.8), we construct the general solution

$$w(\xi) = \frac{pu_1(\xi) + qu_2(\xi)}{ru_1(\xi) + su_2(\xi)} \tag{3.14}$$

of the Schwarz equation ([26]–[30])

$$\{w;\xi\} = \frac{w'''(\xi)}{w'(\xi)} - 1,5\left(\frac{w'(\xi)}{w'(\xi)}\right)^2 = R(\xi), \tag{3.15}$$

where

$$R(\xi) = 2q(\xi) - p'(\xi) - 0,5[p(\xi)]^2 =$$

$$= \sum_{j=1}^{m} \Big\{0,5\big[1 - (\alpha_{1j} - \alpha_{2j})^2\big]\,(\zeta - a_j)^{-2} + c_j^*(\xi - a_j)^{-1}\Big\}, \tag{3.16}$$

$$\alpha_{1j} - \alpha_{2j} = \nu_j, \quad j = 1, \dots, m,$$

$$c_j^* = 2c_i - \beta_j \sum_{k=1,\, k\neq j}^{m} \beta_k (a_j - a_k)^{-1}, \tag{3.17}$$

$$\beta_k = 1 - a_{1k} - a_{2k}, \quad k = 1, \dots, m,$$

p, q, r and s are constants of integration of (3.14) which satisfy the condition $ps - rq \neq 0$, $\pi\nu_j$ is the interior angle at the vertex B_j of the circular polygon.

Using (3.14), we map conformally the half-plane $\mathrm{Im}(\zeta) > 0$ (or $\mathrm{Im}(\zeta) < 0$) onto the domain $S(w)$ with the boundary $\ell(w)$. Expanding the functions $R(\zeta)$ into the serie of powers of $1/\zeta$, we obtain

$$R(\zeta) = \sum_{k=1}^{\infty} M_k \zeta^{-k}. \tag{3.18}$$

Since the point $\zeta = \infty$ is the image of the nonsingular point of the boundary $\ell(\sigma)$, the conditions ([1]–[31])

$$M_1 = \sum_{k=1}^{m} c_k^* = 0, \tag{3.19}$$

$$M_2 = \sum_{k=1}^{m} \big[a_k c_k^* + 0,5(1 - \nu_k^2)\big] = 0, \tag{3.20}$$

$$M_3 = \sum_{k=1}^{m} \left[a_k^2 c_k^* + a_k(1-\nu_k^2)\right] = 0 \tag{3.21}$$

should be fulfilled. From the condition (3.19) follows (3.11), and vice versa. We can obtain the conditions (3.20) and (3.21) in somewhat different way and in another form. Taking into account (3.12), the conditions (3.19)–(3.21) allow one to define three parameters c_j, $j = 1, 2, 3$, of the parameters c_j, $j = 1, \dots, m$. Moreover, we choose arbitrarily three of the parameters $t = a_j$, $j = 1, \dots, m$ and fix them according to the Riemann theorem. Therefore $R(\zeta)$ defined by the formula (3.16) will depend on $2(m-3)$ unknown parameters a_j, c_j, $j = 1, \dots, m-3$. The equation (3.18) near the point $\xi = a_j$ can be rewritten as [26–31]

$$(\zeta - a_j)^2 u''(\xi) + (\xi - a_j) p_j(\xi) u'(\xi) + q_j(\xi) u(\xi) = 0, \tag{3.22}$$

where

$$p_j(\xi) = p_{0j} + \sum_{n=1}^{\infty} p_{nj}(\xi - a_j)^n, \tag{3.23}$$

$$p_{nj} = (-1)^{n-1} \sum_{k=1,\, k\neq j}^{m} \beta_k (a_j - a_k)^{-n}, \quad \beta_k = 1 - \alpha_{1k} - al_{2k}, \tag{3.24}$$

$$q_j = \alpha_{1j}\alpha_{2j} + c_j(\xi - a_j) + \sum_{n=2}^{\infty} q_{nj}(\xi - a_j)^n, \tag{3.25}$$

$$q_{nj} = (-1)^{n-2} \sum_{k=2,\, k\neq j}^{m} \big[\alpha_{1k}\alpha_{2k}(n-1) + {} + c_k(a_j - a_k)\big](a_j - a_k)^{-n}, \quad n = 2, 3, \dots, \tag{3.26}$$

$$q_{0j} = \alpha_{1j}\alpha_{2j}, \quad q_{1j} = c_j, \quad j = 1, \dots, m. \tag{3.27}$$

4. Local Solutions

Local solutions of the equation (3.32) for the points $\xi = a_j$, $j = 1, \dots, m$, are sought in the form

$$u_j(\xi) = (\xi - a_j)^{\alpha_j} \widetilde{u}_j(\xi), \quad \widetilde{u}_j(\xi) = 1 + \sum_{n=1}^{\infty} \gamma_{nj}(\xi - a_j)^n, \tag{4.1}$$

where γ_{0j}, $n = 1, \dots, \infty$, $j = 1, \dots, m$, are defined by the recurrence formulas ([26]–[31])

$$f_{0j}(\alpha_j) = \alpha_j(\alpha_{j-1}) + p_{nj}\alpha_j + q_{0j} = 0, \tag{4.2}$$

$$\gamma_{1j} f_{0j}(\alpha_j + 1) + f_{1j}(\alpha_j) = 0, \tag{4.3}$$

$$\gamma_{2j} f_{0j}(\alpha_j + 2) + \gamma_{1j} f_{1j}(\alpha_j + 1) + f_{2j}(\alpha_j) = 0, \tag{4.4}$$

. . . .

where

$$f_n(\alpha_j) = \alpha_{1j} p_{nj} + q_{nj}. \tag{4.5}$$

If the difference $\alpha_{1j} - \alpha_{2j}$, $j = 1, \dots, m$, is noninteger, then using the formulas (4.3)–(4.5), we construct the linearly independent solutions (3.32),

$$\begin{aligned} &u_{ki}(\xi) = (\xi - a_j)^{\alpha_{kj}} \widetilde{u}_{kj}(\xi), \\ &\widetilde{u}_{kj}(\xi) = 1 + \sum_{n=1}^{\infty} \gamma_{nj}^k (\xi - a_j)^n, \quad k = 1, 2, \quad j = 1, \dots, m. \end{aligned} \tag{4.6}$$

However, if $\alpha_{1j} - \alpha_{2j} = n$, $n = 0, 1, 2$, then $u_{1j}(\xi)$ is constructed by the formulas (4.3)–(4.5), while $u_{2j}(\xi)$ by the Frobenius method ([24], [26]–[31]). Note that for $\alpha_{1j} - \alpha_{2j} = 0$, the function $u_{2j}(\zeta)$ has the form

$$u_{2j}(\xi) = u_{1j}(\xi) \ln(\xi - a_j) + (\xi - a_j)^{\alpha_{1j}} \sum_{n=1}^{\infty} \gamma_{nj}^2 (\xi - a_j)^n, \tag{4.7}$$

where

$$\gamma_{nj}^2 = \Big\{ \frac{d\gamma_{1j}(\alpha_j)}{d\alpha_j} \Big\}_{\alpha_j = \alpha_{2j}}.$$

If $\alpha_{1j} - \alpha_{2j} = n$, $n = 1, 2$, then for the construction of $u_{2j}(\xi)$ we have to differentiate the equality

$$u_{2j}(\xi) = (\xi - a_j)^{\alpha_j} \Big[\alpha_j - \alpha_{2j} + \sum_{n=1}^{\infty} \gamma_{nj}^2 (\xi - a_j)^n \Big] \tag{4.8}$$

with respect to α_j, then take $\alpha_j \to \alpha_{2j}$, we obtain

$$\begin{aligned} u_{2j}(\xi) = (\xi - a_j)^{\alpha_{2j}} &\Big[\sum_{n=1}^{\infty} \lim_{\alpha_j \to \alpha_{2j}} \gamma_{nj}(\alpha_i)(t - a_j)^n \Big] \ln(t - a_j) + \\ &+ (t - a_j)^{\alpha_{2j}} \Big\{ 1 + \sum_{n=1}^{\infty} \Big[\frac{d\gamma_{1j}(\alpha_j)}{d\alpha_j} \Big]_{\alpha_j = \alpha_{2j}} (t - a_j)^n \Big\}. \end{aligned} \tag{4.9}$$

P. Ya. Polubarinova–Kochina has proved that a solution for the cut end $u_{2j}(\xi)$, where $\alpha_{1j} - \alpha_{2j} = 2$, does not involve a logarithmic term. Moreover, for such points she also obtained an algebraic equation connecting the parameters a_j, c_j, $j = 1, \dots, m$. To construct $u_{2j}(\xi)$ uniquely, we suggested in our works the following method. For the point $t = a_j$, the equality (4.4) fails to be fulfilled since

$$f_{0j}(\alpha_j + 2) = 0, \quad \alpha_j \to \alpha_{2j}. \tag{4.10}$$

For the equality (4.4) to take place as $\alpha_j \to \alpha_{2j}$, it will be necessary and sufficient to require the condition

$$\gamma_{1j} f_1(\alpha_j + 1) + f_2(\alpha_j) = 0, \quad \alpha_{1j} \to \alpha_{2j} + 2. \tag{4.11}$$

After simplification, (4.11) takes the form ([26]–[31])

$$q_{2j} + q_{1j}^2 + q_{1j}p_{1j} = 0. \tag{4.12}$$

To construct $u_{2j}(\xi)$ uniquely, it suffices to construct $\gamma_{2j}^2(\alpha_{2j})$ and then make use of the formulas (4.3)–(4.5) ([26]–[39]). Indeed, suppose $\alpha_{1j} \neq \alpha_{2j}$. Then using (4.4), we find $\gamma_{2j}(\alpha_j)$ and obtain

$$\gamma_{2j}(\alpha_j) = -\frac{\gamma_{1j}(\alpha_j)f_{1j}(\alpha_j+1) + f_{2j}(\alpha_j)}{f_0(\alpha_j+2)}\,. \tag{4.13}$$

In the formula (4.13) we remove uncertainty and then pass to the limit $\alpha_j \to \alpha_{2j}$. We have

$$\gamma_{2j}^2 = -0,5\big[p_{1j}(p_{1j} + 2q_{1j}) + p_{2j}\big]. \tag{4.14}$$

Next we define local solutions near the point $t = \infty$. The functions $p(\xi)$ and $q(\xi)$ near the point $t = \infty$ can be represented in the form

$$p(\xi) = \xi^{-1}\sum_{n=0}^{\infty} p_{n\infty}\xi^{-4}, \quad q(\xi) = \xi^{-2}\sum_{n=0}^{\infty} q_{n\infty}\xi^{-n}, \tag{4.15}$$

where

$$p_{n\infty} = \sum_{k=1}^{m} \beta_k a_k^n, \quad p_{0\infty} = 6, \tag{4.16}$$

$$q_{n\infty} = \sum_{k=1}^{m} \big[\alpha_{1k}\alpha_{2k}(n+1) + c_k a_k\big]a_k^n. \tag{4.17}$$

Local solutions near the point $\xi = \infty$ have the form

$$u_\infty(\xi) = \xi^{-\infty}\sum_{n=1}^{\infty} \gamma_{n\infty}\xi^{-(\alpha_\infty+n)}, \tag{4.18}$$

where $\gamma_{n\infty}$, $n = 1, \dots, \infty$, are defined by the formulas

$$f_{0\infty}(\alpha_\infty) = \alpha_\infty(\alpha_\infty+1) - p_{0\infty}\alpha_\infty + q_{0\infty} = 0, \tag{4.19}$$

$$\alpha_{1\infty}f_{0\infty}(\alpha_\infty+1) - p_{1\infty}\alpha_\infty + q_{1\infty} = 0, \tag{4.20}$$

$$\alpha_{2\infty}f_{0\infty}(\alpha_\infty+2) + \gamma_{1\infty}(\alpha_\infty+1) - p_{2\infty}\alpha_\infty + q_{2\infty} = 0, \tag{4.21}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

where

$$f_{k\infty} = q_{k\infty} - (\alpha_\infty + k)p_{k\infty}. \tag{4.22}$$

Since $t = \infty$ is the image of the nonangular point, the equation (4.19) should have the roots $\alpha_{1\infty} = 3$, $\alpha_{2\infty} = 2$. Consequently,

$$q_{0\infty} = \sum_{k=1}^{m} \big[\alpha_{1k}\alpha_{2k} + a_k c_k\big] = 6. \tag{4.23}$$

As far as $\alpha_{1\infty}-\alpha_{2\infty}=1$, the equations (4.20)–(4.22) allow one to define only one solution $u_{1\infty}(\xi)$. To find $u_{2\infty}(\xi)$ as $\alpha_\infty\to\alpha_{2\infty}$, it is necessary and sufficient that the condition

$$q_{1\infty}-p_{1\infty}\alpha_{2\infty}=0 \tag{4.24}$$

takes place. To determine $\gamma^2_{1\infty}$, we act as follows. By virtue of (4.20), for $\alpha_{1\infty}\neq\alpha_{2\infty}$, we find $\gamma_{1\infty}$ and obtain

$$\gamma_{1\infty}=\frac{p_{1\infty}\alpha_\infty-q_{1\infty}}{f_{0\infty}(\alpha_\infty+1)}\,. \tag{4.25}$$

Since the numerator and the denominator in (4.25) vanish as $\alpha_\infty\to\alpha_{2\infty}$, we have to remove uncertainty. We uniquely obtain ([26]–[31])

$$\gamma^2_{1\infty}=p_{1\infty}. \tag{4.26}$$

Having defined $\gamma^2_{1\infty}$, we can find the remaining $\gamma^2_{n\infty}$, $n=2,\dots,\infty$, by using the formulas (4.20)–(4.22). Consequently, $u_{2\infty}(\xi)$ is defined uniquely. Finally, we obtain

$$u_{k\infty}(\xi)=\xi^{-\alpha_{k\infty}}+\sum_{n=1}^{\infty}\gamma^k_{n\infty}\xi^{-\alpha_{k\infty}-n},\ \ k=1,2. \tag{4.27}$$

The system (3.19), (3.20), (3.21) coincides respectively with the system (3.11), (4.23), (4.24), and vice versa.

Local solutions $u_{kj}(\xi)$, $k=1,2$, $j=1,\dots,m$, contain multi-valued functions of which we choose one-valued branches

$$\begin{aligned}
&\exp\big[\beta_{kj}\ln(\xi-a_j)\big]>0,\quad t>a_j,\\
&\Big\{\exp\big[\alpha_{kj}\ln(\xi-a_j)\big]\Big\}^{-1}=\exp\big[i\pi\alpha_{kj}\big]\Big\{\exp\ln(a_j-\xi)\Big\},\ \ a_j-\xi>0,\\
&\Big\{\exp\big[\alpha_{kj}\ln(\xi-a_j)\big]\Big\}^{-1}=\exp\big[-i\pi\alpha_{kj}\big]\Big\{\exp\big[\alpha_{kj}\ln(a_j-\xi)\big]\Big\},\\
&\qquad\qquad a_j-t>0.
\end{aligned}$$

For the equation (3.8), in the neighborhood of every singular point $\xi=a_j$, $j=1,\dots,m+1$, and in the neighborhood of the points $t=a^*_1=(a_j+a_{j+1})/2$, $j=1,\dots,m-1$, we construct respectively $u_{kj}(\xi)$, $k=1,2$, $j=1,\dots,m+1$ and $\gamma_{kj}(\xi)$, $k=1,2$, $j=1,\dots,m-1$.

A solution of (3.6) is sought by means of the matrix $TX(\xi)$, where $\chi(\xi)$ is a solution of (3.12). If $\chi(\xi)$ is a solution of (3.12), then $TX(\xi)$ is likewise a solution of (3.12), where the constant matrix is defined as

$$T=\begin{pmatrix}p&q\\r&s\end{pmatrix},\quad \det T\neq0, \tag{4.28}$$

p, q, r and s are constants of integration of the equation (3.14).

5. Fundamental Matrices

The local fundamental matrices $\Theta_j(\xi)$, $\sigma_j(\xi)$, $\Theta^*_j(\xi)$, $\overline{\Theta}_j(\xi)$, where $\overline{\Theta}_j(\xi)$ is the matrix, complex-conjugate to the matrix $\Theta_j(\xi)$, are defined as follows:

$$\Theta_j(\xi)=\begin{pmatrix} u_{1j}(\xi) & u'_{1j}(\xi)\\ u_{2j}(\xi) & u'_{2j}(\xi)\end{pmatrix},\quad a_j<\xi<a_{j+1},\ \ j=1,\dots,j-1, \tag{5.1}$$

$$\Theta^*_j(\xi)=\begin{pmatrix} u^*_{1j}(\xi) & u'^*_{1j}(\xi)\\ u^*_{2j}(\xi) & u'^*_{2j}(\xi)\end{pmatrix},\quad a_{j-1}<\xi<a_j, \tag{5.2}$$

$$\sigma_j(\xi)=\begin{pmatrix} \sigma_{1j}(\xi) & \sigma'_{1j}(\xi)\\ \sigma_{2j}(\xi) & \sigma'_{2j}(\xi)\end{pmatrix},\quad \xi=\frac{a_j+a_{j+1}}{2}=a^*_j,\ j=1,\dots,m-1, \tag{5.3}$$

$$\Theta^*_j(\xi)=\vartheta^{\pm}_j\Theta^*_j(\xi),\ \ a_{j-1}<\xi<a_j, \tag{5.4}$$

$$\Theta_\infty(\xi)=\begin{pmatrix} u_{1\infty}(\xi) & u'_{1\infty}(\xi)\\ u_{2\infty}(\xi) & u'_{2\infty}(\xi)\end{pmatrix}, \tag{5.5}$$

$$\begin{aligned}\vartheta^{\pm}_j&=\begin{pmatrix}\exp(\pm i\pi\alpha_{1j}) & 0\\ 0 & \exp(\pm i\pi\alpha_{2j})\end{pmatrix},\\ \overline{\vartheta}^{\,+}_j&=\vartheta^-_j,\ \ \alpha_{1j}-\alpha_{2j}\neq n,\ \ n=0,1,2,\end{aligned} \tag{5.6}$$

while if $\alpha_{1j}-\alpha_{2j}=n$, $n=0,1,2$, we have

$$\vartheta^{\pm}_j=\exp\big[\pm i\pi\alpha_{2j}\big]\begin{pmatrix}1 & 0\\ \pm i\pi & 1\end{pmatrix},\ \ n=0,2, \tag{5.7}$$

$$\vartheta^{\pm}_j=\exp\big[\pm i\pi\alpha_{2j}\big]\begin{pmatrix}-1 & 0\\ \mp\pi i & 1\end{pmatrix},\ \ n=1. \tag{5.8}$$

One Essential Remark. The fact that the series $u_{kj}(\xi)$, $k=1,2$, $j=1,\dots,m$, converge weakly this makes the process of calculations difficult. To remove this drawback, we act as follows ([26]–[31]). We replace the series $u_{kj}(\xi)$, $k=1,2$, $j=1,\dots,m+1$, by strongly and uniformly convergent functional series. Towards this end, it suffices to write the series $u_{kj}(\xi)$, $k=1,2$, $j=1,\dots,m+1$, in a somewhat different form:

$$\begin{aligned}u_{kj}(\xi)&=(\xi-a_j)^{\alpha_{kj}}\widetilde{u}_{kj}(\xi-a_j),\\ \widetilde{u}_{kj}(\xi-a_j)&=1+\sum_{n=1}^{\infty}\gamma^k_{nj}(\xi-a_j),\ \ k=1,2;\ \ j=1,\dots,m,\end{aligned} \tag{5.9}$$

$$u_{k\infty}(\xi)=\xi^{-\alpha_{k\infty}}\Big(1+\sum_{n=1}^{\infty}\gamma^k_{n\infty}(\xi)\Big), \tag{5.10}$$

where γ^k_{nj}, $\gamma^k_{n\infty}$ are defined through $f_{nj}(\alpha_j)$ and $f_{k\infty}(\alpha_j)$ as follows:

$$f_{nj}\big[(\xi-a_j),\beta_k\big]=\alpha_{kj}p_{nj}(\xi-a_j)+q_{mj}(\xi-a_j), \tag{5.11}$$

$$p_{nj}(\xi-a_j)=-\sum_{k=0,\,k\neq j}^{m}\beta_j\Big(\frac{\xi-a_j}{a_j-a_k}\Big)^n,\quad n=1,2,\dots, \tag{5.12}$$

$$q_{1j}(\xi-a_j)=c_j(\xi-a_j), \tag{5.13}$$

$$q_{nj}(\xi-a_j)=(-1)^{n-2}\sum_{k=1,\,k\neq j}^{m}\big[\alpha_{1k}\alpha_{2k}(n-1)+c_k(a_j-a_k)\big]\times$$
$$\times\Big(\frac{\xi-a_j}{a_j-a_k}\Big)^n,\quad n=1,2,\dots, \tag{5.14}$$

$$\Big|\frac{\xi-a_j}{a_j-a_k}\Big|<1,\quad k\neq j,$$

$$\begin{aligned}p_{n\infty}(\xi)&=\sum_{k=1}^{\infty}\beta_k\Big(\frac{a_k}{\xi}\Big)^n,\\ q_{n\infty}(\xi)&=\sum_{k=1}^{\infty}\big[\alpha_{1j}\alpha_{2j}(n+1)+c_ka_k\big]\Big(\frac{a_k}{\xi}\Big)^n,\quad n=01,2,\dots.\end{aligned} \tag{5.15}$$

The local matrix $\Theta_j^-(\xi)$ is complex conjugate to the matrix $\Theta_j^+(\xi)$. The real matrices $\Theta_j(\xi)$, $\Theta_j^*(\xi)$ are local solutions of the system of equations (3.22). Suppose that the elements of these matrices converge on some part of the interval $a_{j-1}<\xi<a_j$, on which the matrices $\Theta_j^*(\xi)$ and $\Theta_{j-1}(\xi)$ are connected by the following matrix identity ([26]–[31]):

$$\Theta_j^*(\xi)=T_{j-1}\Theta_{j-1}(\xi),\quad j=m,m-1,\dots,2, \tag{5.16}$$

from which the matrices T_{j-1} are defined uniquely. Assume also that the domains of convergence of the matrices $\Theta_j^*(\xi)$ and $\Theta_{j-1}(\xi)$ do not intersect. In this case, we construct at the point $\xi=a_j^*=(a_{j-1}+a_j)/2$ the matrix $\sigma_j(\xi)$ which converges in the interval $a_{j-1}<\xi<a_j$. It is seen that one can always pass from the matrix $\Theta_j^*(\xi)$ to the matrix $\Theta_{j-1}(\xi)$ with the following sequence:

$$\Theta_j^*(\xi)=T_{a_j}\sigma_j(\xi),\quad \sigma_j(\xi)=T_{j-1}^*(\xi)\Theta_{j-1}(\xi). \tag{5.17}$$

It follows from the above-said that $\Theta_m(\xi)$ can be analytically continued along the whole axis ξ.

To define the functions $\omega_0'(\xi)$ and $z'(\xi)$ in the interval $(-\infty,+\infty)$, we consider the matrices ([26]–[31])

$$\chi^{\pm}(\xi)=T\Theta_m^*(\xi),\ \ \xi>a_m;\quad \Theta_m^+(\xi)=\Theta_m^-(\zeta),\ \ a_m<\xi<+\infty. \tag{5.18}$$

From (5.18) we have $T=\overline{T}$.

We continue the matrix (5.18) along the real axis ξ and use the notation

$$\chi^*(\xi) = \chi(\xi), \quad \vartheta_j^+ = \vartheta_j.$$

We obtain

$$\begin{aligned}
&\chi(\xi) = T\vartheta_m \Theta_m^*(\xi), \;\; a_{m-1} < \xi < a_m, \\
&\chi(\xi) = T\vartheta_m T_{m-1}\Theta_{m-1}(\xi), \;\; a_{m-1} < \xi < a_m, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\chi(\xi) = T\vartheta_m T_{m-1}\vartheta_{m-1}\Theta_{m-1}^*(\xi)\cdots T_1\vartheta_1\Theta_1^*(\xi), \;\; \xi < a_1, \\
&\chi(\xi) = T\vartheta_m T_{m-1}\vartheta_{m-1}\cdots T_1\vartheta_1 T_{-\infty}\Theta_\infty(\xi), \;\; -\infty < \xi < \infty,
\end{aligned} \tag{5.19}$$

where

$$\begin{aligned}
\Theta_m^*(\xi) &= T_{m-1}\Theta_{m-1}(\xi), \;\; a_{m-1} < \xi < a_m, \\
\Theta_{m-1}^*(\xi) &= T_{m-2}\Theta_{m-2}(\xi), \;\; a_{m-2} < \xi < a_{m-1}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\Theta_2^*(\xi) &= T_1\Theta_1(\xi), \;\; a_1 < \xi < a_2, \\
\Theta_1^*(\xi) &= T_{-\infty}\Theta_\infty(\xi), \;\; -\infty < \xi < a_1, \\
\Theta_m(\xi) &= T_m\Theta_\infty(\xi), \;\; a_m < \xi < +\infty.
\end{aligned} \tag{5.20}$$

(5.20) allows one to determine the matrices $T_1, T_2, \ldots, T_{m-1}, T_{-\infty}, T_{+\infty}$. Substituting the matrices (5.19) into the boundary conditions (3.6) and then multiplying successively every matrix equality from the left by $[\Theta_j^*(\xi)]^{-1}$, $j = m, m-1, \ldots, 1$, we obtain the system of matrix equations ([26]–[31])

$$\begin{aligned}
T\vartheta_m &= g_{m-1}T\vartheta_m^-, \;\; \xi = a_m, \\
T\vartheta_m T_{m-1}\vartheta_{m-1} &= g_{m-2}T\overline{\vartheta}_m T_{m-1}\overline{\vartheta}_{m-1}, \;\; t = a_{m-1}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
T\vartheta_m T_{m-1}\vartheta_{m-1}\cdots T_1\vartheta_1 &= T\overline{\vartheta}_m T_{m-1}\overline{\vartheta}_{m-1}\cdots T_1\overline{\vartheta}_1, \;\; \xi = a_1.
\end{aligned} \tag{5.21}$$

The number of matrix equations is m. Every matrix equation gives two real equations. Consequently, we obtain the system consisting of $2m$ equations with respect to the parameters p, q, r, s, a_j, c_j, $j = 1, \ldots, m$. From the system (5.20) we define the elements of the matrices T_j, $j = 1, \ldots, m-1$, and substitute them in (5.21).

According to Riemann's theorem, we can choose arbitrarily three of the parameters $\xi = a_j$, $j = 1, \ldots, m$, and fix them. Thus we obtain the system of equations (3.11), (4.23, (4.25).

Suppose that one of the vertices of the circular polygon has a cut with the angle 2π at the cut end. If to that point on the contour $\ell(\sigma)$ there corresponds a regular nonangular point, then instead of two equations we have only one, (4.12). Under such an assumption we will have a system of $2(m+1)$ equations with respect to $2m+1$ parameters (a_j, $j = 1, \ldots, m-3$, c_j, $j = 1, \ldots, j$, p, q, r, s).

From the system (3.19), (4.23), (4.24), (4.12) we can define four accessory parameters and then substitute them in the remaining equations.

For the sake of simplicity, we assume that on the plane of complex velocity there is a circular pentagon whose one vertex has a cut with the angle 2π at the cut end. In this case, the homogeneous problem (3.6) is reduced to a system of three higher transcendent equations. It is assumed that such a system of equations has a solution.

If we denote $v_1(\xi)$ and $v_2(\xi)$, where

$$v_1(\xi) = pu_1(\xi) + qu_2(\xi), \tag{5.22}$$

$$v_2(\xi) = ru_1(\xi) + su_2(\xi) \tag{5.23}$$

are the components of the vector $\Phi_(\xi)$, or what comes to the same thing, the components of the first row of the matrix $\chi(\xi)$, then by the formula

$$w(\xi) = \frac{v_1(\xi)}{v_2(\xi)} \tag{5.24}$$

we obtain the general solution (3.14). The components $\omega'(z)$ and $z'(\xi)$ of the vector $\Phi'(\xi)$ are defined by the equalities

$$d\omega_0(\xi) = v_1(\xi)\chi_1(\xi)d\xi, \quad -\infty < \xi < +\infty, \tag{5.25}$$

$$d\sigma(\xi) = v_2(\xi)\chi_1(\xi)d\xi, \quad -\infty < \xi < +\infty, \tag{5.26}$$

where $v_1(\xi)\chi_1(\xi)$, $v_2(\xi)\chi_1(\xi)$ satisfy the boundary conditions (3.1) and those at the singular points $\xi = e_j$, $j = 1, \dots, n$, $\xi = \infty$. The integration of (5.25) and (5.26) in the intervals $(-\infty < \xi)$, (e_j, ξ), $j = 1, \dots, n$, provides us with

$$\omega_0(\xi) = \int_{-\infty}^{\xi} v_1(\xi)\chi_1(\xi)\, d\xi + \omega(-\infty), \tag{5.27}$$

$$\sigma(\xi) = \int_{-\infty}^{\xi} v_2(\xi)\chi_1(\xi)\, d\xi + \sigma(-\infty), \tag{5.28}$$

$$\omega(\xi) = \int_{e_j}^{\xi} v_1(\xi)\chi_1(\xi)\, d\xi + \omega(e_j+), \tag{5.29}$$

$$\sigma(\xi) = \int_{e_j}^{\xi} v_2(\xi)\chi_1(\xi)\, d\xi + \sigma(e_j, +0). \tag{5.30}$$

Considering (5.29) and (5.30) for $\xi = e_{j+1}$, we obtain a system of equations with respect to the removable singular points $\xi = e_{j+1}$ and to another unknown parameters. The equations (5.29) and (5.30) allow one to determine the parametric equation of the depression curve.

6. One Essential Remark

Consider one simplest integral Fredholm equation of the second kind ([33]–[39])

$$u(x) - \lambda \int_a^b k(x,t)u(t)\,dt = f(x), \tag{6.1}$$

where the unknown function $u(x)$ depends on the real variable x which changes in the same interval $[a,b]$ as the integration variable t; this requirement refers to all classes of integral equations we deal with in the present work. The interval may be finite or infinite. The functions $k(x,t)$ and $f(x)$ are assumed to be known and defined almost everywhere respectively in the square $a \le x \le b$, $a \le t \le b$ and in the interval $[a,b]$. The function $k(x,t)$ is called the kernel of the integral equation. It is assumed that the kernel $k(x,t)$ of Fredholm's equation satisfies the inequality

$$\int_a^b \int_a^b |k(x,t)|^2\,dx\,dt < \infty, \tag{6.2}$$

while the free term of Fredholm's equation satisfies the inequality

$$\int_a^b |f(x)|^2\,dx < \infty. \tag{6.3}$$

It is necessary to consider Fredholm's equations of more general type.

Let Ω^- be a measurable set in the space of any number of variables, x and t be the points of that set, and μ be a nonnegative measure defined in Ω. The equation

$$u(x) - \lambda \int_\Omega k(x,t)u(t)\,d\mu(t) = f(x) \tag{6.4}$$

is likewise called the Fredholm equation whose kernel $k(x,t)$ and free term $f(x)$ satisfy respectively the inequalities

$$\int_\Omega \int_\Omega |k(x,t)|^2\,d\mu(x)\,d\mu(t) < 0, \quad \int_\Omega |f(x)|^2\,d\mu(x) < \infty. \tag{6.5}$$

The kernel $k(x,t)$ satisfying (6.5) is called the Fredholm one.

The unknown function $u(x)$ is quadratically summable in (a,b), and hence belongs to the functional space $L_2(a,b)$. A solution of the equation (6.4) belongs to the space $L_2(\mu,\Omega)$ of functions which are quadratically summable with respect to μ. The inequalities (6.3) and (6.5) imply that the free term of the equation belongs to the same space. The parameter λ may take both real and complex values.

Denote the volume element by dx, and the integral (6.5) by B_k^2:

$$\int\limits_\Omega\int\limits_\Omega |k(x,t)|^2\,dx\,dt = B_k^2. \tag{6.6}$$

As is known, Fredholm's equation has either finite, or countable set of characteristic numbers; if there is a countable set of numbers, then they tend to infinity. But there are kernels which have no characteristic numbers at all, for example, Volterra kernels. A complete characteristic of such kernels is given in the following Lalesko's theorem. Let $k(x,t)$ be a Fredholm kernel and $k_n(x,t)$ be its iteration. For the kernel $k(x,t)$ to have no characteristic numbers, it is necessary and sufficient that

$$A_n = \int\limits_\Omega k_n(x,x)\,dx = 0, \quad n = 3,4,5,\dots . \tag{6.7}$$

Note that the numbers A_n are called the traces of the kernel $k(x,t)$. Lalesko has proved his theorem for bounded kernels, and a general proof has been given by S. N. Krachkovskiĭ ([33]–[39]).

The determinant and Fredholm's minors are represented as a quotient of two entire functions of λ. Note that the poles of the resolvent, the characteristic numbers of the kernel $k(x,t)$, do not depend on x and t. Thus the resolvent should be of the form

$$\Gamma(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)}\,, \tag{6.8}$$

where $D(x,t;\lambda)$ and $D(\lambda)$ are entire functions of λ. If we succeed in constructing these functions, then we will be able to find the resolvent, and a solution of the integral equation will be constructed by the well-known formula. For the numerator and the denominator of the fraction in (6.8) we give representations in the form of the so-called Fredholm series

$$\begin{gathered} D(x,t;\lambda) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\,B_n(x,t)\lambda^n, \\ D(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\,C_n\lambda^n, \end{gathered} \tag{6.9}$$

where the formulas

$$\begin{gathered} C_0 = 1, \quad B_0(x,t) = k(x,t), \quad C_n = \int\limits_\Omega B_{n-1}(x,x)\,dx, \quad n>0, \\ B_n = C_n k(x,t) - n\int\limits_\Omega k(x,\tau)B_{n-1}(\tau,t)\,d\tau \end{gathered} \tag{6.10}$$

allow one to calculate recursively the coefficients $B_n(x,t)$ and C_n.

Below, we will need the well-known formula ([33]–[39])

$$\frac{D'(\lambda)}{D(\lambda)} = -\sum_{n=1}^{\infty} A_n \lambda^{n-1}, \tag{6.11}$$

where

$$A_n = \int_{\Omega} k_n(x,x)\, dx, \quad n = 1,2,3,\dots, \tag{6.12}$$

are the traces of the kernel $k(x,t)$ mentioned above.

If the kernel is not continuous having second order discontinuities, then the integrals (6.12) defining the coefficients $c_1, c_2, c_3, \dots$ from the formulas (6.10), make no sense. For example, when the kernel $k(x,t)$ contains as a multiplier Green's function $G[P,Q]$ of the Dirichlet problem for harmonic functions which is symmetric with respect to P and Q, equals to zero on the boundary C and is analytic at all points P of the domain D except the points $P = Q$ where it has logarithmic singularity, the kernel $k(x,t)$ will have logarithmic singularity as well. Then the integral $\int_{\Omega} k(x,x)\, dx$ defining the coefficient c_1 makes no sense. This difficulty can be disregarded by putting, for example, the density $c_1 = 0$.

The iterated kernel $k_2(s,t)$ has the form

$$k_2(s,t) = \int_a^b k(s,t_1)k(t_1,t)\, dt_1. \tag{6.13}$$

The integral $k_2(s,t)$ has sense for any s and t from $[a,b]$ since in the unfavorable case, when s and t coincide, we have the following estimate of the integrand:

$$\left| k(s,t_1 k(t_1,s) \right| \le \frac{M_1}{|s-t_1|^{\varepsilon_1}}\,, \quad \varepsilon_1 > 0. \tag{6.14}$$

It is proved that the function $k_2(s,t)$ is continuous in the square $a \le x \le b$, $a \le t_1 \le b$. The functions

$$k_n(s,t) = \int_a^b k(s,t_1)k_{n-1}(t_1,t)\, dt_1, \quad n = 1,2,3,\dots\,, \tag{6.15}$$

$$\left| k(s,t_1 k_{n-1}(t_1,t) \right| < \frac{M_{n-1}}{|s-t_1|^{\varepsilon_1}}\,, \quad \varepsilon_{n-1} > 0, \tag{6.16}$$

are estimated analogously. The integral $k_n(s,t)$, $n = 1,2,3\dots$, makes sense for any positions s and t from $[a,b]$, and the estimates of the integrand have the form (6.16). Thus we have to put

$$A_n = \int\limits_{\Omega} k_n(s,s)\,ds = 0, \quad n = 1,2,3,\dots,n, \tag{6.17}$$

and then $k_n(x,x) = 0$, $n = 1,2,3,\dots$, $c_n = 0$, $n = 1,2,3,\dots,n$. Taking into account (6.17), we obtain from (6.11) that

$$D'(\lambda) = 0, \tag{6.18}$$

and from (6.18) we have

$$D(\lambda) = 1. \tag{6.19}$$

Consequently, the kernel of the integral equation (2.43) has no characteristic numbers. Analogously, one can prove that the considered in our work [39] kernel of the integral equation (3.35) has no characteristic numbers.

References

1. N. E. Kochin, I. A. Kibel', and N. V. Roze, Theoretical hydromechanics. (Translated from the Russian) *Interscience Publishers John Wiley & Sons, Inc. New York–London–Sydney,* 1964; Russian original: *Moscow,* 1955.
2. J. Happel and H. Brenner, Low Reynolds number hydrodynamics with special applications to particulate media. *Prentice-Hall, Inc., Englewood Cliffs, N.J.,* 1965; Russian transl.: *Mir, Moscow,* 1976.
3. M. A. Lavrent'ev and B. V. Shabat, Methods of the theory of functions of a complex variable. (Russian) *Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow*, 1958.
4. A. V. Bitsadze, Equations of mathematical physics. *Nauka, Moscow,* 1982.
5. I. N. Vekua, Generalized analytic functions. (Russian) *Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow*, 1959.
6. M. A. Lavrent'ev and B. V. Shabat, Problems of hydrodynamics and their mathematical models. (Russian) *Nauka, Moscow,* 1973.
7. I. N. Vekua, New Methods for Solving Elliptic Equations. (Russian) *OGIZ, Moscow–Leningrad*, 1948.
8. C. Miranda, Equazioni alle derivate parziali di tipo ellittico. (Italian) *Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.), Heft 2. Springer-Verlag, Berlin–Göttingen–Heidelberg,* 1955; Russian transl.: *Izd. Inostr. Lit., Moscow,* 1957.
9. A. Wangerin, Reduction der Potentialgleichung fur gewisse Rotationskorper auf eine gewohnliche Differentialgleichung. *Preisschr. der Jabl. Ges. Leipzig*, 1875.
10. G. Bateman and A. Erdelyi, Higher transcendental functions. Elliptic and modular functions. Lame and Mathieu functions. (Translated from the English into Russian) *Nauka, Moscow,* 1967.
11. M. M. Smirnov, Equations of mixed type. (Russian) *Vyssh. Shkola, Moscow,* 1985.
12. V. F. Piven', The method of axisymmetric generalized analytic functions in the investigation of dynamic processes. (Russian) *Prikl. Mat. Mekh.* **55** (1991), No. 2, 228–234; English transl.: *J. Appl. Math. Mech.* **55** (1991), No. 2, 181–186 (1992).
13. V. M. Radygin and O. V. Golubeva, Application of functions of a complex variable to problems of physics and technology. Textbook. (Russian) *Vyssh. Shkola, Moscow,* 1983.
14. P. Ya. Polubarinova-Kochina, The theory of underground water motion. 2nd ed. (Russian) *Moscow, Nauka*, 1977.

15. P. Ya. Polubarinova-Kochina, Circular polygons in filtration theory. (Russian) *Problems of mathematics and mechanics*, 166–177, *"Nauka", Sibirsk. Otdel., Novosibirsk*, 1983.
16. P. Ya. Polubarinova-Kochina, Analytic theory of linear differential equations in the theory of filtration. Mathematics and problems of water handling facilities. *Collection of scientific papers,* 19–36. *Naukova Dumka, Kiev,* 1986.
17. Ya. Bear, D. Zaslavskii, and S. Irmey, Physical and mathematical foundations of water filtration. (Translated from English) *Mir, Moscow*, 1971.
18. N. I. Muskhelishvili, Singular integral equations. Boundary value problems in the theory of function and some applications of them to mathematical physics. 3rd ed. (Russian) *Nauka, Moscow*, 1968; English transl.: *Wolters-Noordhoff Publishing, Groningen,* 1972.
19. N. P. Vekua, Systems of singular integral equations and certain boundary value problems. 2nd ed. (Russian) *Nauka, Moscow*, 1970.
20. E. L. Ince, Ordinary Differential Equations. *Dover Publications, New York,* 1944. Russian transl.: *ONTI, State Scientific Technical Publishing House of Ukrtaine, Kharkov,* 1939.
21. A. Hurwitz and R. Courant, Theory of functions. (Translation from German) *Nauka, Moscow*, 1968.
22. G. N. Goluzin, Geometrical theory of functions of a complex variable. 2nd ed. (Russian) *Nauka, Moscow*, 1966.
23. V. V. Golubev, Lectures in analytical theory of differential equations. 2nd ed. (Russian) *Gostekhizdat, Moscow-Leningrad*, 1950.
24. E. A. Coddington and N. Levinson, Theory of ordinary differential equations. *McGraw-Hill Book Company, Inc., New York-Toronto-London,* 1955.
25. W. von Koppenfels and F. Stallmann, Praxis der konformen Abbildung. *Die Grundlehren der mathematischen Wissenschaften,* Bd. 100, *Springer-Verlag, Berlin-Göttingen-Heidelberg,* 1959; Russian transl.: *Izd. Inostr. Lit., Moscow,* 1963.
26. A. P. Tsitskishvili, Conformal mapping of a half-plane on circular polygons. (Russian) *Trudy Tbiliss. Univ. Mat. Mekh. Astronom.* **185**(1977), 65–89.
27. A. P. Tsitskishvili, On the conformal mapping of a half-plane onto circular polygons with a cut. (Russian) *Differentsial'nye Uravneniya* **12**(1976), No. 1, 2044–2051.
28. A. Tsitskishvili, Solution of the Schwarz differential equations. *Mem. Differential Equations Math. Phys.* **11**(1997), 129–156.
29. A. R. Tsitskishvili, Construction of analytic functions that conformally map a half plane onto circular polygons. (Russian) *Differentsial'nye Uravneniya* **21**(1985), No. 4, 646–656.
30. A. Tsitskishvili, Solution of some plane filtration problems with partially unknown boundaries. *Mem. Differential Equations Math. Phys.* **15**(1998), 109–138.
31. A. Tsitskishvili, Solution of Spatial Axially Symmetric Problems Of The Theory Of Filtration With Partially Unknown Boundaries. *Mem. Differential Equations Math. Phys.* **39**(2006), 105–140.
32. G. M. Položiĭ, The theory and application of p-analytic and and (p, q)-analytic functions. Generalization of the theory of analytic functions of a complex variable. (Russian) *Second edition, revised and augmented. Izdat. "Naukova Dumka", Kiev*, 1973.
33. V. I. Smirnov, The course of higher mathematics. T. II. 11th edition. *Gos. Izdat. Tekhniko-Teoretich. Lit. Moscow–Leningrad*, 1952.
34. V. I. Smirnov, The course of higher mathematics. T. III, Part 2, 5th edition. *Gos. Izdat.Tekhniko-Teoretich. Lit. Moscow–Leningrad*, 1952.

35. V. I. Smirnov, The course of higher mathematics. T. IV, 2nd edition. *Gos. Izdat. Tekhniko-Teoretich. Lit. Moscow–Leningrad* 1952.
36. A. N. Tikhonov and A. A. Samarskiĭ, The equations of mathematical physics. (Russian) 2d ed. *Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow*, 1953.
37. S. G. Mikhlin, Integral Equations and their Applications to some Problems of Mechanics, Mathematical Physics and Engineering. (Russian) 2d ed. *Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad*, 1949.
38. P. P. Zabreiko, A. I. Koshelev, M. A. Krasnoselski, S. G. Mikhlin, L. S. Rakovshchik, and V. Ja. Stetsenko, Integral equations. *"Nauka" Publ. House, Glav. Redak. Fiz.-Mat. Lit. Moscow*, 1968.
39. A. Tsitskishvili, The exact mathematical method of solution of spatial axisymmetric problems of the theory of filtration with partially unknown boundaries, and its application to the hole hydraulics. *Proc. A. Razmadze Math. Inst.* **142**(2006), 67–107.

Chapter V

A General Method of Constructing the Solutions of Spatial Axisymmetric Stationary with Partially Unknown Boundaries Problems of the Jet and Filtration Theories

Abstract. We consider a general mathematical method of constructing the solutions of spatial axisymmetric stationary with partially unknown boundaries problems of the jet and filtration theories. The x-axis coincides with the symmetry axis, and the distance to the x-axis is denoted by y. The use is made of the right coordinate system. Of infinitely many half-planes we arbitrarily select one passing through the symmetric axis. But for the sake of effectiveness sometimes it is more convenient to take two symmetric half-planes lying in one plane. The boundary of the domain under consideration consists of the known and unknown parts. The known ones consist of straight lines and their portions, while the unknown parts consist of curves. Every portion of the boundary is assigned two boundary conditions. The unknown functions (the velocity potential, the flow function) and their arguments on every portion of the boundary must satisfy two inhomogeneous boundary conditions.

The system of differential equations with respect to the velocity potential and flow function is reduced to a normal equation. Unknown functions are represented as sums of holomorphic and generalized analytic functions.

One problem of the jet theory and one problem of the filtration theory are solved.

1. Axisymmetric Flows

If the velocity components u_x and u_y are functions of only x and y, whereas the velocity component u_z is equal to zero, then the motion takes place in the planes parallel to the plane x, y; the motion is the same in all such planes. This implies that there is a direction to which all velocities of the field are perpendicular. The investigation of the plane stationary liquid motion under the above assumptions is, as is known, characterized by certain analytic peculiarities, and many interesting problems can be solved effectively ([1]–[37]).

But, as is known, if the boundaries of the problems under consideration are partially unknown and the boundary conditions are mixed, then the solution of such problems becomes more complicated. The flow function in terms of which many problems are formulated is, as usual, introduced in the plane case, but it is very difficult to introduce it in the spatial case. In the plane problems, the velocity potential and the flow function form analytic functions, and the theory of such functions is well developed both from the qualitative and quantitative points of view ([1]–[6]). As it can be seen below, there exist spatial axisymmetric problems whose solution reduces to the solution of plane problems ([23]–[25]).

The solution of spatial axisymmetric problems with partially unknown boundaries present great mathematical difficulties. Such problems are encountered in the theory of filtration, in the theory of jet flows, and in many parts of mathematical physics such, for example, as the mathematical theory of hydromechanics and some other sections of mechanics. The conditions are different in each case. For example, the liquid in the theory of jet flows is weightless, ideal and incompressible, capillary forces and vortices are absent, and the flow is stationary. In the problem of filtration the liquid has weight. Below we will describe a general method of solution of spatial axisymmetric with partially unknown boundaries problems of the jet and filtration theories.

The liquid motion is said to be spatial and axisymmetric, if all velocity vectors lie in half-planes passing through a straight line which is called the axis of symmetry, and the picture of the field of velocities is the same for all meridional half-planes. However, from the mechanical point of view the difference between the half-planes exists, and this is connected with the direction of velocity which can be determined according to the physical sense of the variables involved. The spatial field of velocities of an axisymmetric motion is completely described by the plane field of any of such half-planes. The symmetry axis is assumed to be the x-axis; the distance to the x-axis is denoted by y, and by u_x and u_y we denote, respectively, the components of the velocity vector $\vec{u}(u_x, u_y)$ which is connected with the velocity potential $\varphi(x,t)$ as follows: $\vec{u}\big[\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\big]$. On the plane, the use is made, as usual, of the right system of coordinates x, y; $z = x + iy$ ([1]–[3]).

The velocity potential φ and the flow function ψ are the functions of only cylindrical coordinates x, y. Due to the axial symmetry, it suffices to study the flow in any arbitrarily taken meridional half-plane with the system of coordinates x, y.

We choose arbitrarily one half-plane passing through the symmetry axis x on which the moving liquid occupies certain simply connected domain $S(z)$, where $z = x + iy$, with the boundary $S(\ell)$; if some part of the boundary $\ell(z)$ of the domain $S(z)$ is unknown, we have to find it.

Here we present another definition of axisymmetry: the flow is axisymmetric, if the flow lines lie in the half-planes passing thorough the given axis; a picture of distribution of the flow lines is the same for every half-plane.

The lines of intersection of a surface and the planes passing through the symmetry axis x are called meridians, whereas the lines of intersection with the planes perpendicular to the x-axis are called parallels.

In the cylindrical system of coordinates x, θ, y, where from the definition of axisymmetry follows $u_\theta = 0$, the equation of continuity has the form

$$\frac{\partial(yu_x)}{\partial x} + \frac{\partial(yu_y)}{\partial y} = 0, \tag{1.1}$$

where $u_x = \frac{\partial\varphi}{\partial x}$ and $u_y = \frac{\partial\varphi}{\partial y}$ are the projections of velocities onto the axes x and y.

As is known, the differential equation of any flow line for an axisymmetric flow, $u_y dx - u_x dy = 0$, multiplied by y, is the full differential of the flow function $d\psi = yu_y dx - yu_x dy$, since

$$u_x = \frac{1}{y}\,\frac{\partial\psi}{\partial y}\,, \quad u_y = -\frac{1}{y}\,\frac{\partial\psi}{\partial x}\,. \tag{1.2}$$

On the other hand,

$$u_x = \frac{\partial\varphi}{\partial x} = \frac{1}{y}\,\frac{\partial\psi}{\partial y}\,, \quad u_y = \frac{\partial\varphi}{\partial y} = -\frac{1}{y}\,\frac{\partial\psi}{\partial x}\,. \tag{1.3}$$

In [25] the reader can find a general method of solution of spatial axisymmetric stationary with partially unknown boundaries problems of filtration with the mixed boundary conditions, where the porous medium is non-deformable, isotropic and homogeneous. Stationary motion of the liquid in the porous medium obeys the Darcy law.

Below we will present some statements of the well-known authors regarding solutions of spatial stationary axisymmetric problems with partially unknown boundaries ([3], [4], [6]).

Everywhere below, when solving the problems of the jet theory, the use will be made of the following assumptions. The liquid is weightless, ideal and incompressible. Capillary forces and vortices are absent, and the flow is stationary [3].

“Solution of spatial jet problems presents great mathematical difficulty. At present we are aware only of the works which are devoted to axisymmetric jet flows. However, even for that simple particular case of spatial problems no one succeeded in creation of a mathematical device which would be as convenient as that of the theory of functions of a complex variable. The authors engaged in the axisymmetric jet flows either restrict themselves to approximate numerical solutions of the problems, or prove theorems of general nature” [3].

"Unfortunately, the methods of the theory of functions of complex variable applied to solution of plane problems have no effective analogue in the axisymmetric case, or, more precisely, analytic methods provide us with little information of physical interest" [6].

"The qualitative theory of solutions of the system of differential equations (1.3) can be constructed rather completely, whereas the quantitative theory is not as well developed as for the solutions of the (Cauchy–Riemann) system, i.e., for analytic functions" [4].

Below, we will give a general method of solution of spatial axisymmetric problems with unknown boundaries both in the theory of filtration and in the theory of jet flows.

Here we cite some rather frequently encountered boundary conditions for spatial axisymmetric problems of filtration.

1. On a free surface, the boundary conditions have the form

$$\varphi(x,y) - kx = const, \tag{1.4}$$

$$\psi(x,y) = const, \tag{1.5}$$

where $k = const$ is the coefficient of filtration;

2. along the boundary of water basins:

$$\varphi(x,y) = const, \tag{1.6}$$

$$a_1x + b_1y + c_1 = 0, \quad a_1, b_1, c_1 = const; \tag{1.7}$$

3. along the leaking intervals:

$$\varphi(x,y) - kx = const, \tag{1.8}$$

$$a_2x + b_2y + c_2 = 0, \quad a_2, b_2, c_2 = const; \tag{1.9}$$

4. along the symmetry axis, when a segment of the symmetry axis x coincides with a portion of the boundary of $S(z)$, the boundary conditions are of the form

$$y = 0, \tag{1.10}$$

$$\psi(x,y) = 0, \tag{1.11}$$

but if the symmetry axis does not coincide with any part of the boundary of the flow domain $S(z)$, then

$$y = const, \quad const \neq 0, \tag{1.12}$$

$$\psi(x,y) = const, \quad const \neq 0; \tag{1.13}$$

5. along nonpermeable boundaries there take place the following boundary conditions:

$$\psi(x,y) = const, \tag{1.14}$$

$$a_3x + b_3y + c_3 = 0, \quad a_3, b_3, c_3 = const; \tag{1.15}$$

6. along the nonpermeable boundary, the velocity vector is directed along the boundary;
7. the velocity vector is perpendicular to the boundaries of water basins;
8. along a free surface (depression curve) we have

$$u_x^2 + u_y^2 - ku_x = 0. \tag{1.16}$$

In our works [23]–[25] it is assumed that on the plane of complex velocity we have circular polygons of particular types. Despite this fact, this class of problem is still wide enough. There exist axisymmetric spatial problems with partially unknown boundaries, when the boundary of the domain does not contain the symmetry axis. But there are problems when the boundary of the domain involves, as is said above, the symmetry axis or its portions.

For circular polygons, in particular for linear ones, we are able to solve plane with partially unknown boundaries problems of filtration. The statement and solution of the corresponding plane with partially unknown boundaries problems of filtration can be found in [2], [12] and [18]–[25].

A flow of a substance moving almost in a constant direction at a distance exceeding many times its cross-section size is called a jet. In order to get a jet, it suffices to make a hole in the reservoir whose local pressure exceeds that of the environment ([3], [5], [6]).

When flowing around an immovable obstacle or a wall protuberance, the flow, as usual, separates and forms the so-called isolated flow lines. The liquid between these flow lines forms a trace; right behind the obstacle the flow is quiet. The traces in the liquid are of dissimilar nature. A trace forms a chain of vortices stretching at a long distance behind the obstacle. The importance of traces is that they are the main source of resistance in the real liquid. As is known, the resistance in nonviscous liquids does not usually arise for subsonic velocities if the flow separation and the associated trace are absent ([3]–[6]).

If a body moves in a liquid with great velocity, the trace becomes gaseous; such a trace is called a cavity. If a ball moves in water at velocity about 8 m/sec or more, we obtain a cavity filled with air. Cavities arising at the velocity 30 m/sec or more are filled with steam ([3]–[6]).

Besides, there are still many questions of practical importance which are connected with formation of jets, traces and cavities ([3]–[6]).

2. Statement of the Problem in the Theory of Jets

The theory of jets considers flows which are bounded partially by rigid walls and unknown free surfaces of constant pressure ([3]–[6]).

The hydrodynamic problem is assumed to be solved if any of the two functions $\varphi(x, y)$ and $\psi(x, y)$ is known. Besides the equations (1.2) and

(1.3), for finding $\varphi(x,y)$ and $\psi(x,y)$ we have the following boundary conditions. The normal velocity on the free and body surfaces is equal to zero ([1]–[7]),

$$\frac{\partial\varphi}{\partial n} = 0, \tag{2.1}$$

where n is the normal directed into the liquid. The flow function ψ on the free and body surfaces is a constant value [3],

$$\psi = const. \tag{2.2}$$

This condition for ψ is equivalent to the condition (2.1). On the boundaries, the constant in (2.2) may take different values.

For example, in Figure 1 we can see one-half of the meridional plane $x0y$ for the problem concerning the flow round a circular cone in a tube. Since the stream function is defined within a constant summand, we can put $\psi = 0$ on the symmetry axis x, on the cone and on the free surface. But the difference between the values of ψ on the flow surface is equal to the liquid discharge between these surfaces divided by 2π; hence on the tube walls $\psi = \pi\mathbf{v}_\infty h^2/(2\pi)$, where h is the tube radius, and $\mathbf{v}_\infty$ is the velocity at infinity of the flow coming from the left ([3]–[6]).

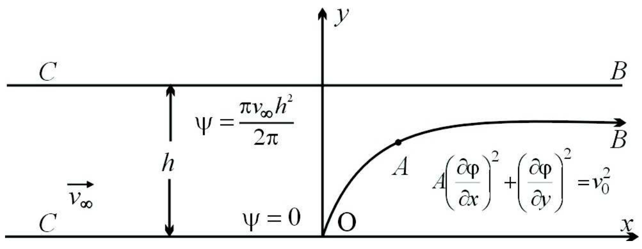

FIGURE 1

The form of the free surfaces is unknown, but here we have the supplementary condition of constancy of the velocity modulus $\mathbf{v}$, which is equivalent to the condition of pressure constancy. This condition can be written as ([3])

$$\frac{1}{\rho}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] = \left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 = \mathbf{v}_0^2, \tag{2.3}$$

where $\mathbf{v}_0$ is equal to $\mathbf{v}$ on the free surface ([3]–[6]).

3. The Stream Function for the Axisymmetric Flows

If the flow is irrotational, then the stream function ψ should satisfy the equation

$$\frac{\partial u_x}{\partial y} = \frac{\partial u_y}{\partial x}\,, \quad \text{then} \quad \frac{\partial}{\partial x}\left(\frac{1}{y}\,\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{y}\,\frac{\partial \psi}{\partial y}\right) = 0. \tag{3.1}$$

Recall that the function $\varphi(x,t)$ is harmonic in the cylindrical coordinate system. Unlike the plane case, the steam function $\psi(x,y)$ is not harmonic. It follows from (1.3) that

$$\frac{\partial \varphi}{\partial x}\,\frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial y}\,\frac{\partial \psi}{\partial y} = 0. \tag{3.2}$$

The system (1.1), (1.3) can be rewritten as follows:

$$\Delta\varphi(x,t) + \frac{1}{y}\,\frac{\partial \varphi}{\partial y} = 0, \tag{3.3}$$

$$\Delta\psi(x,t) - \frac{1}{y}\,\frac{\partial \psi}{\partial y} = 0, \tag{3.4}$$

where Δ is the Laplace operator.

We write the system (3.3), (3.4) in the form

$$\frac{\partial^2 \varphi}{\partial x^2} + 4\alpha\,\frac{\partial^2 \varphi}{\partial \alpha^2} + 4\,\frac{\partial \varphi}{\partial \alpha} = 0, \tag{3.5}$$

$$\frac{\partial^2 \psi}{\partial x^2} + 4\alpha\,\frac{\partial^2 \psi}{\partial \alpha^2} = 0, \tag{3.6}$$

where $\alpha = y^2$.

It can be seen from (3.5) and (3.6) that the given system for $\alpha = y^2 \neq 0$ is elliptic. Along the $0x$-axis, as $\alpha \to 0$, we have

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{4}\,\frac{\partial \varphi}{\partial \alpha} = 0, \tag{3.7}$$

$$\frac{\partial^2 \psi}{\partial x^2} = 0. \tag{3.8}$$

Along the symmetry axis $0x$, we have

$$\lim_{y\to 0}\frac{\partial \varphi}{\partial y} = 0, \quad \lim_{y\to 0}\frac{\partial \psi}{\partial y} = 0, \quad \lim_{y\to 0}\frac{1}{y}\,\frac{\partial \varphi}{\partial y} = \frac{\partial^2 \varphi}{\partial y^2}\,, \tag{3.9}$$

$$\lim_{y\to 0}\frac{1}{y}\,\frac{\partial \psi}{\partial y} = \frac{\partial^2 \varphi}{\partial y^2}\,. \tag{3.10}$$

4. Application of Analytic and Generalized Analytic Functions to Solution of Axisymmetric Problems

We map conformally the half-plane $\mathrm{Im}(\zeta) \geq 0$ (or $\mathrm{Im}(\zeta) < 0$) of an auxiliary complex plane $\zeta = \xi + i\eta$ onto the domain $S(z)$, where $z(\zeta) = x(\xi,\eta)+iy(\xi,\eta)$. A part of the boundary $S(\ell)$ of the domain $S(z)$ is unknown and should be defined. On the plane $\zeta = \xi + i\eta$, the system (1.3) takes the form

$$\frac{\partial\varphi}{\partial\xi} = \frac{1}{y(\xi,\eta)}\,\frac{\partial\psi}{\partial\eta}\,, \tag{4.1}$$

$$\frac{\partial\psi}{\partial\eta} = -\frac{1}{y(\xi,\eta)}\,\frac{\partial\psi}{\partial\xi}\,. \tag{4.2}$$

It can be seen from (4.1), (4.2) that $\varphi(\xi,\eta)$ and $\psi(\xi,\eta)$ are mutually connected, and this fact should always be taken into consideration.

We rewrite the system (4.1), (4.2) as follows:

$$\Delta\varphi(\xi,\eta) + \frac{1}{y}\,\frac{\partial y}{\partial\xi}\,\frac{\partial\varphi}{\partial\xi} + \frac{1}{y}\,\frac{\partial y}{\partial\eta}\,\frac{\partial\varphi}{\partial\eta} = 0, \tag{4.3}$$

$$\Delta\psi(\xi,\eta) - \frac{1}{y}\,\frac{\partial y}{\partial\xi}\,\frac{\partial\psi}{\partial\xi} - \frac{1}{y}\,\frac{\partial y}{\partial\eta}\,\frac{\partial\psi}{\partial\eta} = 0. \tag{4.4}$$

Suppose that we have solved the plane problem, i.e., we have constructed analytic functions mapping conformally the half-plane $\mathrm{Im}(\zeta) \geq 0$ (or $\mathrm{Im}(\zeta) < 0$) of the plane $\zeta = \xi + i\eta$ onto the circular polygon. For general discussion we assume that there is a circular polygon with m vertices. To find an analytic function in the general case, we have to solve a nonlinear third order Schwartz differential equation. Its solution is reduced to the solution of a Fuchs class differential equation. The Schwartz equation, and hence the corresponding Fuchs class equation, contains $2(m-3)$ essential unknown parameters. The general solution of the Schwartz equation involves additionally six parameters of integration. We write the system of higher $2(m-3)$ transcendent equations and also the system of six equations for finding the integration parameters of the Schwartz equation. Next, we construct solutions $\varphi(\xi,\eta)$ and $\psi(\xi,\eta)$ for the system (4.3) and (4.4) with regard for (4.1), (4.2) ([18]–[25]).

Introduce the notation for three analytic functions:

$$\begin{gathered} z(\zeta) = x(\xi,\eta) + iy(\xi,\eta), \quad \omega_0(\zeta) = \varphi_0(\xi,\eta) + i\psi_0(\xi,\eta), \\ w_0(\zeta) = \omega_0'(\zeta)/z'(\zeta), \end{gathered} \tag{4.5}$$

$$\Delta x(\xi,\eta) = 0, \quad \Delta y(\xi,\eta) = 0, \quad \Delta\varphi_0(\xi,\eta) = 0, \quad \Delta\psi_0(\xi,\eta) = 0, \tag{4.6}$$

which map conformally the half-plane $\mathrm{Im}(\zeta) \geq 0$ respectively onto the domain $S(z(\zeta))$ of liquid motion, onto the domain of the complex potential

$\varphi_0(\xi,\eta) + i\psi_0(\xi,\eta) = \omega_0(\zeta)$, and onto the domain of the complex velocity $S(\omega_0'(\zeta)/z'(\zeta))$. The above functions are unknown and to be defined.

Below, we will consider the problem of solvability of the system of equations (4.1), (4.2).

A solution of (4.3), (4.4) will be sought with regard for (4.1) and (4.2) in the form

$$\varphi(\xi,\eta) = \varphi_0(\xi,\eta) + \varphi_1(\xi,\eta), \tag{4.7}$$

$$\psi(\xi,\eta) = \psi_0(\xi,\eta) + \psi_1(\xi,\eta), \tag{4.8}$$

where $\varphi_0(\xi,\eta)$, $\psi_0(\xi,\eta)$ are self-conjugate harmonic functions satisfying all boundary conditions. Substituting (4.7) and (4.8) into (4.3) and (4.4), we obtain

$$\Delta\varphi_1(\xi,\eta) + \frac{1}{y}\frac{\partial y}{\partial \xi}\frac{\partial \varphi_1}{\partial \xi} + \frac{1}{y}\frac{\partial y}{\partial \eta}\frac{\partial \varphi_1}{\partial \eta} = -\left[\Delta\varphi_0 + \frac{1}{y}\frac{\partial y}{\partial \xi}\frac{\partial \varphi_0}{\partial \xi} + \frac{1}{y}\frac{\partial y}{\partial \eta}\frac{\partial \varphi_0}{\partial \eta}\right], \tag{4.9}$$

$$\Delta\psi_1(\xi,\eta) - \frac{1}{y}\frac{\partial y}{\partial \xi}\frac{\partial \psi_1}{\partial \xi} - \frac{1}{y}\frac{\partial y}{\partial \eta}\frac{\partial \psi_1}{\partial \eta} = -\left[\Delta\psi_0 - \frac{1}{y}\frac{\partial y}{\partial \xi}\frac{\partial \psi_0}{\partial \xi} - \frac{1}{y}\frac{\partial y}{\partial \eta}\frac{\partial \psi_0}{\partial \eta}\right]. \tag{4.10}$$

In the right-hand sides of (4.9) and (4.10) we retain $\Delta\varphi_0 = 0$ and $\Delta\psi_0 = 0$ deliberately.

We transform the unknown functions $\varphi_1(\xi,\eta)$, $\psi_1(\xi,\eta)$, $\varphi_0(\xi,\eta)$ and $\psi_0(\xi,\eta)$ as follows:

$$\varphi_1(\xi,\eta) = y^{-1/2}(\xi,\eta)\varphi_2(\xi,\eta), \quad \psi_1 = y^{1/2}(\xi,\eta)\psi_2(\xi,\eta), \tag{4.11}$$

$$\varphi_0(\xi,\eta) = y^{-1/2}(\xi,\eta)\varphi_2^*(\xi,\eta), \quad \psi_0(\xi,\eta) = y^{1/2}(\xi,\eta)\psi_2^*(\xi,\eta). \tag{4.12}$$

After transformation, the system (4.9), (4.10) takes the form

$$\Delta(\varphi_1 + \varphi_2^*) = -\frac{1}{4}\,\rho_1(\varphi_2 + \varphi_2^*), \tag{4.13}$$

$$\Delta(\psi_2 + \psi_2^*) = \frac{3}{4}\,\rho_1(\psi_2 + \psi_2^*), \tag{4.14}$$

where

$$\rho_1 = \left(\frac{1}{y}\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{1}{y}\frac{\partial y}{\partial \eta}\right)^2. \tag{4.15}$$

As is said above, the hydrodynamic problem is assumed to be solved if either of the functions $\varphi(x,y)$ and $\psi(x,y)$ is found with regard for (4.13). Next, on the plane ζ we have to take into consideration (4.1) and (4.2).

Using Green's formula, we can obtain from (4.13) and (4.14) the following Fredholm integral equations of second kind:

$$\varphi_2(\xi,\eta)+\frac{1}{4}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x,y)\rho_1(x,y)\varphi_2(x,y)\,dx\,dy=f_1(\xi,\eta), \tag{4.16}$$

$$\psi_2(\xi,\eta)-\frac{3}{4}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x,y)\rho_1(x,y)\psi_2(x,y)\,dx\,dy=f_2(\xi,\eta), \tag{4.17}$$

where

$$f_1(\xi,\eta)=-\,\varphi_2(\xi,\eta)- \\ -\frac{1}{4}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x,y)\rho_1(x,y)\varphi_2^*(x,y)\,dx\,dy, \tag{4.18}$$

$$f_2(\xi,\eta)=-\,\psi_2^*(\xi,\eta)+ \\ +\frac{3}{4}\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x,y)\rho_1(x,y)\psi_2^*(x,y)\,dx\,dy \tag{4.19}$$

and

$$G(\xi,\eta;x,y)=\frac{1}{4\pi}\,\ln\frac{(\xi-x)^2+(\eta+y)^2}{(\xi-x)^2+(\eta-y)^2}\,.$$

Solutions of the integral equations (4.16) and (4.17) will be sought by using the method of successive approximations in the form of the following series:

$$\varphi_2(\xi,\eta)=\sum_{n=0}^{\infty}\lambda^n\varphi_{2(n)}(\xi,\eta), \tag{4.20}$$

$$\psi_2(\xi,\eta)=\sum_{n=0}^{\infty}\mu^n\psi_{2(n)}(\xi,\eta), \tag{4.21}$$

where $\lambda=\frac{1}{4}$, $\mu=\frac{3}{4}$.

Substituting the series (4.20) and (4.21) respectively into the integral equations (4.16) and (4.17), and then equating the coefficients at the same degrees of the parameters λ and μ, we will obtain

$$\varphi_{2(0)}(\xi,\eta)=f_1(\xi,\eta), \tag{4.22}$$

. . . .

$$\varphi_{2(n)}(\xi,\eta)=\iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x,y)\rho_1(x,y)\varphi_{2(n-1)}(x,y)\,dx\,dy, \tag{4.23}$$

. . . .

$$\psi_{2(0)}(\xi,\eta)=f_2(\xi,\eta), \tag{4.24}$$

. . . .

$$\psi_{2(n)}(\xi,\eta) = \iint\limits_{\mathrm{Im}(\zeta)\geq 0} G(\xi,\eta;x,y)\rho_1(x,y)\psi_{2(n-1)}(x,y)\,dx\,dy, \tag{4.25}$$

. . . .

$n = 1, 2, 3, \dots$.

5. On the Solution of Some Fredholm Integral Equations

Consider the simplest Fredholm integral equation of the second kind [32]

$$u(x) - \lambda \int\limits_a^b K(x,t)u(t)\,dt = f(x), \tag{5.1}$$

where the unknown function $u(x)$ depends on the real variable x which varies in the same interval $[a,b]$ as the integration variable t. This requirement concerns without exception to all classes of integral equations under consideration. The interval $[a,b]$ may be finite or infinite. The functions $K(x,t)$ and $f(x)$ are assumed to be given and defined almost everywhere, respectively, in the square $a \leq x \leq b$, $a \leq t \leq b$ and in the interval $a \leq x \leq b$.

The function $K(x,t)$ is said to be the kernel of the integral equation. The kernel $K(x,t)$ of the Fredholm equation satisfies the inequality

$$\int\limits_a^b \int\limits_a^b |K(x,t)|^2\,dx\,dt < \infty \tag{5.2}$$

and the free term $f(x)$ satisfies the inequality

$$\int\limits_a^b |f(x)|^2\,dx < \infty. \tag{5.3}$$

We consider Fredholm equations of more general type. Let Ω be a measurable set in the space of an arbitrary number of variables, x and t be points of that set, and μ be a nonnegative measure defined on Ω [32].

The equality

$$u(x) - \lambda \int\limits_\Omega K(x,t)u(t)\,d\mu(t) = f(x), \tag{5.4}$$

whose kernel $K(x,t)$ and free terms $f(x)$ satisfy, respectively, the inequalities

$$\int\limits_\Omega \int\limits_\Omega |K(x,t)|^2\,d\mu(x)\,d\mu(t) < \infty, \quad \int\limits_\Omega |f(x)|^2\,d\mu(x) < \infty, \tag{5.5}$$

is also called a Fredholm equation.

The kernel $K(x,t)$ satisfying (5.5) is called the Fredholm kernel. We denote the volume element by dx and the integral (5.5) by B_K^2:

$$\int\limits_{\Omega}\int\limits_{\Omega} |K(x,t)|^2\,dx\,dt = B_K^2. \tag{5.6}$$

The unknown function $u(x)$ is quadratically summable in (a,b), and, consequently, belongs to the space $L_2(a,b)$. A solution of the equation (5.4) belongs to the space $L_2(\mu,\Omega)$ of functions which are quadratically summable in Ω in measure μ. The inequalities (5.3) and (5.5) mean that the free term of the equation belongs to the same space. The parameter λ may take both real and complex values.

The parameters λ and μ of the integral equations (4.16) and (4.17) are less than unity, hence the convergence of the series (4.20) and (4.21) is guaranteed.

As is known, the Fredholm equation of the second kind has either finite, or countable set of characteristic numbers. But there are kernels having no characteristic numbers at all, as, for example, Volterra kernels. A complete characteristic of such kernels is given in the following Lalesko theorem. Let $K(x,t)$ be a Fredholm kernel, and $K_n(x,t)$ be its iterated kernel. For the kernel $K(x,t)$ to have no characteristic numbers, it is necessary and sufficient that

$$A_n = \int\limits_{\Omega} K_n(x,t)\,dx = 0, \quad n = 3,4,\dots\ , \tag{5.7}$$

where the numbers A_n are called traces of the kernel $K(x,t)$. Lalesko has proved his theorem for the case of bounded kernels, while a general proof has been given by S. Krachkovskiĭ (see [31], [32]).

The Fredholm determinant and minors are represented as quotients of two entire functions of λ, the poles of the resolvent, i.e., the characteristic numbers of the kernel $K(x,t)$, not depending on x and t. Thus the resolvent should have the form

$$R(x,t;\lambda) = D(x,t;\lambda)/D(\lambda), \tag{5.8}$$

where $D(x,t;\lambda)$ and $D(\lambda)$ are entire functions of λ ([31], [32]).

For the numerator and denominator of the fraction in (5.8) we give the representations in the form of the following series ([31], [32]):

$$D(x,t;\lambda) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\,B_n(x,t)\lambda^n, \quad D(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\,c_n\lambda^n, \tag{5.9}$$

where

$$c_0 = 1, \quad B_0(x,t) = K(x,t), \quad c_n = \int\limits_{\Omega} B_{n-1}(x,x)\,dx, \ \ n > 0, \tag{5.10}$$

$$B_n(x,t) = c_n - n \int\limits_{\Omega} K(x,t) B_{n-1}(\tau,t)\,d\tau, \tag{5.11}$$

which makes it possible to calculate the coefficients $B_n(x,t)$ and c_n recursively.

Below we will need the well-known formula ([31], [32])

$$D'(\lambda)/D(\lambda) = -\sum_{n=1}^{\infty} A_n \lambda^{n-1}, \tag{5.12}$$

where

$$A_n = \int\limits_{\Omega} K_n(x,x)\,dx, \quad n = 1, 2, 3, \dots\ , \tag{5.13}$$

are the above-mentioned traces of the kernel $K(x,t)$.

If the kernel $K(x,t)$ is noncontinuous and, more so, has discontinuities of the second kind, then the integrals in (5.10) defining the coefficients $c_1, c_2, \dots$ become meaningless.

The Fredholm kernel may sometimes have Green's function $G(P,Q)$ as a multiplier. As is known, this function is defined as a harmonic function symmetric with respect to P and Q, equal on the boundary to zero, and analytic at all points P of the domain D, except for the points $P = Q$ at which it has logarithmic singularity.

The kernel $K(x,t)$ may have logarithmic singularity. Then the integral

$$\int\limits_{\Omega} K(x,x)\,dx \tag{5.14}$$

defining the coefficient c_1 becomes meaningless. This difficulty can be overcome successfully by putting, for example, $c_1 = 0$ ([31], [32]).

The iterated kernel $K_2(s,t)$ has the form

$$K_2(s,t) = \int\limits_{\Omega} K(s,t_1) K(t_1,t)\,dt_1. \tag{5.15}$$

The integral $K_2(s,t)$ is meaningful for any positions of s and t in $[a,b]$ because in the most unfavorable case, when s and t coincide, the integrand admits the following estimate ([**?**, 27]-[30]):

$$|K(s.t_1) K(t_1,t)| \le \frac{M_1}{|s - t_1|^{\varepsilon_1}}, \quad \varepsilon_1 > 0. \tag{5.16}$$

It is proved that $K_2(s,t)$ is a function, continuous in the square $a \le x \le b$, $a \le t_1 \le b$, and the functions

$$K_n(s,t) = \int\limits_{\Omega} K(s,t_1)K_{n-1}(t_1,t)\,dt_1, \quad n = 1,2,3,\dots, \tag{5.17}$$

are estimated analogously:

$$|K(s,t_1)K_{n-1}(t_1,t)| \le \frac{M_{n-1}}{|s-t_1|^{\varepsilon_{n-1}}}, \quad \varepsilon_{n-1} > 0. \tag{5.18}$$

The integral $K_n(s,t)$, $n = 1,2,\dots$, is meaningful for any positions of s and t in $[a,b]$, and the estimates of the integrands have the form (5.18). Consequently, we have to put

$$K_n(s,s) = 0, \quad n = 1,2,\dots, \tag{5.19}$$

$$A_n = \int\limits_{\Omega} K_n(s,s)\,ds = 0, \quad n = 1,2,\dots. \tag{5.20}$$

Then

$$c_n = 0, \quad n = 1,2,\dots,n,\dots. \tag{5.21}$$

Taking into account (5.19), from (5.12) we get

$$D'(\lambda) = 0, \tag{5.22}$$

and in its turn, from (5.22) it follows that

$$D(\lambda) = 1. \tag{5.23}$$

Consequently, the kernel of the integral equation (5.4) has no characteristic numbers. In a complete analogy we can prove that the kernel of the integral equation (3.36), considered by us in [24], has no characteristic numbers.

6. Spatial Axisymmetric Jet Flows with Partially Unknown Boundaries

Below, the use will frequently be made of the works [3], [6]. Let us consider the stationary axisymmetric flow of an ideal, weightless, incompressible liquid. Let the x-axis coincide with the symmetry axis. The velocity potential $\varphi(x,y)$ and the flow function $\psi(x,y)$ are the functions of only cylindrical coordinates x and y, where y is the distance to the axis x. Owing to the axial symmetry, it suffices to study the flow in an arbitrarily chosen meridional half-plane with the coordinate system x,y ([1]–[6]). By $w(x,y) = \varphi(x,y) + i\psi(x,y)$ we denote the complex potential, and by $z = x + iy$ the complex coordinate. As is known, these functions should satisfy the conditions (1.2) and (1.3).

In Figure 1 we can see one half of the meridional plane $x0y$ for the problem of flow round a circular cone in a circular tube. Since the flow function $\psi(x, y)$ is defined to within a constant summand, we can put $\psi(x, y) = 0$ along the symmetry axis x both on the cone and on the free surface. But the difference between the values of ψ on the flow surfaces is equal to the liquid discharge between these surfaces divided by 2π, and hence on the tube walls $\psi = \pi \mathbf{v}_\infty h^2/(2\pi)$, where h is the tube radius, and $\mathbf{v}_\infty$ is velocity at infinity coming from the left [3].

The form of the free surfaces is unknown, but the supplementary condition for steady pressure is given. This condition can be written in the form (2.3), where v_0 is equal to v on the free surface [3].

To solve the problem, we map conformally the domains of variation of $\frac{dw}{(v_0 dz)}$ and w onto the semi-circle of unit radius (Figure 2) of the parametric variable t ($|t| \le 1$, $\mathrm{Im}(t) \ge 0$), where $t = \xi + i\eta$. Having chosen arbitrarily three points on the mapped contour according to the Riemann theorem, we assume that to the singular points a_1, a_2, a_3, a_4, a_5 there correspond the points $\xi = a_1 = 0$, $a_2 = 1$, $a_3 = -1$, $a_4 = -h_0$ and $a_5 = -\frac{1}{h_0}$, where a_4 and a_5 are the source, and a_3 is the sink. The complex potential can be written either as

$$w(t) = \frac{q}{\pi} \ln \Big\{ \big[(t - a_4)(t - 1/a_4)\big] / (t - a_3)^2 \Big\}, \tag{6.1}$$

or as

$$w(t) = \frac{q}{\pi} \ln \Big\{ \big[(\xi - a_4) + i\eta\big] \big[(\xi - 1/a_4) + i\eta\big] / \big[(\xi - a_3) + i\eta\big]^2 \Big\}. \tag{6.2}$$

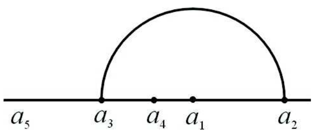

FIGURE 2

In the hodograph domain, the filtration velocity $dw/(v_0 dx)$ does get equal to infinity and it vanishes only at the point $\xi = a_1$.

Analyzing the behavior of the function $dw/(v_0 dz)$ [3], we obtain

$$\frac{dw}{(v_0 dx)} = t^\mu, \quad t > 0. \tag{6.3}$$

We can easily see that the formula (6.3) is valid. Inside the upper half of the semi-circle $|t| \le 1$, the function t^μ is holomorphic. On the circumference we have the equality $|t|^\mu = 1$. On the real axis $0 < t \le 1$, the function t^μ takes real positive values. Moving in the upper half-plane t around the point $\xi = a_1$ counterclockwise, we can see that the argument t^μ on OA

$(-1 \le t \le 0)$ is equal to $\pi\mu$, that is, the boundary conditions are fulfilled everywhere.

To see that the formula (6.2) is valid, it suffices to verify that the boundary conditions are fulfilled. Suppose that $\eta = 0$. Then

$$\begin{aligned} w(t) &= \frac{q}{\pi} \ln\Big\{\big[(\xi - a_4)(\xi - 1/a_4)\big]\big/(\xi - a_3)^2\Big\} = \\ &= \frac{q}{\pi} \ln\Big\{\big[(\xi + h_0)(\xi + 1/h_0)\big]\big/(\xi + 1)^2\Big\}. \end{aligned} \tag{6.4}$$

It follows from (6.4) that $\psi = q$. Since the expression $(t - a_4)(t - 1/a_4)$ in the interval $(\frac{1}{a_4}, a_4)$ is negative, we have $\operatorname{Im} w(t) = q$, $a_3 < \xi < a_3$, and it is positive in the intervals $t < \frac{1}{a_4}$, $t > a_4$. This implies that in the interval $a_4 < \xi < a_2$

$$\operatorname{Im} w(t) = 0, \quad a_4 < \xi < a_2. \tag{6.5}$$

Assuming on the arc $t = e^{i\alpha}$, we obtain

$$\operatorname{Im} w(t) =$$

$$= \frac{q}{\pi} \operatorname{Im} \ln \Big[(e^{i\alpha} - a_4)(e^{-i\alpha} - a_4)\Big(\frac{1}{-a_4}\Big)\Big]\Big/(e^{-\alpha/2} + e^{-i\alpha/2})^2 = 0. \tag{6.6}$$

Now find the velocity v_{a_4} of the flow in the vessel at infinity:

$$\frac{v_{a_4}}{v_0} = \Big(\frac{dw}{v_0 dx}\Big)_{a_4} = h_0^{\mu}. \tag{6.7}$$

Thus the value h_0 defines the velocity in the vessel at infinity. Obviously, $q = hv_{a_4}$, whence according to (6.7) we obtain

$$q = h \cdot v_0 h_0^{\mu}. \tag{6.7$_1$}$$

From (6.4) and (6.3) we can find $z(t)$. Thus we have

$$z(t) = \frac{e^{i\pi\mu}}{v_0} \int_0^t t^{-\mu} w'(t)\, dt. \tag{6.8}$$

When $t \to a_2$, the equality (6.8) allows us to obtain the formulas

$$z(a_2) = \frac{e^{i\pi\mu}}{v_0} \int_0^{a_2} t^{-\mu} w'(t)\, dt, \quad a_2 = 1. \tag{6.9}$$

When $t < 0$, we define $z(t)$ by the formula

$$z(t) = -\frac{1}{v_0} \int_t^0 (-t)^{-\mu} w'(t)\, dt, \quad t < 0. \tag{6.10}$$

It follows from (6.9) that

$$x(a_2) = \cos(\pi\mu)\,\frac{1}{v_0}\int\limits_0^{a_2} t^{-\mu} w'(t)\,dt, \tag{6.11}$$

$$y(a_2) = \sin(\pi\mu)\,\frac{1}{v_0}\int\limits_0^{a_2} t^{-\mu} w'(t)\,dt, \tag{6.12}$$

$$\sqrt{[x(a_2)]^2 + [y(a_2)]^2} = \frac{1}{v_0}\int\limits_0^{a_2} t^{-\mu} w'(t)\,dt. \tag{6.13}$$

Using the formula (6.10) and moving around the singular point $\zeta = a_4$ on an infinitesimal semi-circumference K with center $t = a_4$, we obtain

$$h = \frac{q}{v_0}\,h_0^{-\mu}, \quad q = hv_0h_0^{\mu}, \tag{6.14}$$

where $h_0 = -a_4$, h is the radius of the cylinder.

The formula (6.14) coincides with (6.7_1).

In calculating the integral (6.10), when integration involves the singular point $\xi = -a_4 = h_0$, we have to apply the principal value of the Cauchy type integral, while when moving around the point $t = a_3 = -1$, we act as follows:

$$\left(\frac{dw}{(v_0dx)}\right)_{a_3} = (+1)^{\mu} = 1. \tag{6.15}$$

At infinity and at the point $a_3 = -1$, the direction of the jets coincides with that of the x-axis.

Next, our main task is to obtain a complete exact solution of the plane problem by means of analytic functions which should be used to obtain a complete solution of the corresponding axisymmetric problem.

Using the functions

$$\ln t = 2i\arccos\frac{1}{\zeta}, \quad \zeta^2 > 1; \quad \ln t = -2i\ln\left|\frac{1+\sqrt{1-\zeta^2}}{\zeta}\right|, \quad \zeta < 1, \tag{6.16}$$

we map conformally the half-plane $\mathrm{Im}(\zeta) \geq 0$ of the auxiliary plane $\zeta = \xi + i\eta$ (Figure 4) onto a triangle (Figure 3), and then, using the function $\ln t$, we map conformally the triangle of the type as in Figure 3 onto the upper semi-circle of unit radius ($|t| \leq 1$, $\mathrm{Im}(t) > 0$).

Thus the functions (6.1), (6.2) and (6.6) are defined in that domain, so we have obtained the solution of the plane problem on a liquid flowing out of a skew-walled vessel (Figure 1). Using the above-obtained functions, we pass to the solution of the spatial problem of flow around the circular cone in the tube. Using the functions (6.1), (6.2) and (6.3), we assume

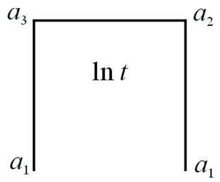

FIGURE 3

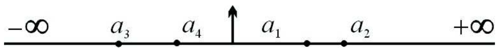

FIGURE 4

that the functions $\varphi_0(\xi,\eta)$, $\psi_0(\xi,\eta)$ are the first approximations of the unknown functions $\varphi(\xi,\eta)$, $\psi(\xi,\eta)$. The functions $\varphi_0(\xi,\eta)$, $\psi_0(\xi,\eta)$, $x(\xi,\eta)$ and $y(\xi,\eta)$ should satisfy all boundary conditions. Thus the above-defined functions $\varphi_0(\xi,\eta)$, $\psi_0(\xi,\eta)$, $x(\xi,\eta)$ and $y(\xi,\eta)$ are pairwise self-conjugate harmonic once. Note that the conditions of compatibility (4.1), (4.2) should be taken into account. The hydrodynamic problem is assumed to be solved if either of the functions $\varphi(x,y)$ and $\psi(x,y)$ is known.

Finally, we proceed to finding the functions $\varphi_2(\xi,\eta)$, $\psi_2(\xi,\eta)$. When solving the integral equation (4.16) or (4.17), we use the method of successive approximations and the fact that the right-hand sides of (4.16) and (4.17) involve the known functions. On the symmetry axis x of the cone and on the free surface we put $\psi = 0$. But the difference between the values of ψ on the flow surfaces is equal to 2π, hence on the tube walls $\psi = \pi v_\infty h^2/(2\pi)$, where h is the tube radius, v_∞ is velocity at infinity of the flow coming from the left.

As is said above, the form of free surfaces is unknown, but there is a complementary condition of constancy of the velocity modulus v which is equivalent to the condition of pressure constancy. This condition can be written in the form (2.3), where v_0 is equal to v on the free surface [3].

7. The Problem on the Ground Water Influx to A Spatial Axisymmetric Basin with Trapezoidal Axial Cross-Section

Under a water permeable ground layer is laid a ground layer of greater (theoretically infinite) water permeability, the pressure on the upper horizontal surface of the lower layer being constant. The depth of the water in the basin is neglected; if water is deep, the solution of the problem becomes more complicated. The basin is given in Figure 5.

In solving this spatial axisymmetric problem the use will be made of the solution of the corresponding plane problem. The plane axisymmetric

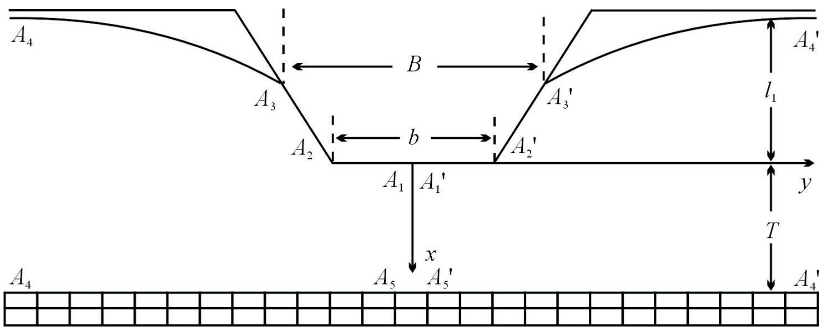

FIGURE 5

problem on the ground water influx to a drainage ditch with trapezoidal cross-section has been solved by V. V. Vedernikov [33], and his investigation was complemented by Yu. D. Sokolov [34]. Here we generalize the problem solved by V. V. Vedernikov [33]. Our generalization consists in the following: under the water permeable ground layer we lay ground layer of greater (theoretically, infinite) water permeability, and the pressure on the upper horizontal surface of the layer is constant. In its turn, we generalize this generalized plane problem to the spatial axisymmetric problem.

We direct the x-axis vertically downwards along the symmetry axis, and the y-axis we direct horizontally; here y is the distance to the x-axis.

Along the whole contour of the domain of liquid motion we have the conditions $\varphi - kx = 0$ and $\varphi - ky = T$. Hence on the Zhukovski's plane we have a strip of length T. To solve the problem under consideration, it is convenient to use Zhukovski's function

$$\theta = \theta_1 + i\theta_2, \quad \theta_1 = \varphi - kx, \quad \theta_2 = \psi - ky. \tag{7.1}$$

The boundaries of the velocity hodograph consists of a circumference arc and straight lines which intersect each other at one point F, where $u = -u_0$, $v_0 = 0$.

The plane case under consideration is represented schematically in Figure 5. Note that

$$\begin{aligned} &\theta = \omega(z) - kz; \quad \omega(z) = \varphi(x,y) + i\psi(x,y), \quad z = x + iy, \\ &\frac{d\theta}{dz} = w - k. \end{aligned} \tag{7.2}$$

The use will be made of the formula

$$u = \frac{dz}{d\theta} = \frac{1}{w-k}\,, \tag{7.3}$$

which corresponds to that function whose domain is obtained after inversion (see Figure 5). We transfer the vertices of this polygonal domain to the

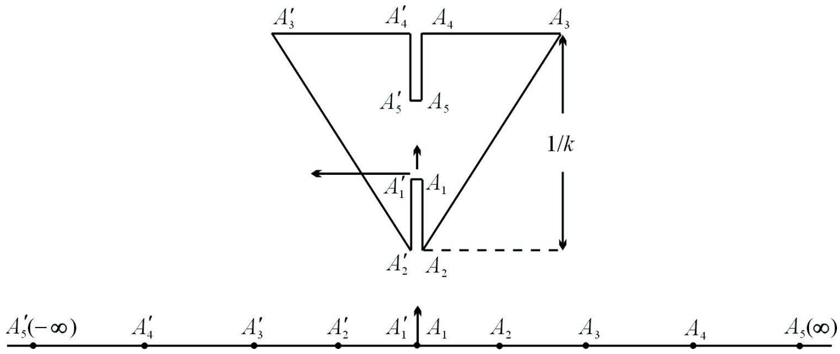

FIGURE 6

points of the plane ζ as in Figure 6, and obtain

$$u(\zeta)=M\int\limits_{a_4}^{\zeta}\zeta(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\,d\zeta+u(a_4), \tag{7.4}$$

where

$$u(a_4)=1/k,\quad M\ \text{ is a real number.} \tag{7.5}$$

From (7.4), it follows that

$$u(a_5)=M\int\limits_{a_4}^{a_5(+\infty)}\zeta(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\,d\zeta+u(a_4), \tag{7.6}$$

where

$$u(a_5)=u_5\ \text{ is a real number.} \tag{7.7}$$

$$u_5=M\int\limits_{a_4}^{a_5(+\infty)}\zeta(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\,d\zeta+\frac{1}{k}, \tag{7.8}$$

$$u(\zeta)=-Mi\int\limits_{a_3}^{\zeta}(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\,d\zeta+u(a_3), \tag{7.9}$$

where

$$u(a_3)=-\frac{1}{k}\ \mathrm{tg}(\pi\alpha)+\frac{i}{k}\,. \tag{7.10}$$

$$u(a_4)=-Mi\int\limits_{a_3}^{a_4}\zeta(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\,d\zeta+u(a_3), \tag{7.11}$$

where $u(a_4)=\frac{1}{k}$.

From (7.11), we have

$$M\int_{a_4}^{a_4}\zeta(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\,d\zeta+\frac{1}{k}\ \mathrm{tg}\,\pi\alpha=0, \quad (7.12)$$

$$u(\zeta)=(-1)Me^{-i\pi\alpha}\times$$
$$\times\int_{a_2}^{\zeta}\zeta(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\,d\zeta+u(a_2), \quad (7.13)$$

where $u(a_2)=0$,

$$u(a_3)=(-1)Me^{-i\pi\alpha}\int_{a_2}^{a_3}\zeta(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\,d\zeta, \quad (7.14)$$

$$u(a_3)=\frac{-i}{k}\ \mathrm{tg}\,\pi\alpha+\frac{1}{k}\,,\quad \frac{1}{k}-M\cos\pi\alpha\times$$
$$\times\int_{a_2}^{a_3}\zeta(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\ d\zeta=0, \quad (7.15)$$

$$u(\zeta)=M\int_{a_1}^{\zeta}\zeta(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\,d\zeta+u(a_1), \quad (7.16)$$

$$M\int_{a_1}^{a_2}\zeta(\zeta^2-a_2^2)^{\alpha-1}(\zeta^2-a_3^2)^{-\frac{1}{2}-\alpha}(\zeta^2-a_4^2)^{-\frac{1}{2}}\,d\zeta-\frac{1}{u_0+k}=0, \quad (7.17)$$

$$u(a_2)=\frac{-1}{u_0+k}\,,\quad u(a_2)=0. \quad (7.18)$$

Of the parameters a_2, a_3 and a_4, we fix one as $a_2=1$, and the parameters u_0, a_3, a_4, M are to be defined by means of the system of equations (7.6), (7.8), (7.12), (7.15) and (7.17).

We now define Zhukovski's function. We have

$$\theta(\zeta)=\frac{T}{\pi}\ \ln\left(\frac{\zeta-a_4}{\zeta+a_4}\right)+T. \quad (7.19)$$

For finding the function $z(\theta)$, we use the following formulas:

$$z(\zeta)=\int_{a_4}^{\zeta}u(\zeta)\theta'(\zeta)\,d\zeta+z(a_4),\quad z(a_5)=\int_{a_4}^{a_5}u(\zeta)\theta'(\zeta)\,d\zeta+z(a_4), \quad (7.20)$$

$$z(\zeta) = \int\limits_{a_3}^{\zeta} u(\zeta)\theta'(\zeta)\,d\zeta + z(a_3), \quad z(a_4) = \int\limits_{a_3}^{a_4} u(\zeta)\theta'(\zeta)\,d\zeta + z(a_3), \qquad (7.21)$$

$$z(\zeta) = \int\limits_{a_2}^{\zeta} u(\zeta)\theta'(\zeta)\,d\zeta + z(a_2), \quad z(a_3) = \int\limits_{a_2}^{a_3} u(\zeta)\theta'(\zeta)\,d\zeta + z(a_2), \qquad (7.22)$$

$$z(\zeta) = \int\limits_{a_1}^{\zeta} u(\zeta)\theta'(\zeta)\,d\zeta + z(a_1), \quad z(a_2) = \int\limits_{a_1}^{a_2} u(\zeta)\theta'(\zeta)\,d\zeta + z(a_1). \qquad (7.23)$$

The system (7.20)–(7.23) allows us to define the coordinates of the leaking interval, and then using the function $\theta(\zeta)$, we find parametric equations of depression curves. In solving the problem (Figure 5) we have considered two symmetric half-planes. Owing to the symmetry, we could have considered arbitrarily one half of the two half-planes. But because of the fact that on the boundary of the hodograph velocity, along the symmetry axis, we have two cuts to the ends of which there correspond two unknown parameters, for their determination we have to write two equations. Determination of another unknown parameters needs another equations, and this exactly has been done in the present work.

References

1. N. E. Kochin, I. A. Kibel', and N. V. Roze, Theoretical hydromechanics. (Translated from the Russian) *Interscience Publishers John Wiley & Sons, Inc. New York–London–Sydney,* 1964; Russian original: *Moscow,* 1955.
2. P. Ya. Polubarinova-Kochina, The theory of underground water motion. 2nd ed. (Russian) *Moscow, Nauka*, 1977.
3. M. I. Gurevich, The theory of jets in an ideal fluid. (Russian) *Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow*, 1961; English transl.: *International Series of Monographs in Pure and Applied Mathematics,* Vol. 93. *Pergamon Press, Oxford–New York–Toronto, Ont.*, 1966.
4. M. A. Lavrent'ev and B. V. Shabat, Methods of the theory of functions of a complex variable. (Russian) *Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow*, 1958.
5. G. Birkhoff, Hydrodynamics: A study in logic, fact and similitude. *Princeton Univ. Press, Princeton, N.J.*, 1960; Russian transl.: *Izdat. Inostr. Lit., Moscow,* 1963.
6. G. Birkhoff and E. H. Zarantonello, Jets, wakes, and cavities. *Academic Press Inc., Publishers, New York,* 1957; Russian transl.: Mir, Moscow, 1964.
7. A. V. Bitsadze, Equations of mathematical physics. *Nauka, Moscow,* 1982.
8. I. N. Vekua, Generalized analytic functions. (Russian) *Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow*, 1959.
9. I. N. Vekua, New methods for solving elliptic equations. (Russian) *OGIZ, Moscow–Leningrad,* 1948.
10. P. Ya. Polubarinova-Kochina, Circular polygons in filtration theory. (Russian) *Problems of mathematics and mechanics*, 166–177, *"Nauka", Sibirsk. Otdel., Novosibirsk*, 1983.

11. P. Ya. Polubarinova-Kochina, Analytic theory of linear differential equations in the theory of filtration. Mathematics and problems of water handling facilities. *Collection of scientific papers,* 19–36. *Naukova Dumka, Kiev,* 1986.
12. Ya. Bear, D. Zaslavskii, and S. Irmey, Physical and mathematical foundations of water filtration. (Translated from English) *Mir, Moscow*, 1971.
13. E. L. Ince, Ordinary Differential Equations. *Dover Publications, New York,* 1944. Russian transl.: *ONTI, State Scientific Technical Publishing House of Ukrtaine, Kharkov,* 1939.
14. A. Hurwitz and R. Courant, Theory of functions. (Translation from German) *Nauka, Moscow*, 1968.
15. V. V. Golubev, Lectures in analytical theory of differential equations. 2nd ed. (Russian) *Gostekhizdat, Moscow-Leningrad*, 1950.
16. E. A. Coddington and N. Levinson, Theory of ordinary differential equations. *McGraw-Hill Book Company, Inc., New York-Toronto-London,* 1955.
17. W. von Koppenfels and F. Stallmann, Praxis der konformen Abbildung. *Die Grundlehren der mathematischen Wissenschaften,* Bd. 100, *Springer-Verlag, Berlin-Göttingen-Heidelberg,* 1959; Russian transl.: *Izd. Inostr. Lit., Moscow,* 1963.
18. A. P. Tsitskishvili, Conformal mapping of a half-plane on circular polygons. (Russian) *Trudy Tbiliss. Univ. Mat. Mekh. Astronom.* **185**(1977), 65–89.
19. A. P. Tsitskishvili, On the conformal mapping of a half-plane onto circular polygons with a cut. (Russian) *Differentsial'nye Uravneniya* **12**(1976), No. 1, 2044–2051.
20. A. Tsitskishvili, Solution of the Schwarz differential equations. *Mem. Differential Equations Math. Phys.* **11**(1997), 129–156.
21. A. R. Tsitskishvili, Construction of analytic functions that conformally map a half plane onto circular polygons. (Russian) *Differentsial'nye Uravneniya* **21**(1985), No. 4, 646–656.
22. A. Tsitskishvili, Connection between solutions of the Schwarz nonlinear differential equation and those of the plane problems of filtration. *Mem. Differential Equations Math. Phys.* **28**(2003), 107–135.
23. A. Tsitskishvili, Solution of Spatial Axially Symmetric Problems Of The Theory Of Filtration With Partially Unknown Boundaries. *Mem. Differential Equations Math. Phys.* **39**(2006), 105–140.
24. A. Tsitskishvili, The exact mathematical method of solution of spatial axisymmetric problems of the theory of filtration with partially unknown boundaries, and its application to the hole hydraulics. *Proc. A. Razmadze Math. Inst.* **142**(2006), 67–108.
25. A. Tsitskishvili, The exact solution with partially unknown boundaries. *Mem. Differential Equations Math. Phys.* **42**(2006), 93–125.
26. G. M. Položiĭ, The theory and application of p-analytic and and $(p,\, q)$-analytic functions. Generalization of the theory of analytic functions of a complex variable. (Russian) *Second edition, revised and augmented. Izdat. "Naukova Dumka", Kiev*, 1973.
27. V. I. Smirnov, The course of higher mathematics. T. II. 11th edition. *Gos. Izdat. Tekhniko-Teoretich. Lit. Moscow–Leningrad*, 1952.
28. V. I. Smirnov, The course of higher mathematics. T. III, Part 2, 5th edition. *Gos. Izdat. Tekhniko-Teoretich. Lit. Moscow–Leningrad*, 1952.
29. V. I. Smirnov, The course of higher mathematics. T. IV, 2nd edition. *Gos. Izdat. Tekhniko-Teoretich. Lit. Moscow–Leningrad* 1952.
30. A. N. Tikhonov and A. A. Samarskiĭ, The equations of mathematical physics. (Russian) 2d ed. *Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow*, 1953.

31. S. G. Mikhlin, Integral Equations and their Applications to some Problems of Mechanics, Mathematical Physics and Engineering. (Russian) 2d ed. *Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad*, 1949.
32. P. P. Zabreiko, A. I. Koshelev, M. A. Krasnoselski, S. G. Mikhlin, L. S. Rakovshchik, and V. Ja. Stetsenko, Integral equations. *"Nauka" Publ. House, Glav. Redak. Fiz.-Mat. Lit. Moscow*, 1968.
33. V. V. Vedernikov, The theory of filtration and its application in irrigation and dreinage. (Russian). Gosstrojizdat, 1939.
34. Yu. D. Sokolov, On the flow of ground water into a drainage ditch of trapezoidal section. (Russian) *Akad. Nauk SSSR. Prikl. Mat. Meh.* **15**(1951), 683–688.
35. J. Happel and H. Brenner, Low Reynolds number hydrodynamics with special applications to particulate media. *Prentice-Hall, Inc., Englewood Cliffs, N.J.,* 1965; Russian transl.: *Mir, Moscow,* 1976.
36. H. Lamb, Hydrodynamics. *Cambridge University Press, Cambridge,* 1932; Russian transl.: *Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad*, 1947.
37. V. I. Aravin and S. N. Numerov, The theory of liquid and gas motion in the nondeformable porous medium. (Russian). Gos. Izdar. Tech.-Teor. Lit. Moscow, 1953.

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