# THE DYNAMICAL CONTACT PROBLEM FOR A HALF-PLANE WITH AN ELASTIC COVER PLATE 

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#### Abstract

The dynamical contact problem for a half-plane reinforced along its boundary by an elastic finite cover plate of small thickness and loaded with horizontal and vertical harmonic forces is considered. To find unknown contact stresses, the problem is reduced to the solution of a system of integro-differential equations. Using the methods of the theory of analytic functions and integral transformations, the system is solved explicitly. The case in which the cover plate is under the action of only normal harmonic load is considered in detail.












A great deal of works are devoted to the investigation of statical contact problems for various domains reinforced by an elastic fastenings or inclusions in the form of cover plates of small thickness. A sufficiently complete bibliography dealing with these questions is contained in monographs [1-2]. The dynamical contact problems for bodies with thin cover plates are considered in [3-6].

In the present work we consider the dynamical contact problem for a half-plane which is reinforced along its boundary by an elastic cover plate of small thickness and excited by harmonic forces.

1. First of all, we consider an auxiliary problem on stationary oscillations of an elastic half-plane whose boundary is under the action of concentrated at the origin unitary horizontal and vertical forces of frequency $\omega$.
[^0]Mathematically, the problem is formulated in the form of the differential Lame equations (with perturbation)

$$
\begin{align*}
& (\lambda+\mu) \frac{\partial \theta}{\partial x}+\mu \Delta u=\rho \frac{\partial^{2} u}{\partial t^{2}}+\rho \varepsilon \frac{\partial u}{\partial t} \\
& (\lambda+\mu) \frac{\partial \theta}{\partial y}+\mu \Delta v=\rho \frac{\partial^{2} v}{\partial t^{2}}+\rho \varepsilon \frac{\partial v}{\partial t} \tag{1.1}
\end{align*}
$$

under the boundary conditions

$$
\begin{align*}
& \sigma_{y}=\left.\left(\lambda \theta+2 \mu \frac{\partial v}{\partial y}\right)\right|_{y=0}=\delta(x) e^{-i \omega t} \\
& \tau_{x y}=\left.\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right|_{y=0}=-\delta(x) e^{-i \omega t} \tag{1.2}
\end{align*}
$$

where $\delta(x)$ is the Dirac function, $u(x, y, t)$ and $v(x, y, t)$ are the projections of the displacement vector onto the coordinate axes, $\lambda$ and $\mu$ are the Lame parameters, $\rho$ is density of the material, and $\varepsilon$ is an arbitrarily small positive number.

Considering stationary oscillations of the elastic half-plate and assuming that

$$
u(x, y, t)=u_{\varepsilon}(x, y) e^{-i \omega t}, \quad v(x, y, t)=v_{\varepsilon}(x, y) e^{-i \omega t}
$$

we obtain the following boundary problem:

$$
\begin{gather*}
\left(\Delta+p_{1}^{2}\right) u_{\varepsilon}+\left(\frac{c_{1}^{2}}{c_{2}^{2}}-1\right) \frac{\partial \theta_{\varepsilon}}{\partial x}=0,  \tag{1.3}\\
\left(\lambda+p_{2}^{2}\right) v_{\varepsilon}+\left(\frac{c_{1}^{2}}{c_{2}^{2}}-1\right) \frac{\partial \theta_{\varepsilon}}{\partial y}=0  \tag{1.4}\\
\left.\left(\lambda \theta_{\varepsilon}+2 \mu \frac{\partial v_{\varepsilon}}{\partial y}\right)\right|_{y=0}=\delta(x),\left.\quad \mu\left(\frac{\partial u_{\varepsilon}}{\partial y}+\frac{\partial v_{\varepsilon}}{\partial x}\right)\right|_{y=0}=-\delta(x)
\end{gather*}
$$

where $\theta_{\varepsilon}(x, y)=\frac{\partial u_{\varepsilon}}{\partial x}+\frac{\partial v_{\varepsilon}}{\partial y}, c_{1}=(\lambda+2 \mu)^{1 / 2} \rho^{-1 / 2}$ is velocity of propagation of extension waves, $c_{2}=\mu^{1 / 2} \rho^{-1 / 2}$ is velocity of propagation of distortion waves, $p_{1}^{2}=\frac{\omega^{2}}{c_{1}^{2}}+\frac{i \omega \varepsilon}{c_{1}^{2}}, p_{2}^{2}=\frac{\omega^{2}}{c_{2}^{2}}+\frac{i \omega \varepsilon}{c_{2}^{2}}\left(p_{1}=k_{1}+i k_{1}^{\prime}, p_{2}=k_{2}+i k_{2}^{\prime}, k_{1}>0\right.$, $\left.k_{1}^{\prime}>0, k_{2}>0, k_{2}^{\prime}>0\right)$.

To solve the boundary value problem $(3,4)$, we use the method of the Fourier integral transformations with respect to the variable $x$ and find those solutions $u_{\varepsilon}(x, y)$ and $v_{\varepsilon}(x, y)$ of the above-mentioned problem which vanish as $x^{2}+y^{2} \rightarrow \infty$. As a result, we obtain a system of ordinary differential equations under the above boundary conditions. Its general solution has the form

$$
\begin{align*}
u_{\varepsilon}^{*}(\alpha, y) & =i \alpha A e^{-\gamma_{1} y}-\gamma_{2} B e^{-\gamma_{2} y}+C e^{\gamma_{1} y}+D e^{\gamma_{2} y} \\
v_{\varepsilon}^{*}(\alpha, y) & =\gamma_{1} A e^{-\gamma_{1} y}+i \alpha B e^{-\gamma_{2} y}+E e^{\gamma_{1} y}+F e^{\gamma_{2} y} \tag{1.5}
\end{align*}
$$

where

$$
\begin{gathered}
u_{\varepsilon}^{*}(\alpha, y)=\int_{-\infty}^{\infty} u_{\varepsilon}(x, y) e^{i \alpha x} d x, \quad v_{\varepsilon}^{*}(\alpha, y)=\int_{-\infty}^{\infty} v_{\varepsilon}(x, y) e^{i \alpha x} d x \\
\gamma_{1}=\sqrt{\alpha^{2}-p_{1}^{2}}, \quad \gamma_{2}=\sqrt{\alpha^{2}-p_{2}^{2}}, \quad \alpha=\sigma+i \tau
\end{gathered}
$$

$A, B, C, D, R, E, F$ are the unknown constants.
It is obvious that $\alpha= \pm p_{1}, \alpha= \pm p_{2}$ are the points of branching of the function $\gamma_{1}(\alpha), \gamma_{2}(\alpha)$. To choose a single-valued analytic branch of that function, we make in a complex plane a cut connecting the points $p_{1}$ and $p_{2}$ with a point at infinity in the upper half-plane, and the points $-p_{1}$ and $-p_{2}$ in the lower half-plane. In the plane, cut as is indicated above, we have the equalities

$$
\gamma_{1}(\alpha)=-i \sqrt{p_{1}^{2}-\alpha^{2}}, \quad \gamma_{2}(\alpha)=-i \sqrt{p_{2}^{2}-\alpha^{2}}
$$

$\gamma_{1}(\alpha), \gamma_{2}(\alpha) \rightarrow|\alpha|$ as $|\sigma| \rightarrow \infty([7])$.
Relying on the above-said, we can show that $u_{\varepsilon}^{*}(\alpha, y)$ and $v_{\varepsilon}^{*}(\alpha, y)$ as $y \rightarrow \infty$ tend to zero if we put $C=D=E=F=0$ in (1.5). When defining the rest constants from the boundary condition for $y=0$, we obtain

$$
\begin{align*}
& u_{\varepsilon}^{*}(\alpha, 0)=\frac{\gamma_{2} p_{2}^{2}}{\mu \Delta(\alpha)}+\frac{i \alpha\left(2 \alpha^{2}-p_{2}^{2}-2 \gamma_{1} \gamma_{2}\right)}{\mu \Delta(\alpha)} \\
& v_{\varepsilon}^{*}(\alpha, 0)=\frac{i \alpha\left(2 \alpha^{2}-p_{2}^{2}-2 \gamma_{1} \gamma_{2}\right)}{\mu \Delta(\alpha)}-\frac{\gamma_{1} p_{2}^{2}}{\mu \Delta(\alpha)} \tag{1.6}
\end{align*}
$$

where $\Delta(\alpha)=4 \alpha^{2} \gamma_{1} \gamma_{2}-\left(2 \alpha^{2}-p_{2}^{2}\right)^{2}$.
On the basis of the previous results, the functions $u_{\varepsilon}^{*}(\alpha, 0)$ and $v_{\varepsilon}^{*}(\alpha, 0)$ are the Fourier transforms of the functions [7]

$$
\begin{align*}
& u_{\varepsilon}^{* *}(x)=\frac{1}{2 \pi} \int_{i \tau-\infty}^{i \tau+\infty} u_{\varepsilon}^{*}(\alpha, 0) e^{-i \alpha x} d x  \tag{1.7}\\
& v_{\varepsilon}^{* *}(x)=\frac{1}{2 \pi} \int_{i \tau-\infty}^{i \tau+\infty} v_{\varepsilon}^{*}(\alpha, 0) e^{-i \alpha x} d x
\end{align*}
$$

and the following inequalities are valid:

$$
\begin{array}{ll}
\left|u_{\varepsilon}^{* *}(x)\right|,\left|v_{\varepsilon}^{* *}(x)\right|<\exp \left(-k_{1}^{\prime}+\delta\right) x & \text { as } \quad x \rightarrow+\infty, \\
\left|u_{\varepsilon}^{* *}(x)\right|,\left|v_{\varepsilon}^{* *}(x)\right|<\exp \left(k_{1}^{\prime}-\delta\right) x \quad & \text { as } \quad x \rightarrow-\infty,
\end{array}
$$

$-k_{1}^{\prime}<\tau<k_{1}^{\prime}, \delta$ is an arbitrarily small positive number.

Thus the amplitudes of horizontal and vertical displacements of the boundary point from the unitary concentrated at the origin horizontal and vertical harmonic force are defined by formulas (1.7), where $u_{\varepsilon}^{*}(\alpha, 0)$ and $v_{\varepsilon}^{*}(\alpha, 0)$ are given by means of (1.6).
2. Let the elastic half-plane be reinforced along its boundary by an elastic fastening in the from of an infinite cover plate of small thickness $h$.

The problem is to determine the law of distribution of contact stresses along the contact line, when harmonic horizontal and vertical forces $\tau_{0} \delta(x) e^{-i \omega t}$ and $p_{0} \delta(x) e^{-i \omega t}$ act on the upper cover side. Assume that the tangential and normal contact stresses act under the cover plate.

Using the D'Alembert principle and Hook's law, the differential equations of cover plate oscillations have the form

$$
\begin{align*}
& \frac{\partial^{2} u^{(1)}(x, t)}{\partial x^{2}}-\frac{\rho_{1}}{E_{1}} \frac{\partial^{2} u^{(1)}(x, t)}{\partial t^{2}}=\frac{1}{E_{1} h_{1}} \tau(x, t)-\frac{\tau_{0} \delta(x) e^{-i \omega t}}{E_{1} h_{1}}  \tag{2.1}\\
& D \frac{\partial^{4} v^{(1)}(x, t)}{\partial x^{4}}-\rho_{1} h_{1} \frac{\partial^{2} v^{(1)}(x, t)}{\partial t^{2}}=p(x, t)-\rho_{0} \delta(x) e^{-i \omega t}
\end{align*}
$$

where $\tau(x, t)$ and $p(x, t)$ are, respectively, the tangential and normal stresses at the point $x$ at the time moment $t$, acting onto the cover plate along the line of joint with the plane, $E_{1}$ is the elasticity modulus of the cover plate, $\rho_{1}$ is density of the material, $u^{(1)}(x, t)$ and $v^{(1)}(x, t)$ are, respectively, horizontal and vertical displacements of points of the cover plate.

We will now pass to the perturbed equation and consider stationary oscillations of the cover plate, assuming that $u^{(1)}(x, t)=u_{\varepsilon}^{(1)}(x) e^{-i \omega t}$, $v^{(1)}(x, t)=v_{\varepsilon}^{(1)}(x) e^{-i \omega t}, \tau(x, t)=\tau_{\varepsilon}(x) e^{-i \omega t}, p(x, t)=p_{\varepsilon}(x) e^{-i \omega t}$. As a result, for the displacement amplitude we obtain the following differential equations:

$$
\begin{align*}
& \frac{d^{2} u_{\varepsilon}^{(1)}(x)}{d x^{2}}+p^{2} u_{\varepsilon}^{(1)}(x)=\frac{1}{E_{1} h_{1}} \tau_{\varepsilon}(x)-\frac{\tau_{0} \delta(x)}{E_{1} h_{1}}  \tag{2.2}\\
& \frac{d^{4} u_{\varepsilon}^{(1)}(x)}{d x^{4}}+\widetilde{p}^{2} v_{\varepsilon}^{(1)}(x)=\frac{p_{\varepsilon}(x)}{D}-\frac{p_{0}}{D} \delta(x)
\end{align*}
$$

where

$$
p^{2}=\frac{\omega^{2}}{c^{2}}+\frac{i \omega \varepsilon}{c^{2}}, \quad \widetilde{p}^{2}=\frac{\omega^{2}}{\widetilde{c}^{2}}+\frac{i \omega \varepsilon}{\widetilde{c}^{2}}, \quad c^{2}=\frac{E_{1}}{\rho_{1}}, \quad \widetilde{c}^{2}=\frac{D}{\rho_{1} h_{1}}
$$

On the other hand, the amplitudes $u_{\varepsilon}^{(2)}(x)$ and $v_{\varepsilon}^{(2)}(x)$ of horizontal and vertical displacements of boundary points of the elastic half-plane from the same amplitudes of stresses $\tau_{\varepsilon}(x)$ and $p_{\varepsilon}(x)$ applied to the half-plane boundary are given, according to (1.6) and the superposition principle, by the
formulas

$$
\begin{align*}
& u_{\varepsilon}^{(2)}(x)=\int_{-\infty}^{\infty} k_{1}(|x-s|) \tau_{\varepsilon}(s) d s+\int_{-\infty}^{\infty} k_{2}(|x-s|) p_{\varepsilon}(s) d s \\
& v_{\varepsilon}^{(2)}(x)=\int_{-\infty}^{\infty} k_{2}(|x-s|) \tau_{\varepsilon}(s) d s+\int_{-\infty}^{\infty} k_{3}(|x-s|) p_{\varepsilon}(s) d s \tag{2.3}
\end{align*}
$$

where

$$
\begin{aligned}
& k_{1}(x)=\frac{1}{2 \pi \mu} \int_{i \tau-\infty}^{i \tau+\infty} \frac{\gamma_{2} p_{2}^{2} e^{-i \alpha x} d \alpha}{\Delta(\alpha)} \\
& k_{2}(x)=\frac{1}{2 \pi \mu} \int_{i \tau-\infty}^{i \tau+\infty} \frac{\alpha\left(2 \alpha^{2}-p_{2}^{2}-2 \gamma_{1} \gamma_{2}\right) e^{-i \alpha x} d \alpha}{\Delta(\alpha)} \\
& k_{3}(x)=\frac{1}{2 \pi \mu} \int_{i \tau-\infty}^{i \tau+\infty} \frac{\gamma_{1} p_{2}^{2} e^{-i \alpha x} d \alpha}{\Delta(\alpha)}
\end{aligned}
$$

On the line connecting the cover plate with the half-plate the contact conditions

$$
\begin{equation*}
u_{\varepsilon}^{(1)}(x)=u_{\varepsilon}^{(2)}(x) \quad \text { and } \quad v_{\varepsilon}^{(1)}(x)=v_{\varepsilon}^{(2)}(x), \quad-\infty<x<\infty \tag{2.4}
\end{equation*}
$$

should be fulfilled. These conditions together with equation (2.2) and formulas (2.3) reduce the problem of finding an amplitude of contact stresses to the solution of a system of integro-differential equations

$$
\begin{gather*}
\left(\frac{d^{2}}{d x^{2}}+p^{2}\right) \int_{-\infty}^{\infty} k_{1}(|x-s|) \tau_{\varepsilon}(s) d s+\int_{-\infty}^{\infty} k_{2}(|x-s|) p_{\varepsilon}(s) d s= \\
=\lambda \tau_{\varepsilon}(x)-\lambda \tau_{0} \delta(x) \\
\begin{array}{c}
\left(\frac{d^{4}}{d x^{4}}+\widetilde{p}^{2}\right) \int_{-\infty}^{\infty} k_{2}(|x-s|) \tau_{\varepsilon}(s) d s+\int_{-\infty}^{\infty} k_{3}(|x-s|) p_{\varepsilon}(s) d s= \\
=\widetilde{\lambda} p_{\varepsilon}(x)-\widetilde{\lambda} p_{0} \delta(x)
\end{array} \tag{2.5}
\end{gather*}
$$

where $\lambda=\frac{1}{E_{1} h_{1}}, \widetilde{\lambda}=\frac{1}{D}$.

Applying to the both parts of equations (2.5) the Fourier transformation and making use of the known properties of convolution, we obtain

$$
\begin{align*}
\tau_{\varepsilon}^{*}(\alpha)= & \lambda \Delta(\alpha) \mu \frac{\left(\alpha^{4}+\widetilde{p}^{2}\right) \gamma_{1} p_{2}^{2}-\widetilde{\lambda} \Delta(\alpha) \mu}{\widetilde{\Delta}(\alpha)} \tau_{0}- \\
& -\widetilde{\lambda} \Delta(\alpha) \mu \frac{\left(\alpha^{2}-p^{2}\right) \alpha\left(2 \alpha^{2}-p_{2}^{2}-2 \gamma_{1} \gamma_{2}\right)}{\widetilde{\Delta}(\alpha)} p_{0}  \tag{2.6}\\
p_{\varepsilon}^{*}(\alpha)= & -\widetilde{\lambda} \Delta(\alpha) \mu \frac{\left(\alpha^{2}-p^{2}\right) \gamma_{2} p_{2}^{2}+\lambda \Delta(\alpha) \mu}{\widetilde{\Delta}(\alpha)} p_{0}- \\
& -\lambda \Delta(\alpha) \mu \frac{\left(\alpha^{4}+\widetilde{p}^{2}\right) \alpha\left(2 \alpha^{2}-p_{2}^{2}-2 \gamma_{1} \gamma_{2}\right)}{\widetilde{\Delta}(\alpha)} \tau_{0}
\end{align*}
$$

where

$$
\begin{aligned}
\widetilde{\Delta}(\alpha) & =\left(\alpha^{2}-p^{2}\right)\left(\alpha^{4}+\widetilde{p}^{2}\right)\left(\gamma_{1} \gamma_{2} p_{2}^{4}+2 \alpha^{4}-\alpha^{2} p_{2}^{2}-2 \gamma_{1} \gamma_{2} \alpha^{2}\right)+ \\
& +\lambda\left(\alpha^{4}+\widetilde{p}^{2}\right) \gamma_{1} p_{2}^{2} \mu \Delta(\alpha)-\widetilde{\lambda}\left(\alpha^{2}-p^{2}\right) \gamma_{2} p_{2}^{2} \mu \Delta(\alpha)-\lambda \widetilde{\lambda} \Delta^{2}(\alpha) \mu^{2}
\end{aligned}
$$

It is not difficult to show that the functions $\tau_{\varepsilon}^{*}(\alpha)$ and $p_{\varepsilon}^{*}(\alpha)$ given by formulas (2.6), have in the strip $-k_{1}^{\prime}<\tau<k_{1}^{\prime}$ no zeros and poles. Therefore they satisfy the conditions of the well-known theorem ([7]) by which

$$
\begin{gather*}
\tau_{\varepsilon}(x)=\frac{1}{2 \pi} \int_{i \tau-\infty}^{i \tau+\infty} \tau_{\varepsilon}^{*}(\alpha) e^{-i \alpha x} d \alpha, \quad p_{\varepsilon}(x)=\frac{1}{2 \pi} \int_{i \tau-\infty}^{i \tau+\infty} p_{\varepsilon}^{*}(\alpha) e^{-i \alpha x} d \alpha  \tag{2.7}\\
-\infty<x<\infty \quad-k_{1}^{\prime}<\tau<k_{1}^{\prime}
\end{gather*}
$$

and

$$
\begin{aligned}
& \left(\left|\tau_{\varepsilon}(x)\right|,\left|p_{\varepsilon}(x)\right|\right)<\exp \left(-k_{1}^{\prime}+\delta\right) x \quad \text { as } \quad x \rightarrow+\infty \\
& \left(\left|\tau_{\varepsilon}(x)\right|,\left|p_{\varepsilon}(x)\right|\right)<\exp \left(k_{1}^{\prime}-\delta\right) x \quad \text { as } \quad x \rightarrow-\infty
\end{aligned}
$$

If $\varepsilon$ tends to zero, the contour if integration in (2.7) turns into the real axis and amplitudes of the unknown contact stresses have in this case the form

$$
\tau(x)=\lim _{\varepsilon \rightarrow 0} \tau_{\varepsilon}(x), \quad p(x)=\lim _{\varepsilon \rightarrow 0} p_{\varepsilon}(x)
$$

3. Referring to the case of an infinite cover plate loaded with only vertical (normal) harmonic force $p_{0} \delta(x) e^{-i \omega t}$, the problem of finding an amplitude of contact normal stresses is reduced to the solution of the following integrodifferential equation

$$
\begin{equation*}
\left(\frac{d^{4}}{d x^{4}}+\widetilde{p}^{2}\right) \int_{-\infty}^{\infty} k_{3}(|x-s|) p_{3}(s) d s=\widetilde{\lambda} p_{\varepsilon}(x)-\widetilde{\lambda} p_{0} \delta(x) \tag{3.1}
\end{equation*}
$$

Applying to the both parts of equation (3.1) the Fourier transformation, we obtain

$$
\left(\frac{\left(\alpha^{4}+\widetilde{p}^{2}\right) \gamma_{1} p_{2}^{2}}{\mu \Delta(\alpha)}-\tilde{\lambda}\right) p_{\varepsilon}^{*}(\alpha)=-\widetilde{\lambda} p_{0}
$$

Investigate now the roots of the functions

$$
\begin{aligned}
& \Delta(\alpha)=4 \alpha^{2} \sqrt{\left(\alpha^{2}-p_{1}^{2}\right)\left(\alpha^{2}-p_{2}^{2}\right)}-\left(2 \alpha^{2}-p_{2}^{2}\right)^{2} \\
& f_{\varepsilon}(\alpha)=\left(\alpha^{4}+\widetilde{p}^{2}\right) \sqrt{\alpha^{2}-p_{1}^{2}} p_{2}^{2}-\widetilde{\lambda}\left(4 \alpha^{2} \sqrt{\left(\alpha^{2}-p_{1}^{2}\right)\left(\alpha^{2}-p_{2}^{2}\right)}-\left(2 \alpha^{2}-p_{2}^{2}\right)^{2}\right)
\end{aligned}
$$

After the change of variables $\alpha^{2}=z^{2} m^{2}\left(m^{2}=\omega^{2}+i \varepsilon \omega, m=m_{1}+i m_{2}\right.$, $m_{1}>0, m_{2}>0, m_{2}(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ ), we obtain

$$
\begin{aligned}
\Delta(z) & =-m^{4}\left[\left(2 z^{2}-\frac{1}{c_{2}^{2}}\right)^{2}-4 z^{2} \sqrt{\left(z^{2}-\frac{1}{c_{1}^{2}}\right)\left(z^{2}-\frac{1}{c_{2}^{2}}\right)}\right] \\
f_{\varepsilon}(z) & =m^{4}\left[\left(m^{2} z^{4}+\frac{1}{\widetilde{c}^{2}}\right) \frac{m_{1}}{c_{2}^{2}} \sqrt{z^{2}-\frac{1}{c_{1}^{2}}}+\widetilde{\lambda}\left(\left(2 z^{2}-\frac{1}{c_{2}^{2}}\right)^{2}\right.\right. \\
& -4 z^{2} \sqrt{\left.\left.\left(z^{2}-\frac{1}{c_{1}^{2}}\right)\left(z^{2}-\frac{1}{c_{2}^{2}}\right)\right)\right]+\frac{i m_{2}}{c_{2}^{2}}\left(m^{2} z^{4}+\frac{1}{\widetilde{c}^{2}}\right) \sqrt{z^{2}-\frac{1}{c_{1}^{2}}} \cdot m^{4} .}
\end{aligned}
$$

The function $\Delta(z)$ has two real roots $z= \pm z_{R}\left(z_{R}>\frac{1}{c_{2}}\right)$. This implies that the function $f_{0}(z)=\lim _{\varepsilon \rightarrow 0} f_{\varepsilon}(z)$ has no real roots in the intervals $\left( \pm \frac{1}{c_{2}}, \pm z_{R}\right)$, since

$$
\begin{aligned}
& f_{0}\left( \pm z_{R}\right)=\frac{\omega^{5}}{c_{2}^{2}}\left(\omega^{2} z_{R}^{4}+\frac{1}{\widetilde{c}^{2}}\right) \sqrt{z_{R}^{2}-\frac{1}{c_{1}^{2}}}>0, \\
& f_{0}\left( \pm \frac{1}{c_{2}}\right)=\frac{\omega^{5}}{c_{2}^{2}}\left(\frac{\omega^{2}}{c_{2}^{4}}+\frac{1}{\widetilde{c}^{2}}\right) \sqrt{\frac{1}{c_{2}^{2}}-\frac{1}{c_{1}^{2}}}+\widetilde{\lambda} \frac{\omega^{4}}{c_{2}^{4}}>0 .
\end{aligned}
$$

As far as $\Delta(z)>0$ for real $z$ satisfying the condition $|z|>z_{R}>0$, there exists $\widetilde{\lambda}_{0}$ such that for $\widetilde{\lambda}<\widetilde{\lambda}_{0}$ the function $f_{0}(z)$ has no real roots and for $\widetilde{\lambda}>\widetilde{\lambda}_{0}$ it has two real roots $\pm z_{\tilde{\lambda}},\left|z_{\tilde{\lambda}}\right|>z_{R}$. If $f_{0}(z)$ has two real roots, then the function $f_{\varepsilon}(z)$ has likewise two roots in a sufficiently narrow strip of the complex plane $z$ containing the real axis.

Passing to the variable $\alpha$, we find that $f_{\varepsilon}(\alpha)$ either has no, or may have roots $\alpha= \pm \alpha_{\tilde{\lambda}}= \pm m z_{\tilde{\lambda}}$, while the function $\Delta(\alpha)$ has the roots $\alpha= \pm \alpha_{R}=$ $\pm m z_{R}$. This implies that $\alpha_{R}$ and $\alpha_{\tilde{\lambda}}$ lie outside of the strip $-k_{1}^{\prime}<\tau<k_{1}^{\prime}$. Consequently, $p_{\varepsilon}(x)$ is defined from the second formula (2.7).

If $\varepsilon$ tends to zero, then the contour of integration turns into the real axis which goes around the points $\sigma_{R}=\lim _{\varepsilon \rightarrow 0} \alpha_{R}, \sigma_{\tilde{\lambda}}=\lim _{\varepsilon \rightarrow 0} \alpha_{\tilde{\lambda}}, k_{2}, k_{1}$ from below,
while the points $-\sigma_{R},-\sigma_{\tilde{\lambda}},-k_{2},-k_{1}$ from above. Then

$$
\begin{aligned}
& p(x)=\lim _{\varepsilon \rightarrow 0} p_{\varepsilon}(x)= \\
& =-\frac{\widetilde{\lambda} \mu p_{0}}{2 \pi} \int_{-\infty}^{\infty} \frac{\left[4 \sigma^{2} \sqrt{\left(\sigma^{2}-k_{1}^{2}\right)\left(\sigma^{2}-k_{2}^{2}\right)}-\left(2 \sigma^{2}-k_{2}^{2}\right)^{2}\right] e^{-i \sigma x} d \sigma}{\left(\sigma^{4}+\widetilde{k}^{2}\right) k_{2}^{2} \sqrt{\sigma^{2}-k_{1}^{2}}-\widetilde{\lambda}\left[\left(2 \sigma^{2}-k_{2}^{2}\right)^{2}-4 \sigma^{2} \sqrt{\left(\sigma^{2}-k_{1}^{2}\right)\left(\sigma^{2}-k_{2}^{2}\right)}\right]} .
\end{aligned}
$$

Here the integral is understood in the sense of the principle Cauchy value when the integrand function has the first order poles at the points $\pm \sigma_{\tilde{\lambda}}$.

The presence of poles in the integrand function corresponds to the existence of stresses coming from the surface waves, and such waves do not arise in the opposite case. In either of the cases the stress $p(x)$ involves always summands from the extension and distortion waves.

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