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DESCRIPTIONS OF ZERO SETS AND PARAMETRIC REPRESENTATIONS OF CERTAIN ANALYTIC AREA NEVANLINNA TYPE CLASSES IN THE UNIT DISK

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Abstract. The complete descriptions of zero sets of certain analytic area Nevanlinna type spaces in the unit disk are given and parametric representations for these scales of spaces are also provided. Our results extend some previously known assertions

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1. Definitions, Preliminaries and Formulation of Problems

Let $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ be the unit disk on the complex plane $\mathbb{C}$, let T be its boundary and let $H(\mathbb{D})$ be the space of all holomorphic functions in $\mathbb{D}$. For $\alpha\geq 0$, we denote by $\widetilde{N}_\alpha^\infty$ the following class of functions

$$\widetilde{N}_\alpha^\infty=\left\{f\in H(\mathbb{D}):\,T(\tau,f)\leq\frac{C_f}{(1-\tau)^\alpha},\,0\leq\tau\leq 1\right\}$$

where $T(\tau,f)$ is the Nevanlinna characteristic of f. It is obvious that if $\alpha=0$, $\widetilde{N}_0^\infty=N$ (N is the standard Nevanlinna class). The following classical result of R. Nevanlinna (see [6])which gives a parametric represenatation of N class is well known. If f is an analytic function in $\mathbb{D}$ then if $f\in N$ if and only if

$$f(z)=C_\lambda z^\lambda B(z,z_k)\exp\left(\int\limits_{-\pi}^{\pi}\frac{d\mu(\theta)}{1-ze^{-i\theta}}\right),z\in\mathbb{D}$$

where C_λ is a complex number, λ is a nonnegative integer, $B(z,z_k)$ is a Blaschke product, $\{z_k\}_{k=1}^\infty$ is any sequence of points in $\mathbb{D}$ satisfying $\sum_{k=1}^\infty(1-|z_k|)<+\infty$, $\mu(\theta)$ is any real measure on $[-\pi,\pi]$.

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Let $\{z_k\}_{k=1}^{\infty}$ is any sequence of points in $\mathbb{D}$ satisfying

$$\sum_{k=1}^{\infty}(1-|z_k|)^{\beta+2} < +\infty,\ \beta > -1 \tag{1}$$

In [1] the following proposition was established.

Proposition A. *Under condition* (1) *the infinite product*

$$\Pi_\beta(z,z_k) = \Pi_{k=1}^{+\infty}(1-\frac{z}{z_k})\exp\left(\frac{-\beta}{\pi}\int\limits_{\mathbb{D}}\frac{(1-|\xi|^2)^\beta \ln|1-\frac{\xi}{z_k}|}{(1-\overline{\xi}z)^{\beta+2}}dm_2(\xi)\right)$$

is analytic and converges absolutely uniformly inside $\mathbb{D}$*, vanishing only on* $\{z_k\}_{k=1}^{\infty}$.

We will also need for exposition several other auxiliary statements. The following statement can be found in [1].

Lemma A. ([1]) *If* $\beta > -1$ *and* $\sum_{k=1}^{\infty}(1-|z_k|)^{\beta+2} < +\infty$*, then for Djrbashian product* $\Pi_\beta(z,z_k)$*,* $z \in \mathbb{D}$*, we have the estimate*

$$\left|\ln|\Pi_\beta(z,z_k)|\right| \le C\sum_{k=0}^{+\infty}\left(\frac{1-|z_k|^2}{|1-z\overline{z_k}|}\right)^{\beta+2}$$

The following theorem was established in [9] recently.

Theorem A. *If* $\alpha \ge 0, \beta > \alpha - 1$*, then the following statements are equivalent*
1) $f \in \widetilde{N}_\alpha^\infty$;
2) f *admits a representation*

$$f(z) = C_\lambda z^\lambda \Pi_\beta(z,z_k)\exp\left(\int\limits_{-\pi}^{\pi}\frac{\psi(e^{i\theta})d\theta}{(1-e^{-i\theta}z)^{\beta+2}}\right),\ z \in \mathbb{D}$$

where C_λ *is a complex number,* λ *is a nonnegative integer.* $\Pi_\beta(z,z_k)$ *is a Djrbashian product of mentioned form.* $\{z_k\}_{k=1}^{\infty}$ *is any sequence of points in* $\mathbb{D}$ *satisfying* $n(\tau) = card\{z_k : |z_k| < \tau\} \le \frac{C}{(1-\tau)^{\alpha+1}}$; $\psi(e^{i\theta})$ *is any real function belonging to* $B_{\beta-\alpha+1}^{1,\infty}$ *where* $B_\gamma^{1,\infty}, \gamma > 0$ *is a standard Besov class on the unit circle* $\mathrm{T} = \{z : |z| = 1\}$*. Theorem A provides complete parametric representations of* $\widetilde{N}_\alpha^\infty$*. One of goals of this paper is to obtain such a representation for larger classes* $N_{\alpha,\beta}^p, N_{\alpha,\beta_1}^\infty$ *spaces which will be* *defined as follows*

$$N^p_{\alpha,\,\beta}=\left\{f\in H(\mathbb{D}):\int\limits_0^1\left(\int\limits_{|z|\le R}\ln^+|f(z)|(1-|z|)^\alpha dm_2(z)\right)^p(1-R)^\beta dR<+\infty\right\}$$

$$N^\infty_{\alpha,\beta_1}=\left\{f\in H(\mathbb{D}):\sup_{0\le R<1}\left(\int\limits_{|z|\le R}\ln^+|f(z)|(1-|z|)^\alpha dm_2(z)\right)(1-R)^{\beta_1}<+\infty\right\}$$

where $\beta_1\ge 0$, $\alpha>-1$, $\beta>-1$, $0<p<\infty$.

Note also various properties of $N^\infty_{\alpha,0}$ were studied before in [1]. In particular complete descriptions of zero sets and parametric representations of $N^\infty_{\alpha,0}$ can be found in [1, 8, 9]. The natural question is to extend also these important results to all $N^\infty_{\alpha,\beta_1}$ classes. The problem of description of zero sets has the following simple formulation. Let $f\in X$, where X is a fixed subspace of $H(\mathbb{D})$, $\{z_k\}_{k=1}^\infty$ is a sequence of points such that $|z_k|<1$ that belongs to certain class of sequences. The problem is to find precisely a class Y of sequences such that there is a function $f, f\in X, f(z_k)=0, k=1,\dots,n, \{z_k\}_{k=1}^\infty\in Y$ and $\{z_k\}_{k=1}^\infty$ is an arbitrary fixed sequence of numbers of Y and the reverse is also true, if $\{z_k\}_{k=1}^\infty$ is a zero set of f then $\{z_k\}_{k=1}^\infty$ is in Y. The problem is known as a problem of description of zero sets (see [1, 5]). Let us note that for many classical analytic classes such as Bergman space A^p_α, this problem is still open (see [1]). On the other hand, the complete description of zero sets of $N^\infty_{\alpha,0}$, $\widetilde{N}^\infty_\alpha$ are known (see [1, 7, 8]). So such a problem for $N^p_{\alpha,\beta}, N^\infty_{\alpha,\beta_1}$ appears naturally. We denote further $n(t)=n_f(t)$ the (finite) numbers of zeros of the analytic function f in the unit disk $|z|\le t<1$. Let also further

$$(NA)_{p,\gamma,v}=\left\{f\in H(\mathbb{D}):\int\limits_0^1\left(\sup_{0<\tau<R}T(f,\tau)(1-\tau)^\gamma\right)^p(1-R)^v dR<+\infty\right\}$$

where $\gamma\ge 0$, $v>-1$, $0<p<\infty$. Let us note also that zero sets of $N^{\infty,\,p}_{\alpha,\,\beta}$ classes were described in [8] for $\beta=0$. Here

$$N^{\infty,\,p}_{\alpha,\,\beta}=\left\{f\in H(\mathbb{D}):\sup_{0\le R<1}\int\limits_0^R\left(\int\limits_{\mathbb{T}}\ln^+|f(|z|\xi)d\xi|\right)^p(1-|z|)^\alpha d|z|(1-R)^\beta<+\infty\right\}$$

We also note that mentioned by us two problems of zero sets and parametric representations have various applications and play an important role in function theory (for example, see [4]). Solutions of various problems for example the existence of radial limits is based also on descriptions of zero sets and parametric representations. Note another such problem is the action of the operator of fractional integration and derivative in classes with $\ln|f(z)|$ (for example, see [2, 3]).

2. Main Results

The goal of this section is to give complete descriptions of zero sets of area Nevanlinna type classes defined above.

Theorem 1. *Let $0 < p < \infty$, $\alpha > -1$, $\beta > -1$. The following are equivalent.*
1) $\{z_k\}_{k=1}^{\infty}$ *is a zero set of a function f, $f \neq 0$, $f \in N^p_{\alpha,\beta}$;*
2)

$$\sum_{k=1}^{\infty} \frac{n_k^p}{2^{k\alpha p} \cdot 2^{k\beta} \cdot 2^{k(2p+1)}} < +\infty, \ \textit{where } n_k = n(1 - \frac{1}{2^k}),\ k \in Z^+. \qquad (2)$$

Moreover if (2) holds then for $t > \frac{\beta+1}{p} + \alpha$ the infinite product $\Pi_t(z, z_k)$ converges uniformly within $\mathbb{D}$ and belongs to $N^p_{\alpha,\beta}$.

Theorem 2. *Let $0 < p < \infty$, $v \in (-1, 0)$, $\gamma \geq 0$. The following are equivalent.*
1) $\{z_k\}_{k=1}^{\infty}$ *is the zero set of a function f, $f \neq 0$, $f \in (NA)_{p,\gamma,v}$;*
2)

$$\sum_{k=1}^{\infty} \frac{n_k^p}{2^k((p\gamma + 1) + v + 1)} < +\infty$$

where $n_k = n(1 - \frac{1}{2^k})$, $k \in Z^+$.

Theorem 3. *Let $0 < p < \infty$, $\alpha > -1$, $\beta > 0$. The following are equivalent.*
1) $\{z_k\}_{k=1}^{\infty}$ *is a zero set of a function f, $f \neq 0$, $f \in N^{\infty,p}_{\alpha,\beta}$;*
2)

$$n(\tau) \leq \frac{C}{(1-\tau)^{\alpha+\beta+p+1}},\ \tau \in (0,1) \qquad (3)$$

where $n(\tau) = card\{z_k : |z_k| < \tau\}$.

Moreover if condition (3) holds then for $t > \alpha+\beta+p$ the product $\Pi_t(z, z_k)$ converges uniformly in $\mathbb{D}$ and belongs to $N^{\infty,p}_{\alpha,\beta}$.

Remark. The proofs and formulation of Theorems 1-3 can be expanded without difficulties to some more general weights $w(1-\tau)$ from so called slowly varied functions from S class see [1, 7, 8, 9] and references there.

Theorems 1-3 give immediately as corollaries the following parametric representation of mentioned above area Nevanlinna type classes, we formulate, as example, the following assertion that gives complete parametric representation of $N^p_{\alpha,\beta}$ class.

Theorem 4. *Let* $0 < p < +\infty$, $t > \frac{\beta+1}{p} + \alpha$, $\alpha > -1$, $\beta > -1$. *Then* $N^p_{\alpha,\beta}$ *classes coincide with the spaces of functions that admits representation of the following type*

$$f(z) = C_\lambda z^\lambda \Pi_{k=1}^{\infty}(1 - \frac{z}{z_k}) \exp(\frac{\alpha+1}{\pi}) \int_0^1 \times$$

$$\times \int_{-\pi}^{\pi} \frac{(1-\rho^2)\ln|1 - \frac{\rho e^{i\varphi}}{z_k}|}{(1-\rho e^{-i\varphi} z)^{t+2}} \rho d\rho d\varphi \cdot [\exp h(z)]$$

where $z \in \mathbb{D}$, C_λ *is a constant,* $\lambda \in Z_+$,

$$\sum_{k=1}^{\infty} \frac{n_k^p}{2^{k\alpha p} \cdot 2^{k\beta} \cdot 2^{k(2p+1)}} < +\infty, \text{ where } n_k = n(1 - \frac{1}{2^k}),\ k \in Z^+ \tag{2}$$

and $h \in H(\mathbb{D})$ *such that*

$$\int_0^1 \left(\int_0^R \left(\int_{-\pi}^{\pi} |h(\tau e^{i\varphi})| d\varphi \right) (1-\tau)^\alpha d\tau \right)^p (1-R)^\beta dR < +\infty$$

where $0 < p < +\infty$, $\alpha > -1$, $\beta > -1$.

As it was shown in [2, 3] such parametric representations are playing a crucial role in problems connected with existence of radial limits and the study of action of operators of fractional derivative in area Nevanlinna-type classes in the unit disk.

Finally we note that the proofs of Theorems 1-3 follow mainly from standard arguments (see, for example, [1, 8, 9] and references there) and are based on Lemma A and classical Jensen formula, but with more careful examination of estimates in it. Complete proofs and various applications of these results to the problem of complete descriptions of closed ideals in mentioned area Nevanlinna spaces in the unit disk and other problems in these classes will be presented by authors elsewhere.

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