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MULTIDIMENSIONAL GENERALIZATION OF THE LIZORKIN THEOREM ON FOURIER MULTIPLIERS

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Abstract. A generalization and sharpening of the Lizorkin theorem concerning Fourier multipliers between L_p and L_q is proved. Some multidimensional Lorentz spaces and an interpolation technique (of Sparr type) are used as crucial tools in the proofs. The obtained results are discussed in the light of other generalizations of the Lizorkin theorem and some open questions are raised.

რეზიუმე. განზოგადებული და დაზუსტებულია ლიზორკინის თეორემა ფურიეს $L^p \rightarrow L^q$ მულტიპლიკატორების შესახებ. დამტკიცება ეყრდნობა ლორენცის მრავალგანზომილებიან სივრცეებს და შპარის ტიპის ინტერპოლების ტექნიკას. ჩამოყალიბებულია ზოგიერთი ღია პრობლემა

1. Introduction

Let F and F^{-1} be the direct and the inverse Fourier transforms, respectively, in $\mathbb{R}^2$ defined by

$$(Ff)(\xi) := \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} f(x) e^{-i\langle \xi, x\rangle} dx,$$

$$(F^{-1}g)(x) := \int\limits_{\mathbb{R}^2} g(\xi) e^{i\langle \xi, x\rangle} d\xi.$$

Let $p = (p_1, p_2)$, $q = (q_1, q_2)$, $1 \leqslant p_i \leqslant q_i \leqslant \infty$, $i = 1, 2$. It is said that φ is a Fourier multiplier from L_p to L_q, briefly, $\varphi \in M_p^q$, if there exists $c_1 > 0$ such that for every function f in Schwartz space S the following inequality holds

$$\|T_\varphi(f)\|_{L_q} \leqslant c_1 \|f\|_{L_p},$$

where $T_\varphi(f) = F^{-1}\varphi F f$.

The set M_p^q of all Fourier multipliers from L_p to L_q is a normed space with the norm

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$$\|\varphi\|_{M_p^q} = \|T_\varphi\|_{L_p \to L_q}.$$

M_p^p is denoted by M_p.

Let $E = \{\varepsilon = (\varepsilon_1, \varepsilon_2) : \varepsilon_i = 0 \text{ or } \varepsilon_i = 1,\ i = 1, 2\}$ be the set of corners in the unit cube in $\mathbb{R}^2$. The measure of Q is denoted by $|Q|$.

The following important sufficient condition certifying that $\varphi \in M_p$ was given by S. Mihlin [18]:

Theorem 1.1. *Let $\varphi(x)$ be a bounded function in $\mathbb{R}^2$ and assume that*

$$|x|^{|\alpha|} |D^\alpha \varphi(x)| \leqslant A \quad (|\alpha| \leqslant L)$$

for some integer $L > 1$. Then $\varphi \in M_p$, $1 < p < \infty$, and

$$\|\varphi\|_{M_p} \leqslant c_2 A,$$

where the constant c_2 depends only on p.

For the case $p \leqslant 2 \leqslant q$ the classes M_p^q were characterized by L. Hörmander [8] as follows:

Theorem 1.2. *Let $1 < p \leqslant 2 \leqslant q < \infty$, $1/s = 1/p - 1/q$, and let f be a measurable function such that*

$$\Big|\xi : |f(\xi)| \geqslant b\Big| \leqslant c_3/b^s.$$

Then $f \in M_p^q$.

Moreover, P. Lizorkin [14] studied the case $1 < p \leqslant q < \infty$ and, in particular, proved the following theorem:

Theorem 1.3. *Let $1 < p \leqslant q < \infty$, $A > 0$, $\beta = \frac{1}{p} - \frac{1}{q}$, $\varepsilon \in E$, $|\varepsilon| = \varepsilon_1 + \varepsilon_2$ and let φ be a continuously differentiable function on $\mathbb{R}^2 \setminus \{0\}$ satisfying the following condition:*

$$\left| \prod_{i=1}^{2} y_i^{\varepsilon_i + \beta} \frac{\partial^{|\varepsilon|} \varphi}{\partial y_1^{\varepsilon_1} \partial y_2^{\varepsilon_2}} \right| \leqslant A.$$

Then $\varphi \in M_p^q$ and

$$\|\varphi\|_{M_p^q} \leqslant c_4 A,$$

where $c_4 > 0$ depends only on p and q.

In this paper we will strengthen Theorem 1.3 above, by introducing some two-dimensional parameters $p = (p_1, p_2)$, $q = (q_1, q_2)$, $1 < p_i < q_i < \infty$, $i = 1, 2$. The paper is organized as follows: The main results are presented and discussed in Section 2. The detailed proofs can be found in Section 3 while Section 4 is reserved for some final remarks, i.e. concerning other generalizations and complements of Theorems 1.2 and 1.3 (see e.g. [4]–[6], [9]–[13], [15]–[17], [21] and the references given there).

2. Main Results

Let f be a Lebesgue measurable function on $\mathbb{R}$ and

$$m(\sigma, f) = \big|\{x \in \mathbb{R} : |f(x)| > \sigma\}\big|$$

be its distribution function. The function

$$f^*(t) = \inf\{\sigma : m(\sigma, f) \leqslant t\}$$

is the usual non-increasing rearrangement of f.

Let $f(x_1, x_2)$ be a measurable function on $\mathbb{R}^2$. In what follows by $f^* = f^{*_1,*_2}(t_1, t_2)$ we mean the non-increasing rearrangement first with respect to x_1 with fixed x_2, and then with respect to x_2. This function is called the non-increasing rearrangement of the function f.

Let $\varepsilon \in E$, then $*_\varepsilon = (*_{\varepsilon_1}, *_{\varepsilon_2})$ is an operator, which is acting by the following rule:

$$f^{*_\varepsilon} = \begin{cases} f(x), & if \;\; \varepsilon = 0 \\ f^*(x), & if \;\; \varepsilon = 1, \end{cases}$$

$$f^{*_{\varepsilon_1}, *_{\varepsilon_2}}(x_1, x_2) = \left(f^{*_{\varepsilon_1}}(x_1)\right)^{*_{\varepsilon_2}}(x_2).$$

Our main result reads:

Theorem 2.1. *Let $p = (p_1, p_2)$, $q = (q_1, q_2)$, $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$, $1 < p_i < q_i < \infty$, $0 \leqslant \alpha_i < 1 - \frac{1}{p_i} + \frac{1}{q_i}$ and $\beta_i = \alpha_i + \frac{1}{p_i} - \frac{1}{q_i}$, $i = 1, 2$. If the continuously differentiable function φ on $\mathbb{R}^2 \setminus \{0\}$ satisfies to the following condition:*

$$\sup_{y_i \in \mathbb{R}^+} \prod_{i=1}^{2} y_i^{\varepsilon_i + (1-\varepsilon_i)\beta_i - \alpha_i} \left(\prod_{j=1}^{2} t_j^{\varepsilon_j \beta_j} \frac{\partial^{|\varepsilon|}\varphi}{\partial t_1^{\varepsilon_1} \partial t_2^{\varepsilon_2}} \right)^{*_{\varepsilon_1}, *_{\varepsilon_2}} (y_1, y_2) \leqslant A, \;\; \forall \varepsilon \in E,$$

then $\varphi \in M_p^q$ and

$$\|\varphi\|_{M_p^q} \leqslant c_5 A,$$

where $c_5 > 0$ depends only on p_i, q_i and $\alpha_i, |\varepsilon| = \varepsilon_1 + \varepsilon_2$.

Remark 2.1. Theorem 2.1 is a strict generalization of (the Lizorkin) Theorem 1.3. In fact, the assumptions in Theorem 2.1 are strictly weaker than those in Theorem 1.3, since

$$\sup_{y_i \in \mathbb{R}^+} \prod_{i=1}^{2} y_i^{\varepsilon_i + (1-\varepsilon_i)\beta_i - \alpha_i} \left(\prod_{j=1}^{2} t_j^{\varepsilon_j \beta_j} \frac{\partial^{|\varepsilon|}\varphi}{\partial t_1^{\varepsilon_1} \partial t_2^{\varepsilon_2}} \right)^{*_{\varepsilon_1}, *_{\varepsilon_2}} (y_1, y_2) \leqslant$$

$$\leqslant \prod_{i=1}^{2} 2^{\varepsilon_i + (1-\varepsilon_i)\beta_i - \alpha_i} \sup_{y_i \in \mathbb{R}} \left| \prod_{i=1}^{2} y_i^{\varepsilon_i + \beta} \frac{\partial^{|\varepsilon|}\varphi}{\partial y_1^{\varepsilon_1} \partial y_2^{\varepsilon_2}} \right|,$$

and there exists a function φ (see (10) on page 13) satisfying the assumptions of Theorem 2.1, but not satisfying the assumptions in Theorem 1.3, i.e.

$$\sup_{y_i\in\mathbb{R}^+}\prod_{i=1}^{2} y_i^{\beta_i-\alpha_i}\varphi(y_1,y_2)<\infty,$$

$$\sup_{y_i\in\mathbb{R}^+} y_1^{\beta_1-\alpha_1}y_2^{1-\alpha_2}\left(t_2^{\beta_2}\varphi'_{t_2}(t_1,t_2)\right)^{*_2}(y_2)<\infty,$$

$$\sup_{y_i\in\mathbb{R}^+} y_1^{1-\alpha_1}y_2^{\beta_2-\alpha_2}\left(t_1^{\beta_1}\varphi'_{t_1}(t_1,t_2)\right)^{*_1}(y_1)<\infty,$$

$$\sup_{y_i\in\mathbb{R}^+}\prod_{i=1}^{2} y_i^{1-\alpha_i}\left(t_1^{\beta_1}t_2^{\beta_2}\varphi''_{t_1,t_2}(t_1,t_2)\right)^{*_1*_2}(y_1,y_2)<\infty,$$

but

$$\sup_{y_i\in\mathbb{R}}\left|\prod_{i=1}^{2} y_i^{1+\frac{1}{p_i}-\frac{1}{q_i}}\varphi''_{y_1y_2}\right|=\infty.$$

For the proof of Theorem 2.1 we need the following embedding theorem of independent interest:

Theorem 2.2. *Let* $1<p=(p_1,p_2)<q=(q_1,q_2)\leqslant\infty$, $0<\tau=(\tau_1,\tau_2)\leqslant\infty$, $\alpha=(\alpha_1,\alpha_2)$, $\beta=(\beta_1,\beta_2)$ *and* $0\leqslant\alpha<1-\frac{1}{p}+\frac{1}{q}$. *Then*

$$L_{p\tau}\hookrightarrow N^{\alpha,\beta,\tau}(L_q),$$

where $\beta=\alpha+\frac{1}{p}-\frac{1}{q}$.

Here $N^{\alpha,\beta,r}(L_q)$ is a version of the net spaces, which was introduced and studied in [20] and [22], defined as follows: for $0\leqslant\alpha=(\alpha_1,\alpha_2)<\infty$, $0<\beta=(\beta_1,\beta_2)<\infty$, $0<q=(q_1,q_2)$, $r=(r_1,r_2)\leqslant\infty$, and $G_t=\{[a_1,b_1]\times[a_2,b_2]:b_i-a_i\geqslant t_i,\ i=1,2\}$, $t\in\mathbb{R}^2$, we say that the function f belongs to the space $N^{\alpha,\beta,r}(L_q)$ if $f\in L_1$ and

$$\|f\|_{N^{\alpha,\beta,r}(L_q)}:=\left(\int_0^\infty\left(\int_0^\infty\left(t_1^{\alpha_1}t_2^{\alpha_2}\times\right.\right.\right.$$

$$\left.\left.\left.\times\sup_{Q_i\in G_{t_i}}\frac{1}{|Q_1|^{\beta_1}|Q_2|^{\beta_2}}\|F^{-1}\chi_Q Ff\|_{L_q}\right)^{r_1}\frac{dt_1}{t_1}\right)^{\frac{r_2}{r_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{r_2}}<\infty,$$

for $0<r<\infty$, and

$$\|f\|_{N^{\alpha,\beta,r}(L_q)}:=\sup_{t_1>0,t_2>0}t_1^{\alpha_1}t_2^{\alpha_2}\sup_{Q_i\in G_{t_i}}\frac{1}{|Q_1|^{\beta_1}|Q_2|^{\beta_2}}\|F^{-1}\chi_Q Ff\|_{L_q}<\infty,$$

for $r=(r_1,r_2)=(\infty,\infty)$, where χ_Q is the characteristic function of Q.

Let $1 \leqslant p = (p_1, p_2) < \infty$, $0 < \tau = (\tau_1, \tau_2) \leqslant \infty$. We say that a function f belongs to the Lorentz space $L_{p\tau}$ (see [22]) if f is measurable on $\mathbb{R}^2$ and for $\tau < \infty$

$$\|f\|_{L_{p\tau}} := \left(\int_0^\infty \left(\int_0^\infty \left(t_1^{\frac{1}{p_1}} t_2^{\frac{1}{p_2}} f^{*_1,*_2}(t_1, t_2) \right)^{\tau_1} \frac{dt_1}{t_1} \right)^{\frac{\tau_2}{\tau_1}} \frac{dt_2}{t_2} \right)^{\frac{1}{\tau_2}} < \infty,$$

and for $\tau = \infty$

$$\|f\|_{L_{p\infty}} := \sup_{t_i > 0} t_1^{\frac{1}{p_1}} t_2^{\frac{1}{p_2}} f^{*_1,*_2}(t_1, t_2) < \infty, i = 1, 2.$$

Moreover, for the proof of Theorem 2.2 we need some embedding and interpolation results also of independent interest. The first one reads:

Proposition 2.1. *Let $0 < \alpha = (\alpha_1, \alpha_2) < 1$, $0 < \beta = (\beta_1, \beta_2) < 1$, $0 < q = (q_1, q_2) \leqslant \infty$.*

a) *If $0 < r = (r_1, r_2) \leqslant \tilde{r} = (\tilde{r}_1, \tilde{r}_2) \leqslant \infty$, then*

$$N^{\alpha,\beta,r}(L_q) \hookrightarrow N^{\alpha,\beta,\tilde{r}}(L_q).$$

b) *If $0 < \sigma < \min\{1 - \alpha, 1 - \beta\}$, $0 < r = (r_1, r_2) \leqslant \infty$, then*

$$N^{\alpha,\beta,r}(L_q) \hookrightarrow N^{\alpha+\sigma,\beta+\sigma,r}(L_q). \tag{1}$$

Let $A = \{A_\varepsilon\}_{\varepsilon \in E}$ be compatible Banach spaces (this means that they are all embedded in a Hausdorff topological vector space) and define the corresponding K-functional as follows:

$$K(\tau, a; A_\varepsilon, \varepsilon \in E) := \inf \left\{ \sum_{\varepsilon \in E} \tau^\varepsilon \|a_\varepsilon\|_{A_\varepsilon} : \quad a = \sum_{\varepsilon \in E} a_\varepsilon, a_\varepsilon \in A_\varepsilon \right\},$$

where $\tau^\varepsilon = \tau_1^{\varepsilon_1} \tau_2^{\varepsilon_2} > 0$. This natural generalization of the Peetre K-functional was introduced by G. Sparr [23] (see also [3]).

Moreover, for $0 < q = (q_1, q_2) < \infty$, $0 < \theta = (\theta_1, \theta_2) < 1$, we define the interpolation spaces $A_{\theta,q}$ as follows:

$$A_{\theta,q} = \left\{ a \in \sum_{\theta \in E} A_\varepsilon : \|a\|_{A_{\theta,q}} = \right.$$

$$\left. = \left(\int_0^\infty \left(\int_0^\infty (\tau_1^{-\theta_1} \tau_2^{-\theta_2} \times \times K(\tau, a; A_\varepsilon, \varepsilon \in E)^{q_1} \frac{d\tau_1}{\tau_1} \right)^{\frac{q_2}{q_1}} \frac{d\tau_2}{\tau_2} \right)^{\frac{1}{q_2}} < \infty \right\},$$

and, for $q = (\infty, \infty)$

$$A_{\theta,\infty} = \left\{ a \in \sum_{\theta \in E} A_\varepsilon : \|a\|_{A_{\theta,\infty}} = \sup_{0 < \tau_i < \infty} \tau_1^{-\theta_1} \tau_2^{-\theta_2} K(\tau, a; A_\varepsilon, \varepsilon \in E) < \infty \right\}.$$

The second auxiliary result reads:

Proposition 2.2. *Let* $\alpha_0 = (\alpha_1^0,\ \alpha_2^0) = (0,0)$, $0 < \alpha_1 = (\alpha_1^1,\ \alpha_2^1) < 1$, $\alpha_\varepsilon = (\alpha_1^{\varepsilon_1},\ \alpha_2^{\varepsilon_2})$, $\varepsilon \in E$, $0 < \beta = (\beta_1, \beta_2) < 1$, $0 < r = (r_1, r_2)$, $q = (q_1, q_2) \leqslant \infty$. *Then*

$$(N^{\alpha_\varepsilon,\beta,\infty}(L_q), \varepsilon \in E)_{\theta,r} \hookrightarrow N^{\alpha,\beta,r}(L_q),$$

where $0 < \theta = (\theta_1, \theta_2) < 1$, $\alpha = \left((1-\theta_1)\alpha_1^1, (1-\theta_2)\alpha_2^1\right)$.

3. Proofs

We make the proofs in the logical order to be used in later proofs.

Proof of Proposition 2.2. Since

$$\varphi(t_1, t_2) = \sup_{Q_i \in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}|Q_2|^{\beta_2}} \|F^{-1}\chi_Q F f\|_{L_q},$$

is a non-increasing function and by using the following inequality

$$t_i^{\alpha_i} \sim t_i^{-w_i}\left(\int\limits_0^{t_i} s_i^{(\alpha_i + w_i)r_i} \frac{ds_i}{s_i}\right)^{\frac{1}{r_i}}$$

when $w_i > 0, i = 1, 2$, we find, that

$$\|f\|_{N^{\alpha,\beta,\tilde{r}}} \sim \left(\int\limits_0^\infty \left(t_2^{-w_2}\left(\int\limits_0^{t_2}\left(\int\limits_0^\infty \left(t_1^{-w_1}\left(\int\limits_0^{t_1} \left(s_1^{\alpha_1+w_1} s_2^{\alpha_2+w_2} \times\right.\right.\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.\left.\left.\times \varphi(s_1, s_2)\right)^{r_1} \frac{ds_1}{s_1}\right)^{\frac{1}{r_1}}\right)^{\tilde{r_1}} \frac{dt_1}{t_1}\right)^{\frac{r_2}{\tilde{r_1}}} \frac{ds_2}{s_2}\right)^{\frac{1}{r_2}}\right)^{\tilde{r_2}} \frac{dt_2}{t_2}\right)^{\frac{1}{\tilde{r_2}}}.$$

Then applying Minkowski inequality to each pare of integrals, taking into account, that $r_i < \widetilde{r_i}$, $i = 1, 2$, we obtain that

$$\|f\|_{N^{\alpha,p,\tilde{r}}} \leqslant c_1 \|f\|_{N^{\alpha,p,r}}$$

Moreover,

$$\|f\|_{N^{\alpha+\sigma,\beta+\sigma,r}(L_q)} = \left(\int\limits_0^\infty \left(\int\limits_0^\infty \left(t_1^{\alpha_1+\sigma_1} t_2^{\alpha_2+\sigma_2}\ \times\right.\right.\right.$$

$$\left.\left.\left.\times \sup_{Q_i \in G_{t_i}} \frac{1}{|Q_1|^{\beta_1+\sigma_1}} \frac{1}{|Q_2|^{\beta_2+\sigma_2}} \|F^{-1}\chi_Q F f\|_{L_q}\right)^{r_1} \frac{dt_1}{t_1}\right)^{\frac{r_2}{r_1}} \frac{dt_2}{t_2}\right)^{\frac{1}{r_2}} \leqslant$$

$$\leqslant \left(\int\limits_0^\infty \left(\int\limits_0^\infty \left(t_1^{\alpha_1} t_2^{\alpha_2} \sup_{Q_i \in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}} \frac{1}{|Q_2|^{\beta_2}} \|F^{-1}\chi_Q F f\|_{L_q}\right)^{r_1} \frac{dt_1}{t_1}\right)^{\frac{r_2}{r_1}} \frac{dt_2}{t_2}\right)^{\frac{1}{r_2}} =$$

$$= \|f\|_{N^{\alpha,\beta,r}(L_q)}.$$

The proof is complete. □

Proof of Proposition 2.1. Make an arbitrary decomposition of f in the form $f = f_0 + f_1 + f_2 + f_3$, $f_0 \in N^{(0,0),\beta,\infty}(L_q)$, $f_1 \in N^{(0,\alpha_2^1),\beta,\infty}(L_q)$, $f_2 \in N^{(\alpha_1^1,0),\beta,\infty}(L_q)$ and $f_3 \in N^{(\alpha_1^1,\alpha_2^1),\beta,\infty}(L_q)$. It is obvious that

$$\sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}\,|Q_2|^{\beta_2}}\|F^{-1}\chi_Q Ff\|_{L_q} \leqslant \prod_{i=1}^{2} 2^{(\frac{1}{q_i}-1)_+}\times$$
$$\times\Big(\sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}\,|Q_2|^{\beta_2}}\|F^{-1}\chi_Q Ff_0\|_{L_q}+$$
$$+\sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}\,|Q_2|^{\beta_2}}\|F^{-1}\chi_Q Ff_1\|_{L_q}+$$
$$+\sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}\,|Q_2|^{\beta_2}}\|F^{-1}\chi_Q Ff_2\|_{L_q}+$$
$$+\sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}\,|Q_2|^{\beta_2}}\|F^{-1}\chi_Q Ff_3\|_{L_q}\Big),$$

where $x_+ = x$, if $x > 0$ and $x_+ = 0$, if $x \leqslant 0$. If we denote $v(\tau_i) = \tau_i^{\frac{1}{\alpha_1^i}}$, $\tau_i > 0, i = 1, 2$, then

$$\sup_{0<t_i<v(\tau_i)} t_1^{\alpha_1^1} t_2^{\alpha_2^1} \sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}\,|Q_2|^{\beta_2}}\|F^{-1}\chi_Q Ff\|_{L_q} \leqslant$$
$$\leqslant \prod_{i=1}^{2} 2^{(\frac{1}{q_i}-1)_+}\Big(\sup_{0<t_i<v(\tau_i)} t_1^{\alpha_1^1} t_2^{\alpha_2^1} \sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}\,|Q_2|^{\beta_2}}\|F^{-1}\chi_Q Ff_0\|_{L_q}+$$
$$+\sup_{0<t_i<v(\tau_i)} t_1^{\alpha_1^1} t_2^{\alpha_2^1} \sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}\,|Q_2|^{\beta_2}}\|F^{-1}\chi_Q Ff_1\|_{L_q}+$$
$$+\sup_{0<t_i<v(\tau_i)} t_1^{\alpha_1^1} t_2^{\alpha_2^1} \sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}\,|Q_2|^{\beta_2}}\|F^{-1}\chi_Q Ff_2\|_{L_q}+$$
$$+\sup_{0<t_i<v(\tau_i)} t_1^{\alpha_1^1} t_2^{\alpha_2^1} \sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}\,|Q_2|^{\beta_2}}\|F^{-1}\chi_Q Ff_3\|_{L_q}\Big) \leqslant$$
$$\leqslant \prod_{i=1}^{2} 2^{(\frac{1}{q_i}-1)_+}\Big(\tau_1\tau_2 \sup_{t_i>0} \sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}\,|Q_2|^{\beta_2}}\|F^{-1}\chi_Q Ff_0\|_{L_q}+$$
$$+\tau_2 \sup_{t_i>0} t_1^{\alpha_1^1} \sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}\,|Q_2|^{\beta_2}}\|F^{-1}\chi_Q Ff_1\|_{L_q}+$$
$$+\tau_1 \sup_{t_i>0} t_2^{\alpha_2^1} \sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}\,|Q_2|^{\beta_2}}\|F^{-1}\chi_Q Ff_2\|_{L_q}+$$

$$+\sup_{t_i>0} t_1^{\alpha_1^1} t_2^{\alpha_2^1} \sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}|Q_2|^{\beta_2}} \|F^{-1}\chi_Q F f_3\|_{L_q}\Big).$$

Taking into account that the representation $f = f_0+f_1+f_2+f_3$ is arbitrary, we obtain that

$$\sup_{0<t_i<v(\tau_i)} t_1^{\alpha_1^1} t_2^{\alpha_2^1} \sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}|Q_2|^{\beta_2}} \|F^{-1}\chi_Q F f\|_{L_q} \leqslant$$

$$\leqslant \prod_{i=1}^{2} 2^{(\frac{1}{q_i}-1)_+} K(\tau, f; N^{\alpha_\varepsilon,\beta,\infty}, \varepsilon\in E).$$

Therefore for $0<r=(r_1,r_2)<\infty$, we get that

$$\left(\int_0^\infty \left(\int_0^\infty \left(\tau_1^{-\theta_1}\tau_2^{-\theta_2} K(\tau, f; N^{\alpha_\varepsilon,\beta,\infty}, \varepsilon\in E)\right)^{r_1} \frac{d\tau_1}{\tau_1}\right)^{\frac{r_2}{r_1}} \frac{d\tau_2}{\tau_2}\right)^{\frac{1}{r_2}} \geqslant$$

$$\geqslant \prod_{i=1}^{2} 2^{(\frac{1}{q_i}-1)_+} \left(\int_0^\infty \left(\int_0^\infty \left(\tau_1^{-\theta_1}\tau_2^{-\theta_2} \sup_{0<t_i<v(\tau_i)} t_1^{\alpha_1^1} t_2^{\alpha_2^1} \times\right.\right.\right.$$

$$\left.\left.\left.\times \sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1}|Q_2|^{\beta_2}} \|F^{-1}\chi_Q F f\|_{L_q}\right)^{r_1} \frac{d\tau_1}{\tau_1}\right)^{\frac{r_2}{r_1}} \frac{d\tau_2}{\tau_2}\right)^{\frac{1}{r_2}} =$$

$$= \prod_{i=1}^{2} 2^{(\frac{1}{q_i}-1)_+} (\alpha_i^1)^{\frac{1}{r_i}} \left(\int_0^\infty \left(\int_0^\infty \left(u_1^{-\theta_1\alpha_1^1} u_2^{-\theta_2\alpha_2^1} \sup_{0<t_i<u_i} t_1^{\alpha_1^1} t_2^{\alpha_2^1} \times\right.\right.\right.$$

$$\left.\left.\left.\times \sup_{Q_i\in G_{u_i}} \frac{1}{|Q_1|^{\beta_1}|Q_2|^{\beta_2}} \|F^{-1}\chi_Q F f\|_{L_q}\right)^{r_1} \frac{du_1}{u_1}\right)^{\frac{r_2}{r_1}} \frac{du_2}{u_2}\right)^{\frac{1}{r_2}} \geqslant$$

$$= \prod_{i=1}^{2} 2^{(\frac{1}{q_i}-1)_+} (\alpha_i^1)^{\frac{1}{r_i}} \left(\int_0^\infty \left(\int_0^\infty \left(u_1^{(1-\theta_1)\alpha_1^1} u_2^{(1-\theta_2)\alpha_2^1} \times\right.\right.\right.$$

$$\left.\left.\left.\times \sup_{0<t_i<u_i} \sup_{Q_i\in G_{u_i}} \frac{1}{|Q_1|^{\beta_1}|Q_2|^{\beta_2}} \|F^{-1}\chi_Q F f\|_{L_q}\right)^{r_1} \frac{du_1}{u_1}\right)^{\frac{r_2}{r_1}} \frac{du_2}{u_2}\right)^{\frac{1}{r_2}} =$$

$$= \prod_{i=1}^{2} 2^{(\frac{1}{q_i}-1)_+} (\alpha_i^1)^{\frac{1}{r_i}} \|f\|_{N^{\alpha,\beta,r}(L_q)},$$

i.e.

$$(N^{\alpha_\varepsilon,\beta,\infty}(L_q), \varepsilon\in E)_{\theta,r} \hookrightarrow N^{\alpha,\beta,r}(L_q),$$

The argument for the case $r=\infty$ is similar. The proof is complete. □

Proof of Theorem 2.2. Let $1 \leqslant r < q \leqslant \infty$. It is known that the characteristic function χ_Q of a parallelogram $Q = Q_1 \times Q_2$ is a Fourier multiplier from L_r to L_q. Moreover, there exists $c_6(r, q) > 0$, such that

$$\left\|F^{-1}\chi_Q F f\right\|_{L_q} \leqslant c_6(r,q) \prod_{i=1}^{2} |Q_i|^{\frac{1}{r_i}-\frac{1}{q_i}} \|f\|_{L_r}, \tag{2}$$

for every $f \in L_r$.

Let $0 \leqslant \alpha_i \leqslant 1 - \frac{1}{p_i}$. Then $\frac{1}{p_i} - \frac{1}{q_i} < \beta \leqslant 1 - \frac{1}{q_i}$ and there exists $p_0 = (p_1^0, p_2^0)$ such that $1 < p_i^0 < p_i$ and $\beta_i = \frac{1}{p_i^0} - \frac{1}{q_i}$, $i = 1, 2$. According to (2) applied with $r = p_0$ we have that

$$\|f\|_{N^{0,\beta,\infty}(L_q)} = \sup_{t_i>0} \sup_{Q\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1} |Q_2|^{\beta_2}} \left\|F^{-1}\chi_Q F f\right\|_{L_q} \leqslant c_7 \|f\|_{L_{p_0}} \tag{3}$$

for every $f \in L_{p_0}$, where $c_7 = c_6(p_0, q)$.

Further let $p_1 = (p_1^1, p_2^1)$, $p_i < p_i^1 < q_i$ and $\alpha_1 = (\alpha_1^1, \alpha_2^1)$, $\alpha_1 = \frac{1}{p_i^0} - \frac{1}{p_i^1}$. Taking into account (2) with $r = p_1$ we have that

$$\sup_{t_i>0} \sup_{Q\in G_{t_i}} \frac{1}{|Q_1|^{\frac{1}{p_1^1}-\frac{1}{q_1}} |Q_2|^{\frac{1}{p_2^1}-\frac{1}{q_2}}} \left\|F^{-1}\chi_Q F f\right\|_{L_q} \leqslant c_8 \|f\|_{L_{p_1}}$$

for every $f \in L_{p_1}$, where $c_8 = c_6(p_1, q)$.

Since

$$\sup_{t_i>0} \sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\frac{1}{p_1^1}-\frac{1}{q_1}} |Q_2|^{\frac{1}{p_2^1}-\frac{1}{q_2}}} \left\|F^{-1}\chi_Q F f\right\|_{L_q} =$$

$$= \sup_{t_i>0} \sup_{Q_i\in G_{t_i}} \frac{|Q_1|^{\alpha_1^1} |Q_2|^{\alpha_2^1}}{|Q_1|^{\beta_1} |Q_2|^{\beta_2}} \left\|F^{-1}\chi_Q F f\right\|_{L_q} \geqslant$$

$$\geqslant \sup_{t_i>0} t_1^{\alpha_1^1} t_2^{\alpha_2^1} \sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1} |Q_2|^{\beta_2}} \left\|F^{-1}\chi_Q F f\right\|_{L_q} = \|f\|_{N^{\alpha_1,\beta,\infty}(L_q)},$$

it yields that

$$\|f\|_{N^{\alpha_1,\beta,\infty}(L_q)} \leqslant c_8 \|f\|_{L_{p_1}} \tag{4}$$

for every $f \in L_{p_1}$. Moreover, according to (2), we have that

$$c_6 \|f\|_{L_{(p_1^0, p_2^1)}} \geqslant \sup_{t_i>0} \sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\frac{1}{p_1^0}-\frac{1}{q_1}} |Q_2|^{\frac{1}{p_2^1}-\frac{1}{q_2}}} \left\|F^{-1}\chi_Q F f\right\|_{L_q} =$$

$$= \sup_{t_i>0} \sup_{Q_i\in G_{t_i}} \frac{|Q_2|^{\alpha_2^1}}{|Q_1|^{\beta_1} |Q_2|^{\beta_2}} \left\|F^{-1}\chi_Q F f\right\|_{L_q} \geqslant$$

$$\geqslant \sup_{t_i>0} t_2^{\alpha_2^1} \sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1} |Q_2|^{\beta_2}} \left\|F^{-1}\chi_Q F f\right\|_{L_q} = \|f\|_{N^{(0,\alpha_2^1),\beta,\infty}(L_q)},$$

and, consequently,

$$\|f\|_{N^{(0,\alpha_2^1),\beta,\infty}(L_q)} \leqslant c_6 \|f\|_{L_{(p_1^0,p_2^1)}}. \tag{5}$$

Now, since

$$\begin{aligned} c_6 \|f\|_{L_{(p_1^1,p_2^0)}} &\geqslant \sup_{t_i>0} \sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\frac{1}{p_1^1}-\frac{1}{q_1}} |Q_2|^{\frac{1}{p_2^0}-\frac{1}{q_2}}} \left\|F^{-1}\chi_Q Ff\right\|_{L_q} = \\ &= \sup_{t_i>0} \sup_{Q_i\in G_{t_i}} \frac{|Q_1|^{\alpha_1^1}}{|Q_1|^{\beta_1} |Q_2|^{\beta_2}} \left\|F^{-1}\chi_Q Ff\right\|_{L_q} \geqslant \\ &\geqslant \sup_{t_i>0} t_1^{\alpha_1^1} \sup_{Q_i\in G_{t_i}} \frac{1}{|Q_1|^{\beta_1} |Q_2|^{\beta_2}} \left\|F^{-1}\chi_Q Ff\right\|_{L_q}, \end{aligned}$$

we conclude that

$$\|f\|_{N^{(\alpha_1^1,0),\beta,\infty}(L_q)} \leqslant c_6 \|f\|_{L_{(p_1^1,p_2^0)}}. \tag{6}$$

Let us denote by I the corresponding embedding operator. According to (3)-(6), we have that

$$I : L_{p_0} \hookrightarrow N^{0,\beta,\infty}(L_q),$$
$$I : L_{p_1} \hookrightarrow N^{\alpha_1,\beta,\infty}(L_q),$$
$$I : L_{(p_1^0,p_2^1)} \hookrightarrow N^{(0,\alpha_2^1),\beta,\infty}(L_q)$$

and

$$I : L_{(p_1^1,p_2^0)} \hookrightarrow N^{(\alpha_1^1,0),\beta,\infty}(L_q).$$

Moreover, in all cases the operator I is bounded. Let $p_\varepsilon = (p_1^{\varepsilon_1},\ p_2^{\varepsilon_2})$, $\alpha_\varepsilon = (\alpha_1^\varepsilon, \alpha_2^\varepsilon)$, $\varepsilon \in E$ and $0 < \theta = (\theta_1, \theta_2) < 1$ be such that $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_0}$. Since

$$\alpha = \beta - \frac{1}{p} + \frac{1}{q} = \frac{1}{p_0} - \frac{1}{p} = (1-\theta)\alpha_1,$$

by the interpolation properties of the spaces L_p (see e.g. [19], [22]) we have that

$$I : L_{p\tau} = (L_{p_\varepsilon}, \varepsilon \in E)_{\theta\tau} \to (N^{\alpha_\varepsilon,\beta,\infty}(L_q), \varepsilon \in E)_{\theta\tau},$$

where the operator I is bounded. Thus,

$$L_{p\tau} \hookrightarrow \left(N^{\alpha_\varepsilon,\beta,\infty}(L_q), \varepsilon \in E\right)_{\theta\tau},$$

and, hence, the statement of the theorem follows by using Proposition 2.2.

When $1 - \frac{1}{p} \leqslant \alpha < 1 - \frac{1}{p} + \frac{1}{q}$ the statement of the theorem follows from Proposition 2.1. Indeed, let $0 < \tilde{\alpha} \leqslant 1 - \frac{1}{p}$. Then, according to what is proved above and by (1), we have that

$$L_{\mathbf{p}\tau} \hookrightarrow N^{\tilde{\alpha},\tilde{\alpha}+\frac{1}{p}-\frac{1}{q},\tau}(L_{\mathbf{q}}) \hookrightarrow N^{\alpha,\beta,\tau}(L_q).$$

The proof is complete. $\square$

Proof of Theorem 2.1. Let $f \in S$ and $0 < a_i < b_i < \infty, i = 1, 2$. Since $\varphi \in AC([a, b])$ and, by integrating by parts and using the Minkowski inequality for sums and integrals, we find that

$$\left\| \int_{a_2}^{b_2} \int_{a_1}^{b_1} \varphi(y_1, y_2)(Ff)(y_1, y_2) e^{i\langle y,z\rangle} dy_1 dy_2 \right\|_{L_q} =$$

$$= \left\| \int_{a_2}^{b_2} \int_{a_1}^{b_1} \varphi(y_1, y_2) \left(\int_0^{y_2} \int_0^{y_1} (Ff)(\xi_1, \xi_2) e^{i\langle \xi,z\rangle} d\xi_1 d\xi_2 \right)''_{y_1,y_2} dy_1 dy_2 \right\|_{L_q} =$$

$$= 2\pi \left\| \sum_{\varepsilon \in E} (-1)^{\varepsilon_1} (-1)^{\varepsilon_2} \int_{a_2}^{b_2} \int_{a_1}^{b_1} D^{1-\varepsilon} \left(D^{\varepsilon} \varphi(y_1, y_2) \cdot \left(F^{-1} \chi_{[0,y_1]\times[0,y_2]} Ff \right) \right) \right\|_{L_q} \leqslant$$

$$\leqslant 2\pi \Bigg(\left\| \varphi(b_1, b_2) F^{-1} \chi_{[0,b_1]\times[0,b_2]} Ff \right\|_{L_q} + \left\| \varphi(a_1, a_2) F^{-1} \chi_{[0,a_1]\times[0,a_2]} Ff \right\|_{L_q} +$$

$$+ \left\| \varphi(a_1, b_2) F^{-1} \chi_{[0,a_1]\times[0,b_2]} Ff \right\|_{L_q} + \left\| \varphi(b_1, a_2) F^{-1} \chi_{[0,b_1]\times[0,a_2]} Ff \right\|_{L_q} +$$

$$+ \left\| \int_{a_2}^{b_2} \varphi'_{y_2}(b_1, y_2) F^{-1} \chi_{[0,b_1]\times[0,y_2]} Ff dy_2 \right\|_{L_q} +$$

$$+ \left\| \int_{a_2}^{b_2} \varphi'_{y_2}(a_1, y_2) F^{-1} \chi_{[0,a_1]\times[0,y_2]} Ff dy_2 \right\|_{L_q} +$$

$$+ \left\| \int_{a_1}^{b_1} \varphi'_{y_1}(y_1, b_2) F^{-1} \chi_{[0,y_1]\times[0,b_2]} Ff dy_1 \right\|_{L_q} +$$

$$+ \left\| \int_{a_1}^{b_1} \varphi'_{y_1}(y_1, a_2) F^{-1} \chi_{[0,y_1]\times[0,a_2]} Ff dy_1 \right\|_{L_q} +$$

$$+ \left\| \int_{a_2}^{b_2} \int_{a_1}^{b_1} \varphi''_{y_1,y_2}(y_1, y_2) F^{-1} \chi_{[0,y_1]\times[0,y_2]} Ff dy_1 dy_2 \right\|_{L_q} \Bigg) \equiv 2\pi \sum_{k=1}^{9} I_k.$$

Moreover, according to (2), we have that

$$I_1 = \prod_{i=1}^{2} a_i^{\frac{1}{p_i} - \frac{1}{q_i}} |\varphi(a_1, a_2)| \frac{1}{\prod_{i=1}^{2} a_i^{\frac{1}{p_i} - \frac{1}{q_i}}} \left\| F^{-1} \chi_{[0,a_1]\times[0,a_2]} Ff \right\|_{L_q} \leqslant Ac_8 \left\| f \right\|_{L_p},$$

where $c_8 = c_6(p, q)$.

Analogously, we find that

$$I_k \leqslant Ac_8 \|f\|_{L_p}, k = 2,3,4.$$

Furthermore, it yields that

$$\begin{aligned} I_5 &= \int_{a_2}^{b_2} b_1^{\beta_1} y_2^{\beta_2} |\varphi'_{y_2}(b_1,y_2)| \frac{1}{b_1^{\beta_1} y_2^{\beta_2}} \left\| F^{-1}\chi_{[0,b_1]\times[0,y_2]} Ff \right\|_{L_q} dy_2 \leqslant \\ &\leqslant \int_0^\infty b_1^{\beta_1} y_2^{\beta_2} |\varphi'_{y_2}(b_1,y_2)| \times \\ &\times \left(\sup_{Q_1\in G_{b_1}} \sup_{Q_2\in G_{y_2}} \frac{1}{|Q_1|^{\beta_1}|Q_2|^{\beta_2}} \left\| F^{-1}\chi_Q Ff \right\|_{L_q} \right) dy_2. \end{aligned} \tag{7}$$

We also note that

$$\sup_{Q_1\in G_{b_1}} \sup_{Q_2\in G_{y_2}} \frac{1}{|Q_1|^{\beta_1}|Q_2|^{\beta_2}} \left\| F^{-1}\chi_Q Ff \right\|_{L_q}$$

is a non-increasing function of y. Hence, by using (7) and the inequality $\int_0^\infty FGdy \leqslant \int_0^\infty F^*G^*dy$, which holds for all nonnegative functions F and G, we have that

$$\begin{aligned} I_5 &\leqslant \int_0^\infty b_1^{\beta_1} \left(z_2^{\beta_2} \varphi'_{z_2}(b_1,z_2) \right)^{*_2} (y_2) \times \\ &\times \left(\sup_{Q_1\in G_{b_1}} \sup_{Q_2\in G_{y_2}} \frac{1}{|Q_1|^{\beta_1}|Q_2|^{\beta_2}} \left\| F^{-1}\chi_Q Ff \right\|_{L_q} \right) dy_2 = \\ &= \int_0^\infty b_1^{\beta_1-\alpha_1} y_2^{1-\alpha_2} \left(z_2^{\beta_2} \varphi'_{z_2}(b_1,z_2) \right)^{*_2} (y_2) \times \\ &\times b_1^{\alpha_1} y_2^{\alpha_2} \left(\sup_{Q_1\in G_{b_1}} \sup_{Q_2\in G_{y_2}} \frac{1}{|Q_1|^{\beta_1}|Q_2|^{\beta_2}} \left\| F^{-1}\chi_Q Ff \right\|_{L_q} \right) \frac{dy_2}{y_2} \leqslant \\ &\leqslant \sup_{y_1\in\mathbb{R}^+} y_1^{\beta_1-\alpha_1} y_2^{1-\alpha_2} \left(z_2^{\beta_2} \varphi'_{z_2}(z_1,z_2) \right)^{*_2} (y_2) \times \\ &\times \int_0^\infty y_1^{\alpha_1} y_2^{\alpha_2} \sup_{Q_1\in G_{y_1}} \sup_{Q_2\in G_{y_2}} \frac{1}{|Q_1|^{\beta_1}|Q_2|^{\beta_2}} \left\| F^{-1}\chi_Q Ff \right\|_{L_q} \frac{dy_2}{y_2} \leqslant \\ &\leqslant A \|f\|_{N^{\alpha,\beta,(\infty,1)}(L_q)}. \end{aligned}$$

Moreover, according to Theorem 2.2 there exists $c_9 > 0$, which depends only on p, q, α, such that

$$I_5 \leqslant Ac_9 \|f\|_{L_{p(\infty,1)}} .$$

Similarly, we find that

$$\begin{aligned} I_6 &\leqslant Ac_9 \|f\|_{L_{p(\infty,1)}} , \\ I_7 &\leqslant Ac_9 \|f\|_{L_{p(1,\infty)}} , \\ I_8 &\leqslant Ac_9 \|f\|_{L_{p(1,\infty)}} , \\ I_9 &\leqslant Ac_9 \|f\|_{L_{p1}} . \end{aligned}$$

Since $L_{p1} \hookrightarrow L_p, L_{p1} \hookrightarrow L_{p(\infty,1)}, L_{p1} \hookrightarrow L_{p(1,\infty)}$ (see [22]), we conclude that there exists $c_{10} > 0$, depending only on p, such that

$$\|f\|_{L_p} \leqslant c_{10} \|f\|_{L_{p1}} .$$

Hence, there exists c_{11}, depending only on p, q and α, such that

$$\left\| \int\limits_{a_2}^{b_2} \int\limits_{a_1}^{b_1} \varphi(y_1, y_2)(Ff)(y_1, y_2) e^{i\langle yz\rangle} dy_1 dy_2 \right\|_{L_q} \leqslant c_{11} A \|f\|_{L_{p1}} \tag{8}$$

for every $f \in S$ and any $0 < a_i < b_i < \infty, i = 1, 2$.

Let the couples of vectors $p_0 = (p_1^0, p_2^0)$, $q_0 = (q_1^0, q_2^0)$ and $p_1 = (p_1^1, p_2^1)$, $q_1 = (q_1^1, q_2^1)$ be such that $1 < p_0 < p < p_1 < \infty$, $1 < q_0 < q < q_1 < \infty$ and

$$\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p_1} - \frac{1}{q_1} = \frac{1}{p} - \frac{1}{q}. \tag{9}$$

Let $E = \{\varepsilon = (\varepsilon_1, \varepsilon_2) : \varepsilon_i \in \{0, 1\}\}$ be the set of corners in the unit cube in $\mathbb{R}^2$, $q_\varepsilon = (q_1^{\varepsilon_1}, q_2^{\varepsilon_2}), p_\varepsilon = (p_1^{\varepsilon_1}, p_2^{\varepsilon_2})$. By (8), we get that there exists $c_{12} > 0$ such that

$$\left\| \int\limits_{a_2}^{b_2} \int\limits_{a_1}^{b_1} \varphi(y_1, y_2)(Ff)(y_1, y_2) e^{i\langle yz\rangle} dy_1 dy_2 \right\|_{L_{q_\varepsilon}} \leqslant c_{12} A \|f\|_{L_{p_\varepsilon 1}}, \varepsilon \in E$$

for every $f \in S$. Choose $\theta = (\theta_1, \theta_2) \in (0, 1)$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. According to (9) we also have that

$$\begin{aligned} \frac{1}{q} &= \frac{1}{p} - \frac{1}{p_0} + \frac{1}{q_0} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} - \frac{1}{p_0} + \frac{1}{q_0} = \\ &= \theta\left(\frac{1}{p_1} - \frac{1}{p_0}\right) + \frac{1}{q_0} = \theta\left(\frac{1}{q_1} - \frac{1}{q_0}\right) + \frac{1}{q_0} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \end{aligned}$$

By interpolation theorem [19] we have that for every $0 < r = (r_1, r_2) \leqslant \infty$

$$\left\| \int_{a_2}^{b_2} \int_{a_1}^{b_1} \varphi(y_1, y_2)(Ff)(y_1, y_2) e^{i\langle yz \rangle} dy_1 dy_2 \right\|_{L_{qr}} \leqslant c_{13} A \left\| f \right\|_{L_{pr}},$$

where $c_{13} > 0$ depends only on p, q, α and r.

If, specifically, $r = p$, then, for some $c_{13} > 0$, which depends only on p, q and α, it yields that

$$\left\| \int_{a_2}^{b_2} \int_{a_1}^{b_1} \varphi(y_1, y_2)(Ff)(y_1, y_2) e^{i\langle yz \rangle} dy_1 dy_2 \right\|_{L_q} \leqslant c_{13} A \left\| f \right\|_{L_p}$$

for all $f \in S$ and for any $0 < a_i < b_i < \infty, i = 1, 2$, because $L_{qp} \hookrightarrow L_q$ if $p < q$.

Since $|\varphi(y_1, y_2)| \leqslant A \prod_{i=1}^{2} |y_i|^{-\frac{1}{p_i} + \frac{1}{q_i}}, y_i \in \mathbb{R} \setminus \{0\}$ and $Ff \in S$, we find that the integral

$$\int_0^\infty \int_0^\infty \varphi(y_1, y_2)(Ff)(y_1, y_2) e^{i\langle yz \rangle} dy_1 dy_2$$

converges absolutely for every $z \in \mathbb{R}^2$.

Hence, for every $z \in \mathbb{R}^2$, it yields that

$$\lim_{n \to \infty} \left| \int_{\frac{1}{n}}^{n} \int_{\frac{1}{n}}^{n} \varphi(y_1, y_2)(Ff)(y_1, y_2) e^{i\langle yz \rangle} dy_1 dy_2 \right| =$$

$$= \left| \int_0^\infty \int_0^\infty \varphi(y_1, y_2)(Ff)(y_1, y_2) e^{i\langle yz \rangle} dy_1 dy_2 \right|,$$

and, by the Fatou theorem,

$$\left\| \int_0^\infty \int_0^\infty \varphi(y_1, y_2)(Ff)(y_1, y_2) e^{i\langle yz \rangle} dy_1 dy_2 \right\|_{L_q} \leqslant$$

$$\leqslant \sup_{n \in N} \left\| \int_{\frac{1}{n}}^{n} \int_{\frac{1}{n}}^{n} \varphi(y_1, y_2)(Ff)(y_1, y_2) e^{i\langle yz \rangle} dy_1 dy_2 \right\|_{L_q} \leqslant c_{13} A \left\| f \right\|_{L_p}.$$

Similarly, we find that

$$\left\| \int_{-\infty}^{0} \int_{-\infty}^{0} \varphi(y_1, y_2)(Ff)(y_1, y_2) e^{i\langle yz \rangle} dy_1 dy_2 \right\|_{L_q} \leqslant c_{13} A \left\| f \right\|_{L_p}.$$

Therefore,

$$\|T_\varphi(f)\|_{L_q} = \left\|F^{-1}\varphi Ff\right\|_{L_q} =$$

$$= \frac{1}{2\pi}\left\|\int\limits_{-\infty}^{\infty} \varphi(y_1,y_2)(Ff)(y_1,y_2)e^{i\langle yz\rangle}dy_1dy_2\right\|_{L_q} \leqslant c_{14}\|f\|_{L_p},$$

for every $f \in S$. This means that

$$\|T_\varphi(f)\|_{L_q} \leqslant c_{14}\|f\|_{L_p}$$

for every $f \in L_p$, i.e. $\varphi \in M_p^q$ and

$$\|\varphi\|_{M_p^q} \leqslant c_{14}.$$

The proof is complete. □

Finally we will present a proof of Remark 2.1.

Proof of Remark 2.1. Let $\gamma = (\gamma_1,\gamma_2)$, $\beta = (\beta_1,\beta_2)$, $1 > \gamma_i > \beta_i$, $i = 1,2$, and

$$\mu_k^i = \left(k^{1-\beta_i} + k^{-\gamma_i} - (k+1)^{-\gamma_i}\right)^{\frac{1}{1-\beta_i}}, k \in N, i = 1,2,$$

where $\beta_i > \frac{1}{p_i} - \frac{1}{q_i}$, and consider

$$\varphi_1(x_1) = \begin{cases} k^{1-\beta_1} + k^{-\gamma_1} - x_1^{1-\beta_1}, & x_1 \in \left[k,\mu_k^1\right], \\ (k+1)^{-\gamma_1}, & x_1 \in \left[\mu_k^1, k+1\right], \end{cases} \qquad k \in N,$$

$$\varphi_2(x_2) = \begin{cases} k^{1-\beta_2} + k^{-\gamma_2} - x_2^{1-\beta_2}, & x_2 \in \left[k,\mu_k^2\right], \\ (k+1)^{-\gamma_2}, & x_2 \in \left[\mu_k^2, k+1\right], \end{cases} \qquad k \in N.$$

We define the function $\varphi = \varphi(x_1,x_2)$ as follows: (10)

$$\varphi(x_1,x_2) = \varphi_1(x_1)\varphi_2(x_2)$$

for $x_i \geqslant 1$, $i = 1,2$,

$$\varphi(x_1,x_2) = 1,$$

for $0 \leqslant x_i \leqslant 1$, $i = 1,2$, and

$$\varphi(x_1,x_2) = \varphi(-x_1,x_2)\varphi(x_1,-x_2)$$

for $x_i < 0$, $i = 1,2$.

The function $\varphi \in AC^{loc}(\mathbb{R}^2\setminus\{0\})$, because it is continuous on $\mathbb{R}^2$, is even by definition and is absolutely continuous.

We now prove that the function φ satisfies the first condition. Let $x_i \in [k,k+1], i = 1,2$. Then

$$\prod_{i=1}^{2}|x_i|^{\frac{1}{p_i}-\frac{1}{q_i}}|\varphi(x_1,x_2)| \leqslant \prod_{i=1}^{2}|x_i|^{\frac{1}{p_i}-\frac{1}{q_i}}|\varphi_1(x_1)|\,|\varphi_2(x_2)| \leqslant$$

$$\leqslant \prod_{i=1}^{2}(k+1)^{\frac{1}{p_i}-\frac{1}{q_i}}\frac{1}{k_i^{\gamma}} \leqslant \prod_{i=1}^{2}\frac{(k+1)^{\frac{1}{p_i}-\frac{1}{q_i}}}{k^{\frac{1}{p_i}-\frac{1}{q_i}}} \leqslant 2.$$

Note that

$$\left|x_2^{\beta_2}\varphi_{x_2}^{'}(x_1,x_2)\right| = \varphi_1(x_1)\begin{cases} 1-\beta_2, & x_2 \in \left(k,\mu_k^2\right), \\ 0, & x_2 \in \left(\mu_k^2, k+1\right), \end{cases} \quad k \in N.$$

Let

$$I_2 = \left\{x_2 : \left|x_2^{\beta_2}\varphi_{x_2}^{'}(x_1,x_2)\right| = 1-\beta_2\right\}.$$

We now prove that the measure of this set

$$|I_2| = \sum_{k=1}^{\infty}\left(\mu_k^2 - k\right) = \sum_{k=1}^{\infty} k\left(\left(1+\frac{1}{k^{1-\beta_2}}\left(k^{-\gamma_2}-(k+1)^{-\gamma_2}\right)\right)^{\frac{1}{1-\beta_2}} - 1\right)$$

is finite.

In fact, by applying the elementary inequality $(1+x)^{\nu}-1 \leqslant \nu 2^{\nu-1}x$ for $\nu \geqslant 1$ and $0 \leqslant x \leqslant 1$ (see e.g. [7]), we obtain that

$$|I_2| \leqslant \frac{2^{\frac{\beta_2}{1-\beta_2}}}{1-\beta_2}\sum_{k=1}^{\infty}\frac{k}{k^{1-\beta_2}}\left(k^{-\gamma_2}-(k+1)^{-\gamma_2}\right).$$

Moreover, since the series $\sum_{k=1}^{\infty} k^{-(1-\beta_2+\gamma_2)}$ converges, by the limit comparison theorem, we find that also the series

$$\sum_{k=1}^{\infty}\frac{k}{k^{1-\beta_2}}\left(k^{-\gamma_2}-(k+1)^{-\gamma_2}\right)$$

converges. Since the function $\left|x_2^{\beta_2}\varphi_{x_2}^{'}(x_1,x_2)\right|$ takes two values namely 0 and $1-\beta$, we have that

$$\left(x_2^{\beta_2}\varphi_{x_2}^{'}(x_1,x_2)\right)^{*_2}(t_2) = \begin{cases} 1-\beta_2, & x_2 \in [0,|I_2|], \\ 0, & x_2 > |I_2|. \end{cases}$$

Hence

$$\sup_{t_i>0} t_1^{\beta_1-\alpha_1}\varphi_1(t_1)t_2^{1-\alpha_2}\cdot\left(x_2^{\beta_2}\varphi_{x_2}^{'}(x_1,x_2)\right)^{*_2}(t) =$$

$$= \sup_{t_1>0} t_1^{\beta_1-\alpha_1}\varphi_1(t_1)\sup_{0<t_2\leqslant 2d_2} t_2^{1-\alpha_2}(1-\beta_2) = 2(1-\beta_2)(|I_2|)^{1-\alpha_2} < \infty,$$

i.e. the second condition holds. The third and fourth conditions can be proved similarly so we omit the details.

On the other hand, if we choose a sequence of arbitrary points $\{x_k\}_{k=1}^{\infty}$, $\{y_k\}_{k=1}^{\infty}$, where $x_k \in \left(k,\mu_k^1\right)$, $y_k \in \left(k,\mu_k^2\right)$, then

$$\lim_{k\to\infty}\left|x_k^{1+\frac{1}{p_1}-\frac{1}{q_1}}y_k^{1+\frac{1}{p_2}-\frac{1}{q_2}}\varphi_{x_1,x_2}^{''}(x_k,y_k)\right| =$$

$$= \lim_{k\to\infty} \left| x_k^{1+\frac{1}{p_1}-\frac{1}{q_1}-\beta_1} y_k^{1+\frac{1}{p_2}-\frac{1}{q_2}-\beta_2} x_k^{\beta_1} \varphi'_{x_1}(x_k) y_k^{\beta_2} \varphi'_{x_2}(y_k) \right| =$$

$$= \lim_{k\to\infty} |x_k|^{1+\frac{1}{p_1}-\frac{1}{q_1}-\beta_1} |y_k|^{1+\frac{1}{p_2}-\frac{1}{q_2}-\beta_2} (1-\beta_1)(1-\beta_2) = \infty.$$

This completes the proof. □

4. Concluding Remarks

Remark 4.1. Already P. Lizorkin himself proved some complementary results of Theorem 1.3, see [13]-[17]. An interesting extension of these results was done by V. Kokilashvili and P. Lizorkin [11] (see also [9]). Moreover, an improvement of (Hörmander's) Theorem 1.2 in terms of spaces of fractional smoothness was presented by O. Besov [4]. Note that in [21] Hörmander's result was generalized to the case concerning anisotropic Lorentz spaces.

Remark 4.2. Two-weighted estimates for Fourier multipliers was first obtained in the important paper [6] by D. E. Edmunds, V. Kokilashvili and A. Meskhi. Their main technique in the proofs consisted of a representation of the operator under consideration in the form of compositions of certain "elementary" transformations.

Remark 4.3. An important argument in our proof is to use interpolation of Sparr type [23] (see also e.g. [3]). In the paper [5] M. Carro used another interpolation technique (Schechter's method) to obtain some other Fourier multiplier results of the type studied in this paper.

Remark 4.4. In this paper we have worked with extension of the Lorentz spaces to a two-dimensional situation [19], [22]. In her PhD thesis [1] S. Barza introduced and studied another multidimensional Lorentz space (see also [2]). We conjecture that our results can be given also in terms of these Lorentz spaces.

Remark 4.5. Partly guided by the result mentioned in Remark 4.2 and other results in [10] it should be interesting to generalize our results to a more general weighted situation. By combining the techniques used in [6] with those in this paper it seems to be possible to further generalize and sharpen the results in [6]. The present authors aim to come back to this question in a forthcoming paper.

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