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ON THE UNIQUENESS OF EXTREMAL FUNCTIONS IN CLASSES OF ENTIRE FUNCTIONS BOUNDED ON THE REAL AXIS

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Abstract. In this paper we investigate the uniqueness problems for:

(1) extremal elements for linear functionals defined in the class (B_σ) of bounded entire functions of finite degree on $R = (-\infty, \infty)$ and

(2) entire functions giving the best approximation in the uniform metric to continuous functions on R using the duality relations for the first time. In both cases, we obtain the solution using the identity properties for entire functions.

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1. Introduction

1.1. Let B_σ denote the class of entire functions of finite degree $\leq \sigma$, bounded on the axis $R = (-\infty, \infty)$. This class was defined by S.N. Bernstein for the first time in [2], where it was proved that

$$|f'(x)| \leq \sigma \|f\|_c = \sigma \sup_{x \in R} |f(x)| \quad (\forall x \in R)$$

for each $f \in B_\sigma$. This inequality reduces to the equality for the function

$$f_0(z) = A \sin(\sigma z + \alpha),$$

where $A = \|f_0\|_c$ and $\alpha \in R$.

After Bernstein"s paper, this inequality was generalized in various directions and transferred onto other functions with various applications (see [1], [3], [4], [6], [13]). In a series of papers, the author has considered all such partial problems from a unified standpoint to show that all of them can be joined into a unique scheme. In the classes of entire functions $W_{\sigma,p}$

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(of order $\rho = 1$ and belonging to L_p on R), $D_{\sigma,\frac{1}{2}}$, $B_{\sigma,\frac{1}{2}}$ (D are the classes of entire functions of order $\rho = \frac{1}{2}$) such general investigations were carried out earlier by the author in [8], [9], [10] and [12]. Investigations involving the uniform metric are complicated due to the absence of a general form for linear functionals in the space $C(-\infty,\infty)$.

Some similar works have been done for the class B_σ using the L_2 space techniques [8]. In [11], the author considered B_σ as a subspace of $C(R)$ and investigated some extremal problems related to the class B_σ (using a uniform metric).

We also note that such a general approach has been used by S.Y. Khavinson [15] for the first time in 1949. Then S.Y. Khavinson, G. Ts. Tumarkin, V. Rogozinskii, A. Macintyre, G. Shapiro, etc. considered extremal problems in the class of analytic functions defined in bounded regions, and obtained very remarkable results (see [16], [17], [18]).

2. In [11], the following classes were defined in order to solve the problem in the class B_σ (see also [10]).

(1) We denote by H_σ the class of entire functions of finite degree $\leq \sigma$.

(2) $V[-\sigma,\sigma] := \left\{ (v : \mathrm{var}(v : [-\sigma,\sigma])) \equiv \overset{\sigma}{\underset{-\sigma}{V}}(v) = \int\limits_{-\sigma}^{\sigma} |dv| < +\infty \right\}$ is the class of functions $v(t)$ with bounded variation on $[-\sigma,\sigma]$. In the case of $V(-\infty,\infty)$ we shall use the notation $V \equiv V_R$.

(3) $M_\sigma = \{f \in H_\sigma : f(z) = \int_{-\sigma}^{\sigma} e^{izt}\varphi(t)dt$, for all $\varphi \in L_1(-\sigma,\sigma)\}$.

(4) $N_\sigma := \{f \in H_\sigma : f(z) = \int_{-\sigma}^{\sigma} e^{izt}dv(t)$, for all $v \in V(-\sigma,\sigma)\}$.

(5) $N_\sigma^+ := \{f \in N_\sigma :$ where v is increasing $\}$.

(6) $N_{\sigma,0} := \{f \in N_\sigma : f(x) \to 0$ as $|x| \to 0\}$.

(7) $W_{\sigma,p} := \{f \in H_\sigma : \|f\|_p^p = \int_R |f(t)|^p dt < \infty,\ p \geq 1\}$.

Let us note the obvious embedding

$$W_{\sigma,p}(1 < p < 2) \subset W_{\sigma,2} \subset M_\sigma \subset N_{\sigma,0} \subset N_\sigma \subset B_\sigma,\ \ w_{\sigma,p}(p \geq 1) \subset B_\sigma,$$

where $M_\sigma \neq N_{\sigma,0}$ is a nontrivial fact ([11], p.46).

3. Our approach is based on a description of annihilators of those classes, which enable us to establish the duality relations, to discover a series of peculiarities of the problems under investigation, and to establish a connection between extremal problems and the best approximation problem.

Definition 1. Let E be a certain linear space and $F \subset E$, E^* is the conjugate space of E. The set of all linear functionals $e \subset E$, each of them vanishing on each element $x \in F$ is called the annihilator of the class F and is denoted by $F^\perp$.

In [11], a) The best approximation by entire functions in the uniform metric of continuous functions on the real axis is associated with a linear extremal problem defined in the classes of entire functions, and

b) an extremal problem in the classes B_σ is related to the best approximation problem of functions of finite varation (measure) and all of them are expressed in the form of duality relations. These kinds of duality relations in classes of entire functions have not been investigated previously. We will investigate the current problem for the first time with particular attention.

In [11] it was proved that (see also [10])

$$M_\sigma^\perp = N_{\sigma,0}^\perp := \left\{ \mu \in V : \widehat{\mu}(x) = \int_{-\infty}^{\infty} e^{izt} d\mu(t) = 0 \quad \text{for} \quad |x| \leq \sigma \right\},$$

and using the last fact, the following duality relations were proved:

1) Let E_σ be one of the classes M_σ or $N_{\sigma,0}$ and $E_\sigma^\perp$ be the annihilator of the class of E_σ. If B_σ is the class of entire function in the class $\sup\limits_{x\in R}|f(x)| \leq 1$ such that $\sup_{x\in R}|f(x)| \leq 1$, then the duality relation

$$\sup_{f\in B_\sigma^1} \left| \int_{-\infty}^{\infty} f(t) d\mu_0(t) \right| = \overset{\mu\in E_\sigma^\perp}{\inf} \int_{-\infty}^{\infty} |d(\mu_0 - \mu)(t)| \tag{1.1}$$

is valid for each $\mu_\sigma(t) \in V_R$.

Furthermore, the extremal elements $f^*(t) \in B_\sigma^1$ and $\mu^*(t) \in E_{\sigma,1}^\perp$ in (1.1) exist.

2) Let C_0 denote the class of continues functions on $R_1 = (-infty, +\infty)$ such that for every f in C_0, $f(x) \to 0$ at $|x| \to \infty$. Then for $\forall w(t) \in C_0/B_\sigma$ the duality relation

$$A_\sigma(w, C) = \inf_{\varphi_\sigma \in B_\sigma} \|w - \varphi_\sigma\|_C = \inf_{\varphi_\sigma \in B_\sigma} \{\max_{t\in R} |w(t) - \varphi_\sigma(t)|\} =$$
$$= \overset{\mu\in E_{\sigma_1,1}^\perp}{\sup} \left| \int_{-\infty}^{\infty} w(t) d\mu(t) \right| \tag{1.2}$$

is valid. Furthermore, the extremal elements $f^*(t) \in B_\sigma^1$ and $\mu^* \in E_{\sigma,1}^\perp$ exist, where

$$E_{\sigma,1}^\perp \equiv \left\{ \mu \in E_\sigma^\perp : \int_{R_1} |d\mu(t) \leq 1| \right\}.$$

3) The property $\mu \perp B_\sigma$ for $\forall \mu \in E_\sigma^\top$ is valid. Here $\mu \perp B_\sigma$ ($\forall \mu \in E_\sigma^\perp$) means that

$$(f, \mu) = \int_{-\infty}^{\infty} f(t) d\mu(t) = 0 \quad \text{for every} \quad f \in B_\sigma \quad \text{and every} \quad \mu \in E_\sigma^\perp.$$

However, the uniqueness of extremal elements has not hitherto been studied. In this paper, we will investigate the uniqueness problem of extremal elements in duality relations (1.1) and (1.2).

We also note that since $L_p(\supset W_{\sigma,p})$ with $1 < p < \infty$ are the strongly normed spaces, such a uniqueness theorem for the classes $W_{\sigma,p}$ $(1 < p < \infty)$ can be easily obtained.

Tchebyshev's criteria and their various generalizations related to the approximation by polynomials problems are well-known. Recently, these kinds of theorems have been proved on the basis on formulas similar to formulas (1.1) and (1.2) (for example, see [14]).

In the case of the approximation by entire functions, S.N. Bernstein's theorem exists ([1], [2], [4], [6] and [13]), but it does not seem to be any remarkable result on this subject in the literature, except this theorem.

There are only a few results relevant to integral norms and no essential results in the uniform norm on this matter at all. We investigate this problem based on formula (1.2) and obtain some results. However, the main difficulty consists in the non-existence of the general form of linear continuous functional in the space of continuous functions defined on R. For this reason, we consider the class C_0 and then solve the problem.

In Section 2, we first attempt to consider the uniqueness problem of extremal functions for the left-hand side of formula (1.1) and then in Section 3, the uniqueness problem of extremal functions for the left-hand side of formula (1.2).

2. On the Uniqueness Problem of the Extremal Function in Formula (1.1)

2.1. If $f^*(t) \in B_\sigma^1$ and $\mu^*(t) \in E_\sigma^\perp$ are the extremal elements in formula (1.1), then we have

$$\left| \int_{-\infty}^{\infty} f^*(t)d\mu_0(t) \right| = \int_{-\infty}^{\infty} |d\mu_0(t) - d\mu^*(t)| = \int_{-\infty}^{\infty} |d(\mu_0 - \mu^*)(t)|. \tag{2.1}$$

If $\mu^*(t) \in E_\sigma^\perp$, then we have $\mu^*(t) \perp B_\sigma$ by item 3) in the previous section. Hence we have the following inequalities:

$$\begin{gathered} \left| \int_{-\infty}^{\infty} f^*(t)d\mu_0(t) \right| = \left| \int_R f^*(t)[d\mu_0(t) - d\mu^*(t)] \right| \le \\ \le \|f^*\|_{C(R_1)} \int_R |d(\mu_0 - \mu^*)(t)|, \\ \left| \int_{-\infty}^{\infty} f^*(t)d\mu_0(t) \right| \le \int_R |d(\mu_0 - \mu^*)(t)|. \end{gathered} \tag{2.2}$$

Comparing equalities (2.1) and (2.2), we can see that the inequality (2.2) transforms to equality (2.1), if and only if the following conditions are satisfied:

1) $\arg[f^*(t)d(\mu_0 - \mu^*(t))] = \beta = \text{const}$;

2) $\|f^*\|_{C(R_1)} = \max\limits_{t\in R_1}|f^*(t)| = 1$;

3) $|f^*(t)| = 1\forall t \in S_v$, where S_v denotes the set of points of growth of measure $v(t) = \mu_0(t) - \mu^*(t)$;

4) $S_v \subset R_{f^*}$ and $R_{f^*} = \{t \in R : |f^*(t)| = \|f^*\|_{C(R)} = 1\}$.

Considering these conditions, it is obviously seen that the formula

$$f^*(t)[d(\mu_0 - \mu^*)(t)] = f^*(t)dv(t) = e^{i\beta}|dv(t)|, \quad \text{for} \quad \forall t \in S \tag{2.3}$$

is valid and hence it is clear that the condition (2.3) is satisfied for $\forall t \in R/S_v$. Thus we have proved the following theorem.

Theorem 1. *$f^*(t) \in B^1_\sigma(\|f^*\|_c = 1)$ and $\mu^*(t) \in E^\perp_\sigma$ are the extremal elements in formula* (1.1), *if and only if the condition* (2.3) *is satisfied for all $t \in R$. Furthermore, we have*

$$f^*(t) = e^{i\beta}\frac{|d(\mu_0 - \mu^*)(t)|}{d(\mu_0 - \mu^*)(t)} \quad (\textit{for some} \quad \beta \in [0, 2)). \tag{2.4}$$

Note. Since the extremal function is of the form (2.4), where β is arbitary, it can be chosen such that

$$\int\limits_{R_1} f^*(t)d\mu_0(t) > 0.$$

2.2. By Theorem 1, we can investigate the uniqueness problem for extremal $f * (t)$.

Suppose that we have two extremal functions f^*_1, f^*_2. In this case, equality (2.3) is valid for both functions. Therefore, we have

$$f^*_1(t)[d\mu_0(t) - d\mu^*(t)] = e^{i\beta}|d\mu_0(t) - d\mu^*(t)| \tag{2.5}$$

and

$$f^*_2(t)[d\mu_0(t) - d\mu^*(t)] = e^{i\alpha}|d\mu_0(t) - d\mu^*(t)|. \tag{2.6}$$

We obtain

$$f^*_1(t) = e^{i\gamma}f^*_2(t) \tag{2.7}$$

for $\forall t \in S_v$ $(v = \mu_0 = \mu^*)$.

This means that all extremal functions are completely distinct with the factor of $e^{i\gamma}$ $(\gamma \in [0, 2\pi))$ at all points t of S_v, and $|f^*_1(t)| = |f^*_2(t)| = 1$ for all $t \in S_v$.

By formula (2.7), we can see that the structure of S_v determines the dependence of the functions $f^*_1(t)$ and $f^*_2(t)$.

Consequently, the uniqueness theorem of extremal functions can be provided using the structure of S_v. Hence, the uniqueness problems of the extremal functions transform to the identity properties of entire transcendental functions. Here two cases may exist.

a) The set S_v $(v = \mu_0 - \mu^*)$ has at least one limit point at $t_0 \neq \pm\infty$. Therefore we can use the property of the uniqueness of analytical functions.

b) The set S_v may have no limit point at $t_0 \neq \pm\infty$.

In [11], it was proved that for each $L > 0$, $\mu^*(t)$ has the growth point t_0 such that $|t_0| > L$ if $\mu^*(t) \in E_\sigma^\perp$ is not constant. This means that the set S_v is not bounded whenever $\mu^*(t) \in E_\sigma^\perp$ is not constant. In the meantime, the existence of measure $\mu(t) \in E_\sigma^\perp$ which is concentrated on a certain sequence $\{t_k\}$, i.e. is a pure jump function with an infinite number of points of growth $\{t_k\}$, was show in [11]. Therefore it is possible to choose a sequence $\tau = \{t_n\}$ in the set S_v $(v = \mu_0 - \mu^*)$ for $\mu^* \in E_\sigma^\perp$ satisfying $t_n \to +\infty$ (or $-\infty$). Then the condition (2.7) holds at all points $\{t_n\} = \tau$.

Suppose now that

$$f^*(tf_1^*(t) - f_2^*(t) \quad \text{and} \quad \gamma = 0.$$

In this case, the condition (2.7) means that $f^*(t_n) = 0$ at all points $\tau\{t_n\}$. Therefore we have to find the conditions which provide that $\forall t : f^*(t) = 0$ follows from $f^*(t_n) = 0$ (see Theorem 2).

Consequently, we now are in a position where the uniqueness theorems for entire transcendental functions can be used. This kind of condition exists in the literature ([5], [7]). Let us now describe some of them. First, we present some required notations we use throughout the paper.

Let $\tau = \{t_n\}$ be a sequence with

$$t_1 \leq t_2 \leq \cdots \leq t_n \leq \cdots .$$

Let $n(R)$ and $N(R)$ denote the density function of the sequence τ and the Nevanlinne function, respectively. Then

$$N(R) = \int_0^R \frac{n(t)}{t} dt = \sum_{|t_n|<R} \ln \frac{R}{|t_n|}.$$

Also, we will use the following notation:

$$M(R) = M_f(R) = \max_{|Z|=R} |f(z)| = \max_{0 \leq t \leq 2\pi} |f(\mathrm{Re}^{it})|;$$

$$S_f(R) = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(\mathrm{Re}^{it})|_d t, \quad h_f(t) = \overline{\lim_{R\to\infty}} \frac{\ln |f(RRe^{it})|}{R}.$$

Theorem 2. *The extremal function $f^*(t)$ for the left-hand side of formula* (1.1) *is unique in the following cases:*

1) *S_v, the set of growth points with measure $v(t) = \mu_0(t) - \mu^*(t)$ has at least one limit point at $t_0 \neq \pm\infty$.*

2) *Let $\tau = \{t_n\} \subset_v v = \mu_0 - \mu^*$ (in the particular case $\tau = \{S_v\}$) and $|t_n| \to \infty$ $(n \to \infty)$. Therewith, for the sequence τ is valid anyone from the conditions the properties identity for entire transcendental functions.*

For example, such conditions are presented below.

Let $f_1^*(t)$ and $f_2^*(t)$ be two extremal functions and $f^*(t) = f_1^*(t) - f_2^*(t)$, $\tau = \{t_n\} \subset S_v$ $(v = \mu_0 - \mu^*)$ and in the particular case $\tau = S_v$ for $|t_n| \to \infty$.

1) $f^*(t_n) = 0$ $n = 1, 2, \dots$ and $\lim\limits_{R\to\infty} [M_{f^*}(R) - N(R)] = -\infty$;

2) $f^*(t_n) = 0$ $n = 1, 2, \dots$ and $\sigma \leq \overline{\lim}_{R\to\infty} \frac{n(R)}{e,R}$ (”$e - \sigma$” conditions);

3) if $t_n = n$ $(n = 0, 1, 2, \dots)$ and $\forall n : f^*(t_n) = 0$, $h_{f^*}\left(\frac{-\pi}{2}\right) + h_{f^*}\left(\frac{\pi}{2}\right) < 2\pi$;

4) if $t_n = n$ $(n = 0, 1, 2, \dots)$ and $\forall n : f^*(t_n) = 0$, besides for $|t| < \frac{\pi}{2}$ the inequality $h_f(t) \leq K(t) = \text{const}\,.\ln(2\cos t) + t.\sin t$, $(\min K(t) = \ln 2)$ is valid;

5) for the same constant θ and c $(0 < \theta < 1)$ the condition $M(R) < N(\theta R) + c(R > R_0)$ is valid; 6) $\lim_{R\to\infty}[S_{f^*}(R) - N(R)] = -\infty$. Then $f^*(t) \equiv 0$, i.e. $f_1^*(t) = f_2^*(t)$.

2.3. Let now $f^*(t)$ be any extremal function. Then for each point $t \in S_v$ we have

$$f^*(t) = e^{i\varphi(t)}|f^*(t)| = e^{i\varphi(t)}.$$

On the other hand, $f^*(t)$ must be an element of B_σ. This implies that in the nontrivial cases $\varphi(t) = \pm\sigma t + \gamma$.

Therefore the extremal functions can be written as

$$f_1^*(t) = e^{i\sigma t + i\gamma} \quad \text{or} \quad f_2^*(t) = e^{-i\sigma t + i\gamma},$$

where $\gamma \in [0, 2\pi)$. On the other hand, in this case, each function in the form

$$f^*(t) = \lambda_1 e^{i\sigma t} + \lambda_2 e^{-i\sigma t} \quad (|\lambda_1| + |\lambda_2| = 1) \tag{2.8}$$

is extremal.

The function $f^*(t)$ in (2.8) becomes $f^*(t) = \sin\sigma t$ if $\lambda_1 = \frac{1}{2i}$, $\lambda_2 = -\frac{1}{2i}$, $(|\lambda_1| + |\lambda_2| = 1)$ and $f^*(t) = \text{const}\,\sigma t$, when $\lambda_1 = \frac{1}{2} = \lambda_2$.

We can write $f^*(t) = \sin(\sigma t + \gamma)$ by combining them. Consequently, if S_v $v = \mu_0 - \mu^*$ satisfies one of the conditions of Theorem 2, the extremal function $f^*(t)$ can be written just as in (2.8).

2.4. Mention now that $\mu_0(t)$ in(1.1) and (1.2) is actually a linear functional in the space C_0, and if the measure $\mu_0(t)$ is chosen appropriately, the functional $l(f)$ determined by $\mu_0(t)$ has a form which was studied by several mathematicians before. For example, $L(f) = f'(x_0)$, $L(f) = Af(x) + Bf'(x)$, $L(f) = f^{(k)}(x_0)$, $L(f)f(x_0 + h) - f(x_0 - h)$ etc. We now focus on one of them and show how the proper solution can be found in the appropriate extremal problem.

Let $\mu_0(t)$ be a pure jump function with jumps

$$d_j = \frac{4\sigma}{\pi^2}\frac{(-1)^j}{(2j+1)^2} \quad (\pm j = 0, 1, 2, \dots)$$

at each points $t_j = (2j+1)\frac{\pi}{2\sigma}$ $(\pm j = 0, 1, 2, \dots)$. Then we have

$$l(f) = \int\limits_{-\infty}^{\infty} f(t)d\mu_0(t) = \frac{4\sigma}{\pi^2}\sum_{j=-\infty}^{\infty}\frac{(-1)^j}{(2j+1)^2} f\Big[(2j+1)\frac{\pi}{2\sigma}\Big] f'(0).$$

If we write $t_j^* = t_j + x_0$ $x_0 \in R$ instead of t_j, then we obtain $l(f) = f'(x_0)$. Hence we may use the interpolation formula

$$f'(x_0) = \frac{4\sigma}{\pi^2}\sum_{j=-\infty}^{\infty}\frac{(-1)^j}{(2j+1)^2} f\Big[(2j+1)\frac{\pi}{2\sigma} + x_0\Big]$$

which is valid for all $f \in B_\sigma$ (see[1]and [4]).

On the other hand, each of $e^{i\sigma t}$, $e^{i\sigma t}$, $\sin\sigma t$, $\cos\sigma t$ are in the class B_σ^1. Since $l(f) = f'(0)$, we have

$$l(e^{\pm i\sigma t}) = \int\limits_{-\infty}^{\infty} e^{\pm} i\sigma t d\mu_0(t) = \widehat{\mu}_0(\pm\sigma) = \pm\sigma;$$

$$l(\sin\sigma t) = l(\cos\sigma t) = \sigma;$$

and consequently¡

$$|l(e^{i\sigma t})| = |l(e^{-i\sigma t})| = |l(\sin\sigma t)| = |l(\cos\sigma t)| = |\widehat{\mu}_0(\pm\sigma)| = \sigma,$$

that is,

$$|l(f^*)| = |\widehat{\mu}_0(\pm\sigma)| = |[f^*(x)]'_{x=0}| = \sigma.$$

This proves that the theorem due S.N. Bernstein on $\sup\limits_{f\in B_\sigma^1}|f'(0)| = \sigma$ can be obtained by using our method.

In addition, the extremal function in S.N. Bernestein's theorem is of the form in (2.8) or

$$f^*(t) = \sin(\sigma t + \gamma).$$

To summarize, if the set S_v of growth points for measure $v = \mu_0 - \mu^*$ satisfies one of the conditions of Theorem 2, then the extremal function is unique for the problem

$$\|l\| = \sup_{f\in B_\sigma^1}|l(f)| = \sup_{f\in B_\sigma^1}|l(f)|\left|\int\limits_R f(td\mu_0(t))\right|.$$

It can be written as in formula (2.8), and also

$$\|l\| = |\widehat{\mu}_0(\pm\sigma)| \tag{2.9}$$

is satisfied.

Consequently, in order to find the norm of $\|l\|$, it is sufficient to find the modulus (absolute value) at the points $\pm\sigma$, of the Fourier transformation of measure $\mu_0(t)$ which determines the functional l.

3. On the Uniqueness Problem for Entire Function Giving the Best Approximation to a Continuous Function under the Uniform Metric

3.1. Let E be a linear normed space, and G be a nonempty subspace of E. In this case, there is a criterion which determines the element giving the best approximation to a given element $w(t) \in E/G$ by means of the E norm-metric in the subspace G.

Theorem 3 ([14], p.18, Theorem 4.2)**.** *The element $x \in G$ gives the best approximation to an element $w(t) \in E/G$, if and only if there exists a functional $f_0 \in E^*$ satisfying the following conditions:*
1) $\|f_0\| = 1$*;*
2) $f_0(w) = f_0(w - x^*) = \|w - x^*\|_E$*;*
3) $f_0 \in G^{\perp}$*, that is* $f_0(x) = 0$ *for all* $x \in G$*.*

In this theorem we suppose $E = C_0 = C_0(R)$, $G = B_\sigma$. Then Theorem 3 means that $\varphi^*_\sigma(t) \in B_\sigma$ is the function which gives the best approximation to a given $w \in C_0/B_\sigma$, if and only if there is a functional $l_0 \in C_0^*$ satisfying the following conditions. In other words, $\mu_0(t)$ is a measure corresponding to l_0 and determines it as a linear functional, if and only if:
1) $\|l_0\| = 1$, that is $\int_R |d\mu_0| = 1$;
2) $l_0(\varphi_\sigma) = 0$ for $\forall \varphi_\sigma(t) \in B_\sigma$, that is $\int_R \varphi_\sigma(t) d\mu_0(t) = 0$ for $\varphi_\sigma(t) \in B_\sigma$ which means that $\mu_0(t) \in E_\sigma^{\perp}$;
3) $l_0(w) = l_0(w - \varphi^*_\sigma) = \|w - \varphi^*_\sigma\|_{C(R)}$. On the other hand, we have

$$l_0(w) = \left| \int_R w(t) d\mu_0(t) \right| = \left| \int_R [w(t) - \varphi^*_\sigma(t)] d\mu_0(t) \right| \le$$
$$\le \int_R |w(t) - \varphi^*_\sigma(t)| \, |d\mu_0(t)| \le \|w - \varphi^*_\sigma\|_{C(R)} = M \qquad (3.1)$$

It can be seen that all inequalities should be transformed to equalities by 1) - 3). This is the case, if and only if the following conditions are satisfied:

a) $\int_R |d\mu_0(t)| = 1$;

b) $\arg\{|w(t) - \varphi^*_\sigma(t)| d\mu_0(t)\} = \beta = \text{const}$;

c) For $\forall t \in S_{\mu_0}$, $|w(t) - \varphi^*_\sigma(t)| = \|w - \varphi^*_\sigma\|_{C(R)} = M$;

d) $S_{\mu_0} \subset R_{w - \varphi_\sigma}$;

e) $\int\limits_R \varphi_\sigma(t)d\mu_0(t) = 0, \quad \forall\varphi_\sigma \in B_\sigma.$

Hence we have

$$\|l_0\| = 1, \quad [w(t) - \varphi_\sigma^*]d\mu_0(t) = e^{i\beta}|w(t) - \varphi_\sigma^*|\,|d\mu_0(t)|$$

and

$$[w(t) - \varphi_\sigma^*]d\mu_0(t) = e^{i\beta}M|d\mu_0(t)| \quad \text{for} \quad \forall t \in S_{\mu_0} \tag{3.2}$$

The last equality holds obviously for all $t \in R\backslash S_{\mu_0}$.

By formula (3.2), taking $\beta = 0$, it follows that

$$\int\limits_R [w(t) - \varphi_\sigma^*]d\mu_0(t) = M\int\limits_R |d\mu_0(t)| = M = \|w - \varphi_\sigma^*\|_{C(R)},$$

that is,

$$l_0(w) = \int\limits_{-\infty}^{\infty} w(t)d\mu_0(t) = \|w - \varphi_\sigma^*\|_{C(R)}. \tag{3.3}$$

This equality means that $\mu_0(t)$ is an extremal element for the right-hand side of formula (1.2). Consequently, the element $\mu^*(t) \in E_{\sigma,1}^{\perp}$ can be employed instead of $\mu_0(t)$ determining the functional l_0 as explained in Theorem 3 (or we can replace $\mu^*(t)$ by $\mu_0(t)$) for all $t \in R$ (in particular, $t \in S_\mu$). Then

$$[w(t) - \varphi_\sigma^*(t)]d\mu^*(t) = e^{i\beta}M|d\mu^*(t)| \quad (\forall\beta \in [0, 2\pi)). \tag{3.4}$$

The extremal elements $\varphi_\sigma^*(t)$ and $\mu^*(t)$ in formula (1.2) are related to equality (3.4). It is necessary that the condition (3.4) satisfied for $\varphi_\sigma^*(t) \in B_\sigma$ and $\mu^*(t) \in E_{\sigma,1}^{\perp}$ to be extremal elements in formula (1.2). It can be easily shown that this condition is sufficient for $\varphi_\sigma^*(t) \in B_\sigma$ to be an entire function, which gives the best approximation to $w(t)$.

Actually, we have for $\forall\varphi_\sigma(t) \in B_\sigma$,

$$M = \int\limits_{-\infty}^{\infty} [w(t) - \varphi_\sigma^*(t)]d\mu^*(t) = \int\limits_{-\infty}^{\infty} [w(t) - \varphi_\sigma(t)]d\mu^*(t) \le$$

$$\le \|w - \varphi_\sigma\|_C \int\limits_{-\infty}^{\infty} |d\mu^*(t)| = \|w - \varphi_\sigma\|_C,$$

that is,

$$\|w - \varphi_\sigma^*\|_C \le \|w - \varphi_\sigma\|_C.$$

Thus we have proved the following theorem.

Theorem 4. *The elements $\varphi^*_\sigma(t) \in B_\sigma$ and $\mu^*(t) \in E^{\perp}_{\sigma,1}$ are extremal for the duality expressions* (1.2), *if and only if these elements satisfy the conditions* $(3,4)$ (*in particular, for all $t \in S_{\mu^*}$*) *at all points $t \in R$ and*

$$w(t) - \varphi^*_\sigma(t) = e^{i\beta} M \frac{|d\mu^*(t)|}{d\mu^*(t)} \tag{3.5}$$

3.2. Consider the uniqueness problem for an entire function $\varphi^*_\sigma(t)$ which gives the best approximation based on Theorem 4.

Assume that we have two extremal functions $\varphi^*_1(t)$ and $\varphi^*_2(t)$ with the extremal measure $\mu^(t) \in E^{\perp}_{\sigma,1}$. Then both of them satisfy the condition (3.4), when $\beta = 0$. Then we have

$$[w(t) - \varphi^*_1(t)]d\mu^*(t) = M|d\mu^*(t)|$$
$$[w(t) - \varphi^*_2(t)]d\mu^*(t) = M|d\mu^*(t)|.$$

From these two conditions it is seen that

$$[\varphi^*_2(t) - \varphi^*_1(t)]d\mu^*(t) = 0.$$

But since the point growths of μ^* satisfies $d\mu^*(t) \neq 0$ at each t, the equality

$$\varphi^*_2(t) = \varphi^*_1(t)$$

holds for all $t \in S_{mu^*}$.

Thus we have proved the following theorem.

Theorem 5. *All entire functions $\varphi^*_\sigma(t)$ which give the best approximation to the given function $w(t) \in C_0 \backslash B_\sigma$ are identical to S^*_μ.*

If

1) *the set S^*_μ has at least one limit point $t_0 \neq \pm\infty$ or*

2) *one of the conditions of the uniqueness theorem* (*for example, see Theorem* 2, 1^0-6^0) *required for entire transcendent al functions is satisfied, then $\varphi^*_\sigma \in B_\sigma$ which gives the best approximation to $w(t) \in C_0 \backslash B_\sigma$ is unique.*

3.3. Consider now separately the uniqueness problems of entire functions that give the best approximation to $w(t)$ and have real values on R_1.

In this case, condition b)(given in the previous conditions (a-e) before formula (3.2)) can be expressed in the form

b) If $\tau = \{t_j\} \subset S_\mu$, then $[w(t_j) - \varphi^*_\sigma(t_j)] = \operatorname{sign} d\mu^*(t_j)$ for all $t_j \in S_\mu$, where $d\mu^*(t_j) = d_j$ denotes the jump of $\mu^*(t)$ at the point t_j.

On the other hand,for all $\varphi_\sigma \in B_\sigma$,

$$\int\limits_{-\infty}^{\infty} \varphi_\sigma(t) d\mu^*(t) = 0,$$

which means that

$$\sum_j \varphi_\sigma(t_j)\mu^*(t_j) = \sum_j \varphi_\sigma(t_j) d_j = 0. \tag{3.6}$$

Let

$$[w(t_j) - \varphi_\sigma^*(t_j)] = (-1)^j M = (-1)^j \|w - \varphi_s g^*\|_C.$$

Then $d_j = (-1)^j |d_j| = (-1)^j \lambda_j$ for $j = 1, 2, \dots$, and the condition in (3.6) becomes

$$\sum_j \varphi_\sigma(t_j) \lambda_j \operatorname{sign}[w(t_j) - \varphi_\sigma^*(t_j)] = 0$$

and

$$\sum_j \frac{\varphi_\sigma(t_j)\lambda_j}{w(t_j) - \varphi_\sigma^*(t_j)} = 0, \quad \forall \varphi_\sigma \in B_\sigma. \tag{3.7}$$

Therefore if $\varphi_\sigma^*(t)$ is an entire function giving the best approximation to $w(t)$, then the condition (3.7) is necessarily satisfied. In order to show that this condition is sufficient for $\varphi_\sigma^*(t_j)$ being an entire function giving the best approximation to $w(t)$, we show that the conditions of a well-known theorem of S. N. Bernstein (see: [2], p. 371-379) can be deduced from the condition (3.7). So we have to show that each sequence $\tau = \{t_j\}$ satisfying the condition in (3.7) in the meaning of Bernstein, is an entire sequence of degree σ.

Note. The sequence $\tau = \{t_j\}$ is said to be an entire sequence of degree σ, if there does not exist an entire function $\varphi_\sigma(t) \in B_\sigma$ such that $(-1)^j \varphi_\sigma(t_j) \leq 1$ $(j = 1, 2, \dots)$.

Assume that there exists a function $\varphi_\sigma^0 \in B_\sigma$ such that $(-1)^j \varphi_\sigma^0(t_j) \leq 1$ $j = 1, 2, \dots$, although the condition in (3.7) for $\forall \varphi_\sigma \in B_\sigma$ is satisfied.Then

$$0 < \frac{\lambda_j}{M} = \frac{\lambda_j}{|w(t_j) - \varphi_\sigma^*(t_j)|} = \frac{(-1)^j \lambda_j}{w(t_j) - \varphi_\sigma^*(t_j)}, \quad \lambda_j = |d_j|.$$

It follows that for $j = 1, 2, \dots,$

$$\frac{(-1)^j \varphi_\sigma^)(t_j)\lambda_j(-1)^j}{w(t_j) - \varphi_\sigma^*(t_j)} \geq \frac{\lambda_j}{M}.$$

Then we have

$$\sum_j \frac{\varphi_\sigma^0(t_j)\lambda_j}{w(t_j) - \varphi_\sigma^*(t_j)} \geq \frac{1}{M} > 0.$$

But this contradicts (3.7).

Thus we have proved the following theorem.

Theorem 6. *Let $\tau = \{t_j\} \subset S_{\mu^*}$, $\mu^* \in E_\sigma^\perp$ and let the difference $w(t) - \varphi_\sigma^*(t)$ take the value $M = \|w - \varphi_\sigma^*\|_C$ by changing the sign at the points $\tau = \{t_j\}$, respectively. Then the function $\varphi_\sigma^*(t) \in B_\sigma$ gives the best approximation to the function $w(t) \in C_0 \backslash B_\sigma$, if and only if the condition (3.7) is satisfied for all $\varphi_\sigma \in B_\sigma$. Furthermore, the condition (3.7) is sufficient for order there to be an entire set of the sequence $\tau = \{t_j\}$ of degree σ.*

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