

**SOME LOCAL APPROXIMATION PROPERTIES OF THE
 CESÀRO MEANS OF TRIGONOMETRIC FOURIER
 SERIES**

D. MAKHARADZE

ABSTRACT. Some approximation properties of the Cesàro means of trigonometric Fourier series are studied.

რეზოუქჟა. ნამრავში დადგენილია ფურიეს ტრიგონომეტრიული მწკრივის $\alpha \geq 1$ რიგის ჩეზარის საშუალოების აპროქსიმაციული ოვა-სებები, როგორც წერტილური, ასევე $[a, b] \subset [-\pi, \pi]$ სუბჟექტები.

1. Assume that $T = [-\pi, \pi]$ and the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ are periodic, of period 2π , where $\mathbb{R} =]-\infty, +\infty[$. If a function $f \in L(T)$, then the symbol $\sigma[f]$ denotes a trigonometric Fourier series of f , i.e.,

$$\sigma[f](x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx,$$

where

$$a_k \equiv a_k(f) = \frac{1}{\pi} \int_T f(t) \cos kt dt,$$

$$b_k \equiv b_k(f) = \frac{1}{\pi} \int_T f(t) \sin kt dt.$$

The symbol $\sigma_n^\alpha(x, f)$ denotes the mean of order α of the series $\sigma[f]$, i.e.,

$$\sigma_n^\alpha(x, f) = \frac{1}{\pi} \int_0^\pi [f(x+t) + f(x-t)] K_n^\alpha(t) dt,$$

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where

$$\begin{aligned} K_n^\alpha(t) &= \frac{1}{A_n^\alpha} \sum_{k=0}^n A_{n-k}^{n-1} D_k(t), \\ D_k(t) &= \frac{1}{2} + \sum_{k=1}^n \cos kt \end{aligned}$$

and

$$A_k^\alpha = 1, \quad A_k^\alpha = \frac{(\alpha+1)\cdots(\alpha+k)}{k!}, \quad k \in \mathbb{N}, \quad \alpha > -1.$$

Denote by Φ the class of all functions $\omega : [0, \pi] \rightarrow R$ with the following properties:

1. ω is continuous on $[0, \pi]$;
2. $\omega \uparrow, t \uparrow$;
3. $\omega(0) = 0$;
4. $\omega(t) > 0, 0 < t \leq \pi$.

Assume further that

$$\varphi(x, t) = f(x+t) + f(x-t) - 2f(x).$$

Remark. 1. In the sequel, the symbols $A(x), A(x, f), A(x, f, \alpha, \eta), \dots$ denote positive finite values depending on the specific parameters.

N. Obreshkoff showed that if $0 < \alpha < 1, 0 < \eta \leq \pi, \beta > \alpha$ and

$$|\varphi(x, t)| \leq A(f, x)t^\alpha, \quad 0 < t \leq \eta,$$

at the point $x \in T$, then

$$|\sigma_n^\beta(x, f) - f(x)| \leq A_1(f, x, \beta, \alpha, \eta) \cdot \frac{1}{n^\alpha}.$$

T. Flett established that if $0 < \alpha < 1, 0 < \eta \leq \pi$ and

$$\int_0^t |\varphi(x, s)| ds \leq A(f, x)t^\alpha, \quad 0 < t \leq \eta,$$

at the point $x \in T$, then

$$|\sigma_n^\alpha(x, f) - f(x)| \leq A_1(f, x, \alpha, \eta) \cdot \frac{1}{n^\alpha}.$$

2. The conclusions generalizing the results of N. Obrechkoff [3] and T. Flett [2] are made.

Theorem 1. *Let $f \in L(T)$, $\omega \in \Phi$, $\alpha \in [1; +\infty[$. If*

$$\int_0^t |\varphi(x, s)| ds \leq A(f, x)t\omega(t), \quad 0 < t \leq \eta \leq \pi,$$

at a point $x \in T$. Then

$$|\sigma_n^\alpha(x, f) - f(x)| \leq A(f, x, \alpha, \eta) \frac{1}{n} \int_{\frac{1}{n}}^{\eta} \frac{\omega(t)}{t^2} dt, \quad n \geq n_0(\eta).$$

Proof. Using the equality $A_k^\alpha = \sum_{j=0}^k A_{k-j}^{\alpha-1}$, $A_k^{\alpha-1} = A_k^\alpha - A_{k-1}^\alpha$, and the Cesàro and Dirichlet expressions for the kernel, we can write

$$\frac{2}{\pi} \int_0^\pi K_n^\alpha(t) dt = 1.$$

Next, using the definition of $\sigma_n^\alpha(x, f)$ we can write

$$\begin{aligned} \sigma_n^\alpha(x, f) - f(x) &= \frac{1}{\pi} \int_0^\pi [f(x+t) + f(x-t) - 2f(x)] K_n^\alpha(t) dt \equiv \\ &\equiv \frac{1}{\pi} \int_0^\pi \varphi(x, t) K_n^\alpha(t) dt. \end{aligned} \quad (1)$$

Let $\alpha = 1$. The following relations are known to be valid:

$$K_n^1(t) \equiv K_n(t) = \frac{1}{2(n+1)} \left[\frac{\sin((n+1)\frac{t}{2})}{\sin \frac{t}{2}} \right]^2, \quad (2)$$

$$0 \leq K_n(t) < n+1, \quad t \in T, \quad (3)$$

$$K_n(t) \leq \frac{A}{nt^2}, \quad 0 < t \leq \pi. \quad (4)$$

Taking inequality (3) into account, by virtue of S. Bernstein's inequality, we obtain

$$|[K_n(t)]'| \leq (n+1)^2. \quad (5)$$

According to equality (1), we have

$$\begin{aligned} \sigma_n^1(x, f) - f(x) &= \frac{1}{\pi} \int_0^{\frac{1}{n}} \varphi(x, t) K_n(t) dt + \frac{1}{\pi} \int_{\frac{1}{n}}^{\eta} \varphi(x, t) K_n(t) dt + \\ &+ \frac{1}{\pi} \int_\eta^\pi \varphi(x, t) K_n(t) dt \equiv \sum_{j=1}^3 U_j(f, x, \eta, n), \quad n \geq n_0(\eta). \end{aligned} \quad (6)$$

Using relation (3) and the condition of the theorem, we obtain

$$|U_1(f, x, n)| \leq A \cdot n \int_0^{\frac{1}{n}} |\varphi(x, t)| dt \leq A(f, x) \omega\left(\frac{1}{n}\right). \quad (7)$$

By virtue of inequality (4), equality (6) implies

$$|U_2(f, x, \eta, n)| \leq \frac{A}{n} \int_{\frac{1}{n}}^{\eta} \frac{|\varphi(x, t)|}{t^2} dt.$$

Applying the formula of integration by parts, we have

$$\begin{aligned} & \frac{1}{n} \int_{\frac{1}{n}}^{\eta} \frac{|\varphi(x, t)|}{t^2} dt = \\ & = \frac{1}{n} \left[\int_0^t |\varphi(x, s)| ds \frac{1}{t^2} \right]_{\frac{1}{n}}^{\eta} + \frac{2}{n} \int_{\frac{1}{n}}^{\eta} \left[\int_0^t |\varphi(x, s)| ds \right] \frac{dt}{t^3}. \end{aligned} \quad (8)$$

Hence, from the conditions of the theorem, we obtain

$$\begin{aligned} & \frac{1}{n} \int_{\frac{1}{n}}^{\eta} |\varphi(x, t)| \frac{dt}{t^2} \leq \\ & \leq A(f, x) \omega\left(\frac{1}{n}\right) + A(f, x, \eta) \frac{\omega(\eta)}{n} + A_1(f, x) \frac{1}{n} \int_{\frac{1}{n}}^{\eta} \frac{\omega(t)}{t^2} dt. \end{aligned}$$

By virtue of relation (8), we can write

$$\begin{aligned} & |U_2(f, x, \eta, n)| \leq \\ & \leq A(f, x) \omega\left(\frac{1}{n}\right) + A(f, x, \eta) \frac{\omega(\eta)}{n} + A_1(f, x) \frac{1}{n} \int_{\frac{1}{n}}^{\eta} \frac{\omega(t)}{t^2} dt. \end{aligned} \quad (9)$$

In view of relations (6) and (4) we conclude that

$$|U_3(f, x, \eta, n)| \leq \frac{A(f, x, \eta)}{n}. \quad (10)$$

It can be easily verified that

$$\omega\left(\frac{1}{n}\right) \leq \frac{A(\eta)}{n} \int_{\frac{1}{n}}^{\eta} \frac{\omega(t)}{t^2} dt. \quad (11)$$

Thus keeping in mind relations (6), (7), (9), (10) and (11), we conclude that the theorem is valid for $\alpha = 1$.

Let now $\alpha > 1$. Taking into account the relations for $D_n(t)$ and $K_n^\alpha(t)$, we obtain

$$\begin{aligned} K_n^\alpha(t) &= \operatorname{Im} \frac{1}{2A_n^\alpha \sin \frac{t}{2}} \sum_{k=0}^n A_{n-k}^{\alpha-1} e^{i(k+\frac{1}{2})t} = \\ &= \operatorname{Im} \left[\frac{e^{i(k+\frac{1}{2})t}}{2A_n^\alpha \sin \frac{t}{2}} \left[\sum_{k=0}^n A_k^{\alpha-1} e^{-ikt} \right] \right], \quad 0 < t < \pi. \end{aligned} \quad (12)$$

According to the well-known Abel transformation, for all real integers $a_0, a_1, a_2, \dots, a_n, \dots$ and $b_0, b_1, \dots, b_n, \dots$ we have

$$\sum_{k=p}^n a_k b_k = \sum_{k=p}^{n-1} (a_k - a_{k+1}) s_k + a_n s_n - a_p s_{p-1}, \quad 0 \leq p < n, \quad (13)$$

where $s_k = b_0 + \dots + b_k$; also, if $a_k \downarrow 0$, $k \uparrow \infty$ and $s_k \leq A$, $k = 0, 1, \dots, n$, then

$$\sum_{k=p}^n a_k b_k \leq 2a_p A. \quad (14)$$

Therefore, using the Abel transformation and the formulas $A_k^\alpha = \sum_{j=0}^k A_{k-j}^{\alpha-1}$, $A_k^{\alpha-1} = A_k^\alpha - A_{k-1}^\alpha$, we obtain

$$\begin{aligned} &\sum_{k=0}^n A_k^{\alpha-1} e^{-ikt} = \\ &= \sum_{k=0}^{n-1} (A_k^{\alpha-1} - A_{k+1}^{\alpha-1}) \frac{1 - e^{-i(k+1)t}}{1 - e^{-it}} + A_n^{\alpha-1} \frac{1 - e^{-i(n+1)t}}{1 - e^{-it}} = \\ &= - \sum_{k=0}^{n-1} A_{k+1}^{\alpha-2} \frac{1 - e^{-i(k+1)t}}{1 - e^{-it}} + A_n^{\alpha-1} \frac{1 - e^{-i(n+1)t}}{1 - e^{-it}} = \\ &= \sum_{k=0}^n A_k^{\alpha-2} \frac{e^{-ikt}}{1 - e^{-it}} + A_n^{\alpha-1} \frac{e^{-i(n+1)t}}{1 - e^{-it}} = \\ &= \frac{1}{(1 - e^{-it})} - A_n^{\alpha-1} \frac{e^{-i(n+1)t}}{1 - e^{-it}} - \sum_{k=n+1}^{\infty} A_k^{\alpha-2} \frac{e^{-ikt}}{1 - e^{-it}}. \end{aligned} \quad (15)$$

If $\alpha \in]1, 2[$, then by virtue of equalities (12) and (15) we write

$$\begin{aligned} K_n^\alpha(t) &= \operatorname{Im} \left[\frac{e^{i(n+\frac{1}{2})t}}{2A_n^\alpha \sin \frac{t}{2}} \frac{1}{(1-e^{-it})^\alpha} - \frac{A_n^{\alpha-1} e^{-i\frac{t}{2}}}{2A_n^\alpha \sin \frac{t}{2} (1-e^{-it})} - \right. \\ &\quad \left. - \frac{e^{i(n+\frac{1}{2})t} \sum_{k=n+1}^{\infty} A_k^{\alpha-2} e^{-ikt}}{2A_n^\alpha \sin \frac{t}{2} \cdot (1-e^{-it})} \right] \equiv \\ &\equiv \operatorname{Im} [R_1(n, t, \alpha) + R_2(n, t, \alpha) + R_3(n, t, \alpha)]. \end{aligned} \quad (16)$$

We have

$$\operatorname{Im} R_1(n, t, \alpha) = \frac{\sin [(n + \frac{1}{2} + \frac{\alpha}{2})t - \frac{\pi}{2}\alpha]}{2^{1+\alpha} A_n^\alpha (\sin \frac{t}{2})^{1+\alpha}}, \quad (17)$$

$$\operatorname{Re} R_1(n, t, \alpha) = \frac{\cos [(n + \frac{1}{2} + \frac{\alpha}{2})t - \frac{\pi}{2}\alpha]}{2^{1+\alpha} A_n^\alpha (\sin \frac{t}{2})^{1+\alpha}}. \quad (18)$$

Taking into account the equality $A_0^\alpha = 1$, $A_k^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}{k!}$, $k \in \mathbb{N}$, $\alpha > -1$, by virtue of (16), we obtain

$$\operatorname{Im} R_2(n, t, \alpha) = \frac{\alpha}{(n + \alpha)(2 \sin \frac{t}{2})^2} \quad (19)$$

and

$$\operatorname{Re} R_2(n, t, \alpha) = 0. \quad (20)$$

Since $\alpha \in]1, 2[$, we have $A_k^{\alpha-2} > 0$ and $A_k^{\alpha-2} \downarrow$, $k \uparrow \infty$. Also,

$$\begin{aligned} &\left| \sum_{k=0}^m \frac{e^{-ikt}}{1-e^{it}} \right| = \\ &= \left| \frac{1-e^{-i(m+1)t}}{(1-e^{-it})^2} \right| = \left| \frac{1-e^{-i(m+1)t}}{4 \sin^2 \frac{t}{2} |e^{i(\frac{\pi}{2}-\frac{t}{2})}|^2} \right| \leq \frac{1}{2 \sin^2 \frac{t}{2}}. \end{aligned} \quad (21)$$

Therefore, if we take into account the expression A_n^α and relations (13), (16) and (21), we will come to

$$|\operatorname{Im} R_3(n, t, \alpha)| \leq \frac{(\alpha-1)\alpha}{2(n-1+\alpha)(n+\alpha) \sin^3 \frac{t}{2}}, \quad (22)$$

$$|\operatorname{Re} R_3(n, t, \alpha)| \leq \frac{(\alpha-1)\alpha}{2(n-1+\alpha)(n+\alpha) \sin^3 \frac{t}{2}}. \quad (23)$$

If $\alpha \in]2, 3[$, then applying again the Abel transformation of the expression

$$\sum_{k=0}^n A_k^{\alpha-2} \frac{e^{-ikt}}{1-e^{-it}} - A_n^{\alpha-1} \frac{e^{-i(n+1)t}}{1-e^{-it}}$$

(obtained by the Abel transformation), we obtain

$$\begin{aligned} & \sum_{k=0}^n A_k^{\alpha-2} \frac{e^{-ikt}}{1-e^{-it}} - A_n^{\alpha-1} \frac{e^{-i(n+1)t}}{1-e^{-it}} = \\ & = \sum_{k=0}^n A_k^{\alpha-3} \frac{e^{-ikt}}{(1-e^{-it})^2} - A_n^{\alpha-2} \frac{e^{-i(n+1)t}}{(1-e^{-it})^2} - A_n^{\alpha-1} \frac{e^{-i(n+1)t}}{1-e^{-it}}. \end{aligned}$$

Generally speaking, if $\alpha \in]1, +\infty[$ is not an integer and $[\alpha] = j$, then by the j -tuple application of the Abel transformation (in the respective sums), we conclude that for the kernels $K_n^\alpha(t)$ and $\tau_n^\alpha(t)$ the relations

$$K_n^\alpha(t) = \varphi_n^\alpha(t) + r_n^\alpha(t), \quad (24)$$

$$\tau_n^\alpha(t) = \frac{1}{2} \operatorname{ctg} \frac{t}{2} + \psi_n^\alpha(t) + \gamma_n^\alpha(t), \quad |t| \in]0, \pi] \quad (25)$$

are fulfilled, where

$$\varphi_n^\alpha(t) = \frac{\sin [(n + \frac{1}{2} + \frac{\alpha}{2})t - \frac{\alpha\pi}{2}]}{A_n^\alpha (2 \sin \frac{t}{2})^{1+\alpha}}, \quad (24')$$

$$\psi_n^\alpha(t) = -\frac{\cos [(n + \frac{1}{2} + \frac{\alpha}{2})t - \frac{\alpha\pi}{2}]}{A_n^\alpha (2 \sin \frac{t}{2})^{1+\alpha}}. \quad (25')$$

Also,

$$\|K_n^\alpha\|_c \leq A(\alpha)n,$$

$$\|\tau_n^\alpha\|_c \leq A(\alpha)n,$$

$$|r_n^\alpha(t)| \leq \frac{A(\alpha)}{nt^2}, \quad \frac{1}{n} \leq t \leq \pi,$$

$$|\gamma_n^\alpha(t)| \leq \frac{A(\alpha)}{nt^2}, \quad \frac{1}{n} \leq t \leq \pi.$$

Remark. The above conclusion is based on formulas (24), (25), (16)–(23) provided that the following facts are taken into account:

a) $\frac{1}{A_n^\alpha} \leq \frac{A(\alpha)}{n^\alpha};$

b) since for the variables $t \in [0, \pi]$

$$\sin \frac{t}{2} \geq \frac{t}{\pi} \quad (26)$$

for each integer $\lambda \geq \alpha + 1$

$$\frac{1}{(\sin \frac{t}{2})^\lambda} \leq \frac{A(\lambda)}{t^\lambda}$$

and also, when $\mu > 1$,

$$\frac{1}{n^\mu t^{\mu+1}} \leq \frac{1}{nt^2}, \quad \frac{1}{n} \leq t \leq \pi;$$

c) it is obvious that for each integer $\alpha \in]0, +\infty[$

$$|[K_n^\alpha(t)]'| \leq A(\alpha)n^2, \quad |[\tau_n^\alpha(t)]'| \leq A(\alpha)n^2. \quad (27)$$

Since the relations $A_k^\alpha = \sum_{j=0}^k A_{k-j}^{\alpha-1}$, $A_k^{\alpha-1} = A_k^\alpha - A_{k-1}^\alpha$, $A(\alpha) \leq k^{-\alpha} A_k^\alpha \leq A_1(\alpha)$, ($A_k^\alpha \sim k^\alpha$) remain valid for nonintegers $\alpha \in]1, +\infty[$, we can state that the theorem is proved for these numbers α , too.

Let $\alpha = 2$. Then, using in the formula $K_n^\alpha(t)$ the expression $A_0^\alpha = 1$, $A_k^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}{k!}$, $k \in \mathbb{N}$, $\alpha > -1$ and relation (2), we obtain

$$\begin{aligned} K_n^2(t) &= \frac{2}{(n+1)(n+2)} \sum_{k=0}^n (n-k+1) D_k(t) = \\ &= \frac{2n}{n+2} K_n(t) + \frac{2}{n+2} K_n(t) - \frac{2}{(n+1)(n+2)} \sum_{k=0}^n k D_k(t). \end{aligned} \quad (28)$$

By the Abel transformation, we conclude that

$$\sum_{k=0}^n k D_k(t) = - \sum_{k=0}^{n-1} (k+1) K_k(t) + n(n+1) K_n(t). \quad (29)$$

Now, applying relations (3)–(5) and (28), we can write

$$0 \leq K_n^2(t) \leq An, \quad t \in T, \quad (30)$$

$$K_n^2(t) \leq \frac{A}{nt^2}, \quad 0 < t \leq \pi, \quad (31)$$

$$|[K_n^2(t)]'| \leq An^2, \quad t \in T. \quad (32)$$

Assume that $\alpha = m \geq 3$, $m \in \mathbb{N}$. According to formula A_n^α , we have

$$A_n^m = \frac{(m+1)\cdots(m+n)}{n!} = \frac{(n+1)\cdots(n+m)}{m!}, \quad (33)$$

$$\begin{aligned} A_{n-k}^{m-1} &= \frac{m(m+1)\cdots(n-k+1)\cdots(n-k+m-1)}{(n-k)!} = \\ &= \frac{(n-k+1)\cdots(n-k+m-1)}{(m-1)!}. \end{aligned} \quad (34)$$

Using now the notation $K_n^\alpha(t)$, we obtain

$$\begin{aligned} K_n^m(t) &= \\ &= \frac{m}{(n+1)\cdots(n+m)} \sum_{k=0}^n (n-k+1)\cdots(n-k+m-1) D_k(t). \end{aligned} \quad (35)$$

Assuming that estimates (30), (31) and (32) hold for the kernel $K_n^j(t)$, $1 \leq j \leq m-1$, and applying the Abel transformation, from relation (35) (with respect to the term containing in the sum the expression of k^{m-1}) it follows that

$$0 \leq K_n^\alpha(t) \leq A(\alpha)n, \quad t \in T, \quad (36)$$

$$\left. \begin{aligned} K_n^\alpha(t) &\leq \frac{A(\alpha)}{nt^2}, & 0 < t \leq \pi, \\ |[K_n^2(t)]'| &\leq A(\alpha)n^2, & t \in T. \end{aligned} \right\} \quad (37)$$

Thus the theorem is proved. \square

Theorem 2. Assume that ω and α satisfy the conditions of Theorem 1. If $[a, b] \subset T$, $b - a > 0$ and

$$\sup_{a \leq x \leq b} \int_0^t |\varphi(x, s)| ds \leq A(f, a, b)t\omega(t), \quad 0 < t \leq \eta \leq \pi, \quad (*)$$

then

$$\sup_{a \leq x \leq b} |\sigma_n^\alpha(x, f) - f(x)| \leq A(f, a, b, \alpha, \eta) \frac{1}{n} \int_{\frac{1}{n}}^\eta \frac{\omega(t)}{t^2} dt, \quad n \geq n_0(\eta).$$

Remark 1. We always bear in mind that $f(x)$ at the point $x \in T$ is a finite natural number. As for the condition $(*)$ of Theorem 2, it guarantees the boundedness of the function f in the interval $[a, b]$. Indeed, it is evident that

$$2f(x) = -[f(x+t) + f(x-t) - 2f(x)] + f(x+t) + f(x-t),$$

whence

$$\begin{aligned} 2|f(x)|\eta &\leq \int_0^\eta |\varphi(x, t)| dt + \int_0^\eta |f(x+t)| dt + \int_0^\eta |f(x-t)| dt \\ &\leq A(f)\eta\omega(\eta) + 2 \int_T^\eta |f(t)| dt, \quad \eta \in]0, \pi]. \end{aligned}$$

Theorem 3. Let $\omega \in \Phi$, $0 < \alpha < 1$, $f \in L(T)$. Assume that the condition

$$\int_0^t |d\varphi(x, s)| \leq A(f, x)\omega(t), \quad 0 < t \leq \eta \leq \pi$$

is fulfilled at a point $x \in T$. Then

$$|\sigma_n^\alpha(x, f) - f(x)| \leq A(f, x, \alpha, \eta) \left[\frac{1}{n^\alpha} + \frac{1}{n} \int_{\frac{1}{n}}^\eta \frac{\omega(t)}{t^2} dt \right], \quad n \geq n_0(\eta).$$

Proof. We have

$$\begin{aligned}
\sigma_n^\alpha(x, f) - f(x) &= \frac{1}{\pi} \int_0^\pi \varphi(x, t) K_n^\alpha(t) dt = \\
&= \frac{1}{\pi} \int_0^{\frac{1}{n}} \varphi(x, t) K_n^\alpha(t) dt + \frac{1}{\pi} \int_{\frac{1}{n}}^\eta \varphi(x, t) K_n^\alpha(t) dt + \frac{1}{\pi} \int_\eta^\pi \varphi(x, t) K_n^\alpha(t) dt \equiv \\
&\equiv \sum_{j=1}^3 V_j(f, x, \alpha, \eta, n).
\end{aligned} \tag{38}$$

Taking into account the inequality $\|K_n^\alpha\|_c \leq A(\alpha)n$, relation (27) and the estimation scheme of $U_1(f, x, n)$, we conclude that (7) is valid for $V_1(f, x, \alpha, n)$, too. Thus

$$|V_1(f, x, \alpha, n)| \leq A(f, x, \alpha) \omega\left(\frac{1}{n}\right). \tag{39}$$

Using the inequalities $|r_n^\alpha(t)| \leq \frac{A(\alpha)}{nt^2}$, $1/n \leq t \leq \pi$, and relations (24'), (26), we obtain

$$|K_n^\alpha(t)| \leq \frac{A(\alpha)}{n^\alpha t^{1+\alpha}}, \quad \frac{1}{n} \leq t \leq \pi. \tag{40}$$

Then (38) implies

$$|V_3(f, x, \alpha, \eta, n)| \leq \frac{A(f, x, \alpha, \eta)}{n^\alpha}. \tag{41}$$

Let us now consider the expression $V_2(f, x, \alpha, \eta, n)$; taking into account (24), from equality (38) we obtain

$$\begin{aligned}
V_2(f, x, \alpha, \eta, n) &= \frac{1}{\pi} \int_{\frac{1}{n}}^\eta \varphi(x, t) \varphi_n^\alpha(t) dt + \frac{1}{\pi} \int_{\frac{1}{n}}^\eta \varphi(x, t) r_n^\alpha(t) dt \equiv \\
&\equiv V_4(f, x, \alpha, \eta, n) + V_5(f, x, \alpha, \eta, n).
\end{aligned} \tag{42}$$

Using the relation $|r_n^\alpha(t)| \leq \frac{A(\alpha)}{nt^2}$, $1/n \leq t \leq \pi$, we conclude that inequalities of type (9) (to the accuracy of the parameter α) remain valid for the expression $V_5(f, x, \alpha, \eta, n)$, too, i.e.,

$$|V_5(f, x, \alpha, \eta, n)| \leq A(f, x, \alpha, \eta) \frac{1}{n} \int_{\frac{1}{n}}^\eta \frac{\omega(t)}{t^2} dt. \tag{43}$$

Let us consider $V_4(f, x, \alpha, \eta, n)$ from equality (42). Relation (24') allows us to conclude that it suffices to establish an estimate of respective order

for the expression

$$V_6(f, x, \alpha, \eta, n) = \frac{1}{\pi A_n^\alpha} \int_{\frac{1}{n}}^{\eta} \varphi(x, t) \frac{\cos(\frac{1}{2} + \frac{\alpha}{2})t}{(2 \sin \frac{t}{2})^{1+\alpha}} \sin nt dt.$$

We have

$$\begin{aligned} V_6(f, x, \alpha, \eta, n) &= \frac{1}{\pi A_n^\alpha} \int_{\frac{1}{n}}^{\eta} \varphi(x, t) \frac{\sin nt}{(2 \sin \frac{t}{2})^{1+\alpha}} dt - \\ &\quad - \frac{2}{\pi A_n^\alpha} \int_{\frac{1}{n}}^{\eta} \varphi(x, t) \frac{\sin^2(\frac{1}{4} + \frac{\alpha}{4})t}{(2 \sin \frac{t}{2})^{1+\alpha}} \sin nt dt. \end{aligned} \quad (44)$$

Taking into account the inequality $A(\alpha) \leq k^{-\alpha} A_k^\alpha \leq A_1(\alpha)$ ($A_k^\alpha \sim k^\alpha$), from relation (44) we write

$$\begin{aligned} &\left| \frac{1}{\pi A_n^\alpha} \int_{\frac{1}{n}}^{\eta} \varphi(x, t) \frac{\sin^2(\frac{1}{4} + \frac{\alpha}{4})t}{(2 \sin \frac{t}{2})^{1+\alpha}} \sin nt dt \right| \leq \\ &\leq \frac{A(\alpha)}{n^\alpha} \int_{\frac{1}{n}}^{\eta} |\varphi(x, t)| dt \leq \frac{A(\alpha)}{n^\alpha} \left[\int_0^{\eta} |\varphi(x, t)| dt + \int_0^{\frac{1}{n}} |\varphi(x, t)| dt \right]. \end{aligned}$$

Relying on the above-said, by virtue of the conditions of the theorem, we obtain

$$\left| \frac{1}{\pi A_n^\alpha} \int_{\frac{1}{n}}^{\eta} \varphi(x, t) \frac{\sin^2(\frac{1}{4} + \frac{\alpha}{4})t}{(2 \sin \frac{t}{2})^{1+\alpha}} \sin nt dt \right| \leq \frac{A(f, x, \alpha, \eta)}{n^\alpha}. \quad (45)$$

Let

$$V_7(f, x, \alpha, \eta, n) \equiv \frac{1}{\pi A_n^\alpha} \int_{\frac{1}{n}}^{\eta} \varphi(x, t) \frac{\sin nt}{(\sin \frac{t}{2})^{1+\alpha}} dt \quad (46)$$

and

$$g(t) \equiv g(t, \alpha, n) \equiv - \int_t^{\pi} \frac{\sin ns}{(\sin \frac{s}{2})^{1+\alpha}} ds. \quad (47)$$

It is obvious that

$$|g(t)| \leq \frac{A(\alpha)}{nt^{1+\alpha}}. \quad (48)$$

Next, in view of the conditions of the theorem, we conclude that

$$\begin{aligned} |\varphi(x, t)| &= |\varphi(x, t) - \varphi(x, 0)| = \\ &= \left| \int_0^t d\varphi(x, t) \right| \leq \int_0^t |d\varphi(x, t)| \leq A(f, x)\omega(t), \quad 0 < t \leq \eta. \end{aligned} \quad (49)$$

According to (46) and (47), we have

$$\begin{aligned} V_7(f, x, \alpha, \eta, n) &= -\frac{1}{\pi A_n^\alpha} \int_{\frac{1}{n}}^\eta \varphi(x, t) g'(t) dt = \\ &= -\frac{1}{\pi A_n^\alpha} [\varphi g]_{\frac{1}{n}}^\eta + \frac{1}{\pi A_n^\alpha} \int_{\frac{1}{n}}^\eta g(t) d\varphi(x, t). \end{aligned} \quad (50)$$

Denote by $\lambda(t) \equiv \lambda(x, t)$ a complete variation for $\varphi(x, s)$ on an interval $[0, t]$. Since

$$\begin{aligned} \sum_{k=1}^n |\varphi(x, t_k) - \varphi(x, t_{k-1})| &\leq \sum_{k=1}^n \left| \int_{t_{k-1}}^{t_k} d\varphi(x, t) \right| \leq \\ &\leq \sum_{k=1}^n \int_{t_{k-1}}^{t_k} |d\varphi(x, t)| \leq \int_0^t |d\varphi(x, t)| \leq A(f, x)\omega(t), \end{aligned}$$

we obtain

$$\lambda(t) \leq A(f, x)\omega(t), \quad 0 < t \leq \eta. \quad (51)$$

Further, in view of (48), (49), (50) and (51) we conclude that

$$\begin{aligned} V_7(f, x, \alpha, \eta, n) &\leq \\ &\leq A(f, x, \alpha)\omega\left(\frac{1}{n}\right) + A(f, x, \alpha, \eta) \frac{1}{n^{1+\alpha}} + \frac{A(\alpha)}{n^{1+\alpha}} \int_{\frac{1}{n}}^\eta \frac{d\lambda(t)}{t^{1+\alpha}}. \end{aligned} \quad (52)$$

On the other hand, we have

$$\int_{\frac{1}{n}}^\eta \frac{d\lambda(t)}{t^{1+\alpha}} = \left[\frac{\lambda(t)}{t^{1+\alpha}} \right]_{\frac{1}{n}}^\eta + (1+\alpha) \int_{\frac{1}{n}}^\eta \frac{\lambda(t)}{t^{2+\alpha}} dt. \quad (53)$$

Hence, taking into account (51), we write

$$\begin{aligned} \int_{\frac{1}{n}}^{\eta} \frac{d\lambda(t)}{t^{1+\alpha}} &\leq A(f, x, \alpha, \eta)\omega(\eta) + \\ &+ A(f, x, \alpha)n^{1+\alpha}\omega\left(\frac{1}{n}\right) + A_1(f, x, \alpha) \int_{\frac{1}{n}}^{\eta} \frac{\omega(t)}{t^{2+\alpha}} dt. \end{aligned} \quad (54)$$

Estimate (52) now implies

$$V_7(f, x, \alpha, \eta, n) \leq A(f, x, \alpha, \eta) \frac{1}{n} \int_{\frac{1}{n}}^{\eta} \frac{\omega(t)}{t^2} dt. \quad (55)$$

Thus by virtue of relations (38), (39), (41)–(46) and (55), we conclude that the theorem is valid. \square

Theorem 4. *Let ω and α satisfy the conditions of Theorem 3. If $[a, b] \subset T$, $b - a > 0$ and*

$$\sup_{a \leq x \leq b} \int_0^t |d\varphi(x, s)| \leq A(f, a, b)\omega(t), \quad 0 < t \leq \eta \leq \pi,$$

then

$$\sup_{a \leq x \leq b} |\sigma_n^\alpha(x, f) - f(x)| \leq A(f, a, b, \alpha, \eta) \left[\frac{1}{n^\alpha} + \frac{1}{n} \int_{\frac{1}{n}}^{\eta} \frac{\omega(t)}{t^2} dt \right].$$

Remark 2. In the above theorems there takes appears

$$\frac{1}{n^\alpha} + \frac{1}{n} \int_{\frac{1}{n}}^{\eta} \frac{\omega(t)}{t^2} dt, \quad (**)$$

if $\omega(t) \leq At^\beta$, $0 < \beta < 1$, then

$$\frac{1}{n} \int_{\frac{1}{n}}^{\eta} \frac{\omega(t)}{t^2} dt \leq \frac{A}{n^\beta}.$$

Namely, when $\beta = \alpha$, then the order of expressions $(**)$ is $n^{-\alpha}$. In the other cases we will have:

(1) if $\beta > \alpha$, then the expression $(**)$ is $n^{-\alpha}$ and

(2) if $\beta < \alpha$, then the expression $(**)$ is of order $n^{-\beta}$. From the corresponding T. Ftett's statement ([1], Theorem 2) it follows that the above estimates cannot be strengthened; the problem on an arbitrary ω remains open.

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Author's address:

Shota Rustaveli Batumi State University
35, Ninoshvili Str., Batumi 6010
Georgia
E-mail: dsm@posta.ge; dsm47@bk.ru