# CAUCHY MEANS INVOLVING CHEBYSHEV FUNCTIONAL 

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#### Abstract

We give exponential convexity for Chebyshev functional. We introduce related means of Cauchy type and give its monotonicity property. Related mean value theorems of Cauchy type are also given.  з   dง.


## 1. Introduction and Preliminaries

A classic result due to Chebyshev $(1882,1883)$ is stated in the following theorem [1, page 195].

Theorem 1.1. Let $f, g:[a, b] \rightarrow \mathbb{R}$ and $w:[a, b] \rightarrow \mathbb{R}^{+}$be integrable functions. We denote by

$$
T(f, g ; w)=\int_{a}^{b} w(x) d x \int_{a}^{b} w(x) f(x) g(x) d x-\int_{a}^{b} w(x) f(x) d x \int_{a}^{b} w(x) g(x) d x
$$

If $f$ and $g$ are monotonic in the same direction, then

$$
\begin{equation*}
T(f, g ; w) \geq 0 \tag{1}
\end{equation*}
$$

provided that the integrals exist.
If $f$ and $g$ are monotonic in opposite directions, then the reverse of the inequality in (1) is valid. In both cases, equality in (1) holds if and only if either $f$ or $g$ is constant almost everywhere.

Remark 1.2 ([1], page 199). (i) Above theorem is also valid when $g$ is an increasing function on $[a, b]$ and for all $x \in(a, b), f$ satisfies

[^0]the condition
\[

$$
\begin{equation*}
\frac{1}{W(x)} \int_{a}^{x} w(t) f(t) d t \leq \frac{1}{W(b)} \int_{a}^{b} w(t) f(t) d t, \text { where } W(x)=\int_{a}^{x} w(t) d t \tag{2}
\end{equation*}
$$

\]

(ii) The condition that $w(t)>0$ can be replaced by

$$
\begin{equation*}
0 \leq W(x) \leq W(b) \text { for } a \leq x \leq b \tag{3}
\end{equation*}
$$

In this paper we give exponential convexity of $T(f, g ; w)$ and related Cauchy means as in [2]. We prove monotonicity property of newly defined mean. Also we give mean value theorems of Cauchy type for Chebyshev functional and its generalized form.

Let us recall the following.
Definition 1. A function $f:(a, b) \rightarrow \mathbb{R}$ is exponentially convex if it is continuous and

$$
\sum_{i, j=1}^{n} v_{i} v_{j} f\left(x_{i}+x_{j}\right) \geq 0
$$

for all $n \in \mathbb{Z}^{+}$and all choices $v_{i} \in \mathbb{R}, i=1, \ldots, n$ such that $x_{i}+x_{j} \in(a, b)$, $1 \leq i, j \leq n$.

Proposition 1.3. Let $f:(a, b) \rightarrow \mathbb{R}$. The following propositions are equivalent
(i) $f$ is exponentially convex
(ii) $f$ is continuous and

$$
\sum_{i, j=1}^{n} v_{i} v_{j} f\left(\frac{x_{i}+x_{j}}{2}\right) \geq 0
$$

for every $v_{i} \in \mathbb{R}$ and for every $x_{i} \in(a, b), 1 \leq i \leq n$.
Corollary 1.4. If $f:(a, b) \rightarrow \mathbb{R}^{+}$is exponentially convex function then $f$ is a log-convex function:

$$
f\left(\frac{x+y}{2}\right) \leq \sqrt{f(x) f(y)} \quad \text { for all } x, y \in(a, b)
$$

2. Main Results

Lemma 2.1. Let $x>0$

$$
h_{p}(x)=\left\{\begin{array}{cl}
\frac{x^{p}}{p}, & p \neq 0 \\
\ln x, & p=0
\end{array}\right.
$$

Then $h_{p}(x)$ is a strictly increasing function.

Proof. Since

$$
h_{p}^{\prime}(x)=x^{p-1}>0,
$$

therefore $h_{p}(x)$ is a strictly increasing function.
Theorem 2.2. Let $f:[a, b] \rightarrow \mathbb{R}$ and $g, w:[a, b] \rightarrow \mathbb{R}^{+}$be integrable functions, where $0<a<b$, such that $f$ and $g$ are increasing functions.
(a) Let $r_{1}, \ldots, r_{m}$ be arbitrary positive real numbers. Then a matrix $A=\left[T\left(f, h_{\frac{r_{i}+r_{j}}{2}} \circ g ; w\right)\right]$, where $1 \leq i, j \leq m$, is a positive semidefinite matrix. Particularly

$$
\operatorname{det}\left[T\left(f, h_{\frac{r_{i}+r_{j}}{2}} \circ g ; w\right)\right]_{i, j=1}^{k} \geq 0 \quad \forall k=1, \ldots, m
$$

(b) A function $p \mapsto T\left(f, h_{p} \circ g ; w\right)$, where $t \in \mathbb{R}^{+}$is an exponentially convex function.
(c) Let $T\left(f, h_{p} \circ g ; w\right)>0$ (i.e. $f$ and $g$ are not constant almost everywhere), then $T\left(f, h_{p} \circ g ; w\right)$ is log-convex function.

Proof. (a) Define a $m \times m$ matrix $M=\left[\frac{h_{\frac{r_{i}+r_{j}}{}}}{}\right]$, where $i, j=1, \ldots, m$, and let $\mathbf{v}=\left(v_{1}, \ldots, v_{m}\right)$ be a nonzero arbitrary vector from $\mathbb{R}^{m}$.

Consider a function

$$
\phi(t)=\mathbf{v} M \mathbf{v}^{\tau}=\sum_{i, j=1}^{m} v_{i} v_{j} h_{\frac{r_{i}+r_{j}}{2}}(t)
$$

Now we have

$$
\phi^{\prime}(t)=\sum_{i, j=1}^{m} v_{i} v_{j} t^{\frac{r_{i}+r_{j}}{2}-1}=\left(\sum_{i=1}^{m} v_{i} t^{\frac{r_{i}-1}{2}}\right)^{2} \geq 0
$$

concluding $\phi(t)$ is an increasing function. Now we apply Theorem 1.1 for increasing function $\psi:=\phi \circ g$

$$
0 \leq T(f, \psi ; w)=\sum_{i, j=1}^{n} v_{i} v_{j} T\left(f, h_{\frac{r_{i}+r_{j}}{2}} \circ g ; w\right)
$$

Therefore matrix $A$ is positive semidefinite matrix.
Specially, we get

$$
\left|\begin{array}{ccc}
T\left(f, h_{r_{1}} \circ g ; w\right) & \cdots & T\left(f, h_{\frac{r_{1}+r_{k}}{2}} \circ g ; w\right)  \tag{4}\\
\vdots & \ddots & \vdots \\
T\left(f, h_{\frac{r_{k}+r_{1}}{2}} \circ g ; w\right) & \cdots & T\left(f, h_{r_{k}} \circ g ; w\right)
\end{array}\right| \geq 0
$$

for all $k=1, \ldots, m$.
(b) Since $\lim _{p \rightarrow 0} T\left(f, h_{p} \circ g ; w\right)=T\left(f, h_{0} \circ g ; w\right)$, it follows that $p \mapsto$ $T\left(f, h_{p} \circ g ; w\right)$ is continuous on $\mathbb{R}$. Now using Proposition 1.3 we have exponential convexity of the function $p \mapsto T\left(f, h_{p} \circ g ; w\right)$.
(c) It follows from the Corollary 1.4.

Theorem 2.3. Let $f:[a, b] \rightarrow \mathbb{R}$ and $g, w:[a, b] \rightarrow \mathbb{R}^{+}$be integrable functions, where $0<a<b$, such that (2) is valid and $g$ is an increasing function. Then $(a),(b)$ and (c) of Theorem 2.2 are valid.

Theorem 2.4. Let $f, w:[a, b] \rightarrow \mathbb{R}$ and $g:[a, b] \rightarrow \mathbb{R}^{+}$be integrable functions, where $0<a<b$, such that $f$ and $g$ are increasing function and condition (3) is valid. Then (a), (b) and (c) of Theorem 2.2 are valid.

If $f$ is a decreasing and $g$ is an increasing function, then we can consider:

$$
U(f, g ; w):=-T(f, g ; w) \geq 0
$$

So we have a following theorem similar to Theorem 2.2.
Theorem 2.5. Let $f:[a, b] \rightarrow \mathbb{R}$ and $g, w:[a, b] \rightarrow \mathbb{R}^{+}$be integrable functions, where $0<a<b$, such that $f$ is a decreasing and $g$ is an increasing functions.
(a) Let $r_{1}, \ldots, r_{m}$ be arbitrary positive real numbers. Then a matrix $A=\left[U\left(f, h_{\frac{r_{i}+r_{j}}{2}} \circ g ; w\right)\right]$, where $1 \leq i, j \leq m$, is a positive semidefinite matrix. Particularly

$$
\operatorname{det}\left[U\left(f, h_{\frac{r_{i}+r_{j}}{2}} \circ g ; w\right)\right]_{i, j=1}^{k} \geq 0 \forall k=1, \ldots, m
$$

(b) A function $p \mapsto U\left(f, h_{p} \circ g ; w\right)$, where $t \in \mathbb{R}^{+}$is an exponential convex function.
(c) Let $U\left(f, h_{p} \circ g ; w\right)>0$ (i.e. $f$ and $g$ are not constant almost everywhere), then $U\left(f, h_{p} \circ g ; w\right)$ is log-convex function.

Let us introduce the following:
Definition 2. Let $f:[a, b] \rightarrow \mathbb{R}$ and $g:[a, b] \rightarrow \mathbb{R}^{+}$be almost everywhere nonconstant, integrable functions and $w:[a, b] \rightarrow \mathbb{R}^{+}$be integrable function where $0<a<b$. Also let $f$ be monotonic and $g$ be an increasing
functions. Then for $s, p \in \mathbb{R}$, we define

$$
I_{s, p}(f, g ; w)= \begin{cases}\left(\frac{p T\left(f, g^{s} ; w\right)}{s T\left(f, g^{p} ; w\right)}\right)^{\frac{1}{s-p}}, & s p(s-p) \neq 0 \\ \left(\frac{T\left(f, g^{p} ; w\right)}{p T(f, \ln g ; w)}\right)^{\frac{1}{p}}, & s=0, p \neq 0 \\ \exp \left(-\frac{1}{s}+\frac{T\left(f, g^{s} \ln g ; w\right)}{T\left(f, g^{s} ; w\right)}\right), & s=p \neq 0 \\ \exp \left(\frac{T\left(f,(\ln g)^{2} ; w\right)}{2 T(f, \ln g ; w)}\right), & s=p=0\end{cases}
$$

Remark 2.6. Note that $\lim _{p \rightarrow s} I_{s, p}(f, g ; w)=I_{s, s}(f, g ; w), \lim _{s \rightarrow 0} I_{s, p}(f, g ; w)=$ $I_{0, p}(f, g ; w)=I_{p, 0}(f, g ; w)$ and $\lim _{s \rightarrow 0} I_{s, s}(f, g ; w)=I_{0,0}(f, g ; w)$.

We shall use a following lemma to prove the monotonicity of $I_{s, p}(f, g ; w)$.
Lemma 2.7. Let $f$ be a log-convex function and assume that if $x_{1} \leq$ $y_{1}, x_{2} \leq y_{2}, x_{1} \neq x_{2}, y_{1} \neq y_{2}$. Then the following inequality is valid:

$$
\begin{equation*}
\left(\frac{f\left(x_{2}\right)}{f\left(x_{1}\right)}\right)^{\frac{1}{x_{2}-x_{1}}} \leq\left(\frac{f\left(y_{2}\right)}{f\left(y_{1}\right)}\right)^{\frac{1}{y_{2}-y_{1}}} \tag{5}
\end{equation*}
$$

Proof. This follows from [1], Remark 1.2.
Theorem 2.8. Let $f:[a, b] \rightarrow \mathbb{R}$ and $g:[a, b] \rightarrow \mathbb{R}^{+}$be almost everywhere nonconstant, integrable functions which are monotonic and $w$ : $[a, b] \rightarrow \mathbb{R}^{+},(0<a<b)$, be integrable function. Let $s, p, u, v \in \mathbb{R}$ such that $s \leq v, p \leq u$. Then we have

$$
\begin{equation*}
I_{s, p}(f, g ; w) \leq I_{v, u}(f, g ; w) \tag{6}
\end{equation*}
$$

Proof. Let $f$ and $g$ be monotonically increasing functions.
Taking $x_{1}=p, x_{2}=s, y_{1}=u, y_{2}=v$ and $f(p)=T\left(f, h_{p} \circ g ; w\right)$, where $p \neq s$ and $u \neq v(p, s, u, v \neq 0)$ in Lemma 2.7, we have

$$
\begin{equation*}
\left(\frac{p T\left(f, g^{s} ; w\right)}{s T\left(f, g^{p} ; w\right)}\right)^{\frac{1}{s-p}} \leq\left(\frac{u T\left(f, g^{v} ; w\right)}{v T\left(f, g^{u} ; w\right)}\right)^{\frac{1}{v-u}} \tag{7}
\end{equation*}
$$

If $f$ be a decreasing function and $g$ be an increasing function, then we consider $f(p)=U\left(f, h_{p} \circ g ; w\right)$ and apply Lemma 2.7 to get

$$
\left(\frac{p U\left(f, g^{s} ; w\right)}{s U\left(f, g^{p} ; w\right)}\right)^{\frac{1}{s-p}} \leq\left(\frac{u U\left(f, g^{v} ; w\right)}{v U\left(f, g^{u} ; w\right)}\right)^{\frac{1}{v-u}}
$$

It follows (7).
From Remark 2.6, we get (7) is also valid for $p=0$ or $s=0$ or $u=0$ or $v=0$ or $p=s$ or $u=v$.

Remark 2.9. Theorem 2.8 can be also proved under assumption (2) on functions $f$ and $g$ or assumption (3) on a function $w$.

## 3. Mean Value Theorems

Lemma 3.1. Let $h \in C^{1}[a, b]$, such that

$$
\begin{equation*}
m \leq h^{\prime}(x) \leq M, x \in[a, b] . \tag{8}
\end{equation*}
$$

Consider the functions $\tilde{h}_{1}, \tilde{h}_{2}:[a, b] \rightarrow \mathbb{R}$ defined as,

$$
\tilde{h}_{1}(x)=M x-h(x)
$$

and

$$
\tilde{h}_{2}(x)=h(x)-m x .
$$

Then $\tilde{h}_{i}(x)$ for $i=1,2$ are monotonically increasing.
Proof. Since

$$
\tilde{h}_{1}^{\prime}(x)=M-h^{\prime}(x) \geq 0
$$

and

$$
\tilde{h}_{2}^{\prime}(x)=h^{\prime}(x)-m \geq 0 .
$$

i.e. $\tilde{h}_{i}$ for $i=1,2$ are monotonically increasing.

Theorem 3.2. Let $f, g:[a, b] \rightarrow \mathbb{R}$ be monotonically increasing, integrable functions such that $f$ and $g$ are nonconstant almost everywhere, $g \in$ $C^{1}[a, b]$ and let $w:[a, b] \rightarrow \mathbb{R}^{+}$be integrable function. If $h \in C^{1}[g(a), g(b)]$ is an integrable function then there exists $\eta \in[g(a), g(b)]$ such that

$$
\begin{equation*}
T(f, h \circ g ; w)=h^{\prime}(\eta) T(f, g ; w) \tag{9}
\end{equation*}
$$

Proof. Let $m=\min _{\tilde{\sim}} h^{\prime}, M=\max h^{\prime}$ and let $\tilde{h}_{1}(x)=M x-h(x), \tilde{h}_{2}(x)=$ $h(x)-m x$. Since $\tilde{h}_{1}$ and $g$ are monotonically increasing functions, $\tilde{h}_{1} \circ g$ is monotonically increasing. Setting $g=\tilde{h}_{1} \circ g$ in Theorem 1.1 we get

$$
T(f, M g-h \circ g ; w) \geq 0,
$$

i.e.

$$
\begin{equation*}
T(f, h \circ g ; w) \leq M T(f, g ; w) \tag{10}
\end{equation*}
$$

Similarly, setting $g=\tilde{h}_{2} \circ g$ in Theorem 1.1 gives us

$$
\begin{equation*}
T(f, h \circ g ; w) \geq m T(f, g ; w) . \tag{11}
\end{equation*}
$$

Combining (10) and (11), we get,

$$
\begin{equation*}
m \leq \frac{T(f, h \circ g ; w)}{T(f, g ; w)} \leq M \tag{12}
\end{equation*}
$$

By Lemma 3.1, there exists $\eta \in[g(a), g(b)]$ such that

$$
\frac{T(f, h \circ g ; w)}{T(f, g ; w)}=g^{\prime}(\eta)
$$

This implies (9).
Moreover (10) is valid if (for example) $h^{\prime}$ is bounded from above and hence (9) is valid.

Of course (9) is obvious if $h^{\prime}$ is not bounded.
If $h=i d$ (i.e. $h(x)=x$ ), then we have a following result.
Corollary 3.3. Let $f, g:[a, b] \rightarrow \mathbb{R}$ and $w:[a, b] \rightarrow \mathbb{R}^{+}$be integrable functions. If $f$ is monotonically increasing such that $f$ is not constant almost everywhere and $g \in C^{1}[a, b]$, then there exists $\eta \in[a, b]$ such that

$$
\begin{equation*}
T(f, g ; w)=g^{\prime}(\eta) T(f, i d ; w) \tag{13}
\end{equation*}
$$

Corollary 3.4. Let $w:[a, b] \rightarrow \mathbb{R}^{+}$be an integrable function and $f, g \in$ $C^{1}[a, b]$. If $f$ is monotonically increasing such that $f$ is not constant almost everywhere and $f, g \in C^{1}[a, b]$, then there exists $\xi, \eta \in[a, b]$ such that

$$
\begin{equation*}
T(f, g ; w)=f^{\prime}(\xi) g^{\prime}(\eta) T(i d, i d ; w) \tag{14}
\end{equation*}
$$

Proof. For $f \in C^{1}[a, b]$, there exists $\xi \in[a, b]$ such that

$$
\begin{equation*}
T(f, i d ; w)=f^{\prime}(\xi) T(i d, i d ; w) \tag{15}
\end{equation*}
$$

Now using (13) and (15), we have (14).
A. M. Ostrowski [4] obtained some very interesting generalization of (1). For example, from the general result which is given in [4], we have the following result.

Corollary 3.5. Let $f$ and $g$ be two monotonically increasing and differentiable functions on $[a, b]$. Let $w$ be a positive integrable function on $[a, b]$. If

$$
f^{\prime}(x) \geq m, \quad g^{\prime}(x) \geq r \quad(\forall x \in[a, b], \quad m, r>0)
$$

then

$$
\begin{equation*}
T(f, g ; w) \geq r T(f, i d ; w) \geq m r T(i d, i d, w)>0 \tag{16}
\end{equation*}
$$

Proof. Since $f$ and $g$ are monotonically increasing and differentiable, therefore (13) implies

$$
\begin{equation*}
T(f, g ; w) \geq r T(f, i d ; w) \tag{17}
\end{equation*}
$$

and

$$
\begin{equation*}
T(f, i d ; w) \geq m T(i d, i d ; w)>0 \tag{18}
\end{equation*}
$$

Combining (17) and (18) we have (16).
Remark 3.6. The above result is an improvement of the result given in [3].

Theorem 3.7. Let $f, g_{1}, g_{2}:[a, b] \rightarrow \mathbb{R}$ and $w:[a, b] \rightarrow \mathbb{R}^{+}$be integrable functions. If $f$ is monotonically increasing such that $f$ is nonconstant almost everywhere and $g_{1}, g_{2} \in C^{1}[a, b]$, then there exists $\eta \in[a, b]$ such that

$$
\begin{equation*}
\frac{T\left(f, g_{1} ; w\right)}{T\left(f, g_{2} ; w\right)}=\frac{g_{1}^{\prime}(\eta)}{g_{2}^{\prime}(\eta)} \tag{19}
\end{equation*}
$$

provided that denominators are non-zero.
Proof. We define function $k \in C^{1}[a, b]$ with

$$
k=c_{1} g_{1}-c_{2} g_{2}
$$

where $c_{1}=T\left(f, g_{2} ; w\right)$ and $c_{2}=T\left(f, g_{1} ; w\right)$.
Then, using Theorem 3.2 for $g=k$, we have

$$
\begin{equation*}
T(f, k ; w)=0=\left(c_{1} g_{1}^{\prime}(\eta)-c_{2} g_{2}^{\prime}(\eta)\right) T(f, i d ; w) \tag{20}
\end{equation*}
$$

Since $T(f, i d ; w)>0$, we have

$$
\frac{c_{2}}{c_{1}}=\frac{g_{1}^{\prime}(\eta)}{g_{2}^{\prime}(\eta)}
$$

and our proof is done.
Remark 3.8. Note that

$$
\begin{equation*}
\eta=\left(\frac{g_{1}^{\prime}}{g_{2}^{\prime}}\right)^{-1}\left(\frac{T\left(f, g_{1} ; w\right)}{T\left(f, g_{2} ; w\right)}\right) \tag{21}
\end{equation*}
$$

This implies $\eta$ is a mean of numbers $a$ and $b$. Especially, when $g$ is a monotonically increasing function and $g_{1}=g^{s}$ and $g_{2}=g^{p}$, we conclude that expression $I_{s, p}(f, g ; w)$, introduced in Definition 2, represents a mean between positive numbers $a$ and $b$.

Remark 3.9. (i) Theorem 3.2 and 3.7 are also valid if $f$ satisfy condition (2) and $g$ is an increasing function.
(ii) Theorem 3.2 and 3.7 are also valid if the condition $w(t)>0$ is replaced by (3).

Corollary 3.10. Let $f_{i}, g_{i}:[a, b] \rightarrow \mathbb{R}$ for $i=1,2$ and $w:[a, b] \rightarrow \mathbb{R}^{+}$ be integrable functions, where $0<a<b$. If $f_{1}$ and $f_{2}$ are monotonically increasing such that $f_{1}$ and $f_{2}$ are not constant almost everywhere and $f_{i}, g_{i} \in C^{1}[a, b]$ for $i=1,2$, then there exist $\xi, \eta \in[a, b]$ such that

$$
\begin{equation*}
\frac{T\left(f_{1}, g_{1} ; w\right)}{T\left(f_{2}, g_{2} ; w\right)}=\frac{f_{1}^{\prime}(\xi) g_{1}^{\prime}(\eta)}{f_{2}^{\prime}(\xi) g_{2}^{\prime}(\eta)} \tag{22}
\end{equation*}
$$

provided that non-zero denominators.

Proof. Since

$$
\frac{T\left(f_{1}, g_{1} ; w\right)}{T\left(f_{2}, g_{2} ; w\right)}=\frac{T\left(f_{1}, g_{1} ; w\right)}{T\left(f_{2}, g_{1} ; w\right)} \cdot \frac{T\left(f_{2}, g_{1} ; w\right)}{T\left(f_{2}, g_{2} ; w\right)}
$$

Now applying Theorem 3.7 two times we get (22).
Let us note that we can use ideas of previous proofs to obtain similar results for generalized form of Chebyshev inequality.

First recall that a real valued function $f: I \times J \rightarrow \mathbb{R}$, where $I \times J=$ $[a, b] \times[c, d]$, is said to be $(1,1)-$ convex function if for all distinct points $x_{0}, x_{1} \in I$ and $y_{0}, y_{1} \in J$, the divided difference $\left[x_{0}, x_{1}\right]\left(\left[y_{0}, y_{1}\right] f\right)$ is nonnegative or alternatively for $\left(x_{1}-x_{0}\right)\left(y_{1}-y_{0}\right)>0$, we have

$$
\left(x_{1}-x_{0}\right)\left(y_{1}-y_{0}\right) \leq f\left(x_{0}, y_{0}\right)+f\left(x_{1}, y_{1}\right)-f\left(x_{0}, y_{1}\right)-f\left(x_{1}, y_{0}\right) .
$$

If the partial derivative $f_{21}$ of $f$ exists, then $f$ is $(1,1)$-convex iff $f_{21}(x, y) \geq$ 0 for all $(x, y) \in I \times J$.

In [5] (see also [1, page 204-205]), the following generalized form of Chebyshev inequality is given for $(1,1)$-convex function.

Theorem 3.11. Let $p: I^{2} \rightarrow \mathbb{R}$ and $q: I \rightarrow \mathbb{R}$, where $I=[a, b]$, be two integrable functions. Then for every (1,1)-convex function $f: I^{2} \rightarrow \mathbb{R}$ the inequality

$$
\begin{equation*}
K(f, p, q):=\int_{a}^{b} q(x) f(x, x) d x-\int_{a}^{b} \int_{a}^{b} p(x, y) f(x, y) d x d y \geq 0 \tag{23}
\end{equation*}
$$

holds iff for every $x, y \in[a, b]$, we have

$$
\begin{equation*}
P(a, y)=Q(y), \quad P(x, a)=Q(x), \quad P(x, y) \leq Q(\max \{x, y\}) \tag{24}
\end{equation*}
$$

where $Q(x)=\int_{x}^{b} q(t) d t$ and $P(x, y)=\int_{x}^{b} \int_{y}^{b} p(s, t) d t d s$.
Lemma 3.12. Let $f: I^{2} \rightarrow \mathbb{R}(I=[a, b])$ has continuous partial derivatives $f_{1}, f_{2}$ and $f_{21}$ such that

$$
\begin{equation*}
m \leq f_{21}(x, y) \leq M, \quad(x, y) \in I^{2} \tag{25}
\end{equation*}
$$

Consider the functions $F$ and $G$ defined as

$$
F(x, y)=M(x-a)(y-a)-f(x, y)
$$

and

$$
G(x, y)=f(x, y)-m(x-a)(y-a),
$$

then $F(x, y)$ and $G(x, y)$ are $(1-1)$-convex functions.
Proof. Since

$$
F_{21}(x, y)=M-f_{21}(x, y) \geq 0
$$

and

$$
G_{21}(x, y)=f_{21}(x, y)-m \geq 0
$$

therefore $F(x, y)$ and $G(x, y)$ are (1-1)-convex functions.
Theorem 3.13. Let $p: I^{2} \rightarrow \mathbb{R}$ and $q: I \rightarrow \mathbb{R}$, where $I=[a, b]$, be two integrable functions such that (24) is satisfied and $K(i, p, q)>0$, where $i(x, y)=(x-a)(y-a)$. If $f: I^{2} \rightarrow \mathbb{R}$ has continuous partial derivatives $f_{1}$, $f_{2}$ and $f_{21}$, then there exists $\xi, \eta \in I$ such that

$$
\begin{equation*}
K(f, p, q)=f_{21}(\xi, \eta) K(i, p, q) \tag{26}
\end{equation*}
$$

Proof. Let $m=\min f_{21}, M=\max f_{21}$ and let $F, G$ defined in Lemma 3.12.
Setting $f=F$ in Theorem 3.11, we get

$$
K(M i-f, p, q) \geq 0
$$

i.e.

$$
\begin{equation*}
K(f, p, q) \leq M K(i, p, q) \tag{27}
\end{equation*}
$$

Similarly, setting $f=G$ in Theorem 3.11 gives us

$$
\begin{equation*}
K(f, p, q) \geq m K(i, p, q) \tag{28}
\end{equation*}
$$

Combining (27)and (28), we get,

$$
\begin{equation*}
m \leq \frac{K(f, p, q)}{K(i, p, q)} \leq M \tag{29}
\end{equation*}
$$

Now by Lemma 3.12, there exists $\xi, \eta \in I$ such that

$$
\frac{K(f, p, q)}{K(i, p, q)}=f_{21}(\xi, \eta)
$$

This implies (26).
Moreover (27) is valid if (for example) $f_{21}$ is bounded from above and hence (26) is valid.

Of course (26) is obvious if $f_{21}$ is not bounded.
Theorem 3.14. Let $p: I^{2} \rightarrow \mathbb{R}$ and $q: I \rightarrow \mathbb{R}$, where $I=[a, b]$, be two integrable functions such that (24) is satisfied. If $f, g: I^{2} \rightarrow \mathbb{R}$ have continuous partial derivatives $f_{1}, f_{2}, f_{21}, g_{1}, g_{2}$ and $g_{21}$, then there exists $\xi, \eta \in I$ such that

$$
\begin{equation*}
\frac{K(f, p, q)}{K(g, p, q)}=\frac{f_{21}(\xi, \eta)}{g_{21}(\xi, \eta)} \tag{30}
\end{equation*}
$$

provided that the denomirators are non-zero.
Proof. We define function $h$, such that

$$
h=c_{1} f-c_{2} g
$$

where $c_{1}=K(g, p, q)$ and $c_{2}=K(f, p, q)$.
Then, using Theorem 3.13 with $f=h$, we have

$$
\begin{equation*}
K(h, p, q)=0=\left(c_{1} f_{21}(\xi, \eta)-c_{2} g_{21}(\xi, \eta)\right) K(i, p, q) \tag{31}
\end{equation*}
$$

Since $K(i, p, q)>0$, we have

$$
\frac{c_{2}}{c_{1}}=\frac{f_{21}(\xi, \eta)}{g_{21}(\xi, \eta)}
$$

and our proof is complete.
Theorem 3.15. Let $p: I^{2} \rightarrow \mathbb{R}$ and $q: I \rightarrow \mathbb{R}$, where $I=[a, b]$, be two integrable functions such that $(24)$ is satisfied and $K(i, p, q)>0$, where $i(x, y)=(x-a)(y-a)$. If $f: I^{2} \rightarrow \mathbb{R}$ has continuous partial derivatives $f_{1}$, $f_{2}$ and $f_{12}$ such that

$$
\begin{equation*}
\left|f_{21}(x, y)\right| \leq M \tag{32}
\end{equation*}
$$

then

$$
\begin{equation*}
|K(f, p, q)| \leq M K(i, p, q) \tag{33}
\end{equation*}
$$

Proof. Since

$$
-M \leq f_{21}(x, y) \leq M
$$

therefore (27) and (28) implies

$$
\begin{gather*}
K(f, p, q) \leq M K(i, p, q)  \tag{34}\\
K(f, p, q) \geq-M K(i, p, q) \tag{35}
\end{gather*}
$$

Combining above two inequalities gives us the required result.
The following result, given in [3], can be derived from above theorem.
Corollary 3.16. Let $f$ and $g$ be two differentiable and integrable functions on $[a, b]$ and let $w$ be a positive integrable function on $[a, b]$ such that (3) holds and

$$
\begin{equation*}
\left|f^{\prime}(x)\right| \leq M, \quad\left|g^{\prime}(x)\right| \leq N \quad(\forall x \in[a, b]) \tag{36}
\end{equation*}
$$

Then

$$
\begin{equation*}
|T(f, g ; w)| \leq M N T(j, j ; w), \text { where } j(x)=x-a \tag{37}
\end{equation*}
$$

Proof. Setting

$$
q(x)=w(x) \int_{a}^{b} w(t) d t, \quad p(x, y)=w(x) w(y) \text { and } f(x, y)=f(x) g(y)
$$

implies

$$
K(f, p, q)=T(f, g ; w), \quad\left|f_{21}(x, y)\right|=\left|f^{\prime}(x) g^{\prime}(y)\right| \leq M N
$$

and condition (24) becomes condition (3).
Hence (33) implies (37).

## Acknowledgement

This research was partially funded by Higher Education Commission, Pakistan. The research of the first author was supported by the Croatian Ministry of Science, Education and Sports under the Research Grant 117-1170889-0888.

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(Received 15.06.2009)
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[^0]:    2000 Mathematics Subject Classification. Primary 26D15, 26D20, 26D99.
    Key words and phrases. Chebyshev functional, exponential convexity, logarithmic convexity, positive semidefinite matrix, Cauchy means, mean value theorems, Generalized Chebyshev functional.

