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# A FORMULA FOR THE NORM OF GENERALIZED SPHERICAL AVERAGING OPERATOR ON WEIGHTED LEBESGUE SPACE

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Abstract. The subject of this work is to define an averaging integral transform and its dual associated with a system of partial differential operators. These operators generalize the spherical average operator and its dual which are investigated by many authors in different points of view. Here we investigate wether or not the generalized spherical operator and its dual are bounded on the weighted Lebesgue space. The operator norms are given precisely.

**რეზიუმე.** ნაშრომში განხილულია გაზოგადებული ინტეგრალური საშუალო და მისი დუალური, რომელიც დაკავშირებულია გარკვეულ კერძოწარმოებულებიან დიფერენციალურ განტოლებათა სისტემასთან. ეს ოპერატორები განაზოგადებენ სფერულ გასაშუალოების ოპერატორს და მის დუალურს. დამტკიცებულია ხსენებული ოპერატორების შემოსაზღვრულობა ლებეგის წონიან სივრცეებში. ცხადი სახით ნაპოვნია ოპერატორების ნორმები.

## 1. Introduction

Consider function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ which are even in the first variable that is $f(r, x) = f(-r, x)$ $r \in \mathbb{R}$, $x \in \mathbb{R}^n$. Define the mapping

$$f \to \mathcal{R}_{\frac{n-1}{2},n}(f) = g$$

by letting $g(r, x)$ be the average of $f$ over a sphere with radius $r$ and center at the point $(0, x)$. Consequently

$$g(r,x) = \int\limits_{S^n} f(r\eta, x + r\xi)d\sigma_n(\eta, \xi), \quad (r, x) \in \mathbb{R} \times \mathbb{R}^n,$$

where $S^n$ is the unit sphere $\{(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n, \ \ \eta^2 + \|\xi\|^2 = 1\}$in $\mathbb{R}^{n+1}$ and $d\sigma_n$ is the surface measure on $S^n$ normalized to have total measure one.

These operators have been extensively studied in [1,8]. In [1] Anderson studied the inverting problem of the mapping $\mathcal{R}_{\frac{n-1}{2},n}$ that is to determine

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the function $f$ when $g$ is known, he also extended the domain of the mapping $\mathcal{R}_{\frac{n-1}{2},n}$ to the class of tempered distributions. In [8] the authors defined and characterized spaces of functions and distributions on which the spherical average is bijective. They also gave inversion formula for the transform $\mathcal{R}_{\frac{n-1}{2},n}$. They arise in connection with the following system of partial differential operators

$$\begin{aligned} D_j =& \frac{\partial}{\partial x_j}, \quad 1 \leq j \leq n, \\ L_{\frac{n-1}{2}} =& \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} - \sum_{j=1}^{n} \frac{\partial^2}{\partial {x_j}^2}, \ (r,x) \in ]0,\infty[\times \mathbb{R}^n. \end{aligned}$$

The first aim of this note is to define and study a weighted average integral transform and its dual which generalize the operator $\mathcal{R}_{\frac{n-1}{2},n}$. and its dual. Precisely, we consider the system of partial differential operators

$$\begin{aligned} D_j &= \frac{\partial}{\partial x_j}, \quad 1 \leq j \leq n, \\ L_\alpha &= \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} - \sum_{j=1}^{n} \frac{\partial^2}{\partial {x_j}^2}, \quad (r,x) \in ]0,\infty[\times \mathbb{R}^n. \end{aligned}$$

We prove that the system of differential equations

$$\begin{cases} D_j u(r,x_1,\ldots,x_n) = -i\lambda_j u(r,x_1,\ldots,x_n), \\ L_\alpha u(r,x_1,\ldots,x_n) = -\mu^2 u(r,x_1,\ldots,x_{n)}, \\ u(0,x_1,x_2,\ldots x_n) = 1; \quad \dfrac{\partial u}{\partial r}(0,x_1,\ldots,x_n) = 0. \end{cases}$$

admits a unique solution denoted $\varphi_{\mu,\lambda}$ We associate to this solution an integral transform $\mathcal{R}_{\alpha,n}$ defined on the space of continuous functions on $\mathbb{R}^{n+1}$, even with respect to the first variable by

$$\mathcal{R}_{\alpha,n}(f)(r,x) = \begin{cases} C_{\alpha,n} r^{-2\alpha} \displaystyle\int_{B_r^+(0,x)} f(u,v)\big(r^2-u^2-\|v-x\|^2\big)^{\alpha-\frac{n+1}{2}} du\, dv \\ \qquad\qquad\qquad\qquad\qquad \text{if} \quad \alpha > \dfrac{n-1}{2}, \\ \displaystyle\int_{S^n} f(r\eta, x+r\xi) d\sigma_n(\eta,\xi) \qquad \text{if} \quad \alpha = \dfrac{n-1}{2}, \end{cases}$$

where

- $B_r^+(0,x) = \big\{(u,v), \ \ u>0; \ \ u^2+\|v-x\|^2 < r^2\big\}$

and

- $C_{\alpha,n}, \ S^n$ and $d\sigma_n$ are defined in section 2.

The dual of the generalized spherical operator ${}^t\mathcal{R}_{\alpha,n}$ is defined on $\mathcal{S}_*(\mathbb{R}^{n+1})$ (the space of infinitely differentiable function on $\mathbb{R}^{n+1}$, rapidly decreasing together with all their derivatives, even with respect to the first variable), by

$$
{}^t\mathcal{R}_{\alpha,n}(g)(r,x)=\begin{cases} C_{\alpha,n}\displaystyle\int_r^{\infty}\int_{B^+_{\sqrt{u^2-r^2}}(0,0)} g(u,v+x)\big(u^2-r^2-\|v\|^2\big)^{\alpha-\frac{n+1}{2}}u\,du\,dv \\ \qquad\qquad\qquad\qquad \text{if}\quad \alpha>\dfrac{n-1}{2}, \\ k_{\alpha,n}\displaystyle\int_r^{\infty}\int_{S^{n-1}} g\big(u,x+\sqrt{u^2-r^2}\big)\big(u^2-r^2\big)^{\frac{n-2}{2}}u\,du\,d\sigma_{n-1}(v) \\ \qquad\qquad\qquad\qquad \text{if}\quad \alpha=\dfrac{n-1}{2}, \end{cases}
$$

where the $k_{\alpha,n}$ is a constant defined in section 2.

In this work we consider the boundedness of the operator $\mathcal{R}_{\alpha,n}$, $\alpha\geq(n-1)/2$, on weighted Lebesgue spaces $L^p\big([0,\infty[\times\mathbb{R}^n, r^\beta dr dx_1 dx_2\dots dx_n\big)$. For convenience we refer to this space as $L^p_\beta$ and denote its norm by

$$
\|f\|_{p,\beta}=\left(\int_{\mathbb{R}^n}\int_0^{\infty}|f(r,x)|^p r^\beta dr\,dx_1\dots dx_n\right)^{\frac{1}{p}},\quad 1\leq p<\infty.
$$

Here $\beta$ can be any real number. The space $L^\infty_\beta\equiv L^\infty$ does not depend on $\beta$ and

$$
\|f\|_{\infty,\beta}\equiv\|f\|_\infty=\operatorname{esssup}\big\{|f(r,x)|;\ \ (r,x)\in[0,\infty[\times\mathbb{R}^n\big\}\ \text{ for }\ p=\infty.
$$

The second aim is to investigate whether or not $\mathcal{R}_{\alpha,n}$ is a bounded operator on $L^p_\beta$, that is whether or not there exists a constant $C$ such that the inequality

$$
\|\mathcal{R}_{\alpha,n}f\|_{p,\beta}\leq C\|f\|_{p,\beta}
$$

holds for all $f\in L^p_\beta$. By using the same technical as [2], this question is completely answered. In addition we provide a formula for the least possible constant $C$ for which the last inequality holds. This is called the operator norm of $\mathcal{R}_{\alpha,n}$ on weighted Lebesgue spaces $L^p_\beta$.

As usual, for $p\in[1,\infty]$ we define $p'$ by $\frac{1}{p}+\frac{1}{p'}=1$.

## 2. Generalized Spherical Average Operator

Consider the system of partial differential operators

$$
\begin{aligned}
&D_j=\frac{\partial}{\partial x_j},\ \ 1\leq j\leq n,\\
&L_\alpha=\frac{\partial^2}{\partial r^2}+\frac{2\alpha+1}{r}\frac{\partial}{\partial r}-\sum_{j=1}^n\frac{\partial^2}{\partial {x_j}^2},\ \ (r,x)\in]0,\infty[\times\mathbb{R}^n.
\end{aligned}
\tag{2.1}
$$

In this section we define an integral operators which are in connection with the system (2.1).

**Theorem 2.1.** *For* $(\mu, \lambda) \in \mathbb{C} \times \mathbb{C}^n$, *the following system of equations*

$$\begin{cases} D_j u(r, x_1, \dots, x_n) = -i\lambda_j u(r, x_1, \dots, x_n), \\ L_\alpha u(r, x_1, \dots, x_n) = -\mu^2 u(r, x_1, \dots, x_n), \\ u(0, \dots, 0) = 1; \quad \dfrac{\partial u}{\partial r}(0, x_1, \dots, x_n) = 0. \end{cases} \tag{2.2}$$

*has a unique infinitely differentiable solution on* $\mathbb{R} \times \mathbb{R}^n$, *even with respect to the first variable given by*

$$\varphi_{\mu,\lambda}(r, x) = j_\alpha\Big(r\sqrt{\mu^2 + \lambda_1^2 + \dots \lambda_n^2}\Big)e^{-i<\lambda,x>}, \tag{2.3}$$

*where*

- $< \lambda, x >= \sum_1^n \lambda_i x_i$.

*and*

- $j_\alpha(s) = \begin{cases} \dfrac{2^\alpha \Gamma(\alpha+1) J_\alpha(s)}{s^\alpha} & \textit{if} \quad s \neq 0, \\ 1 & \textit{if} \quad s = 0. \end{cases}$

*with* $J_\alpha$ *is the Bessel function of first kind and order* $\alpha$.

*Proof.* Let $u$ be the solution of the system (2.2) and let's put $v(r, x) = u(r, x)e^{i<\lambda/x>}$, $(r, x) \in [0, \infty[\times\mathbb{R}^n$. Since for all $1 \le i \le n$, $\frac{\partial v}{\partial x_i}(r, x) = 0$ then we have $u(r, x) = h(r)e^{-i<\lambda/x>}$ where $h$ is the solution of the cauchy problem

$$\begin{cases} l_\alpha h(r) = -(\mu^2 + \lambda_1^2 \dots + \lambda_n^2)h(r) \\ h(0) = 1; h'(0) = 0. \end{cases}$$

with $l_\alpha$ is the Bessel differential operator defined by

$$l_\alpha = \frac{\partial^2}{\partial^2 r} + \frac{2\alpha + 1}{r}\frac{\partial}{\partial r}.$$

So by [9],

$$h(r) = j_\alpha\Big(r\sqrt{\mu^2 + \lambda_1^2 + \dots + \lambda_n^2}\Big).$$

This completes the proof of theorem 2.1. □

**Theorem 2.2.** *The function* $\varphi_{(\mu,\lambda)}$ *given by relation* (2.3) *has the following Mehler type integral representation: for* $(\mu, \lambda) \in \mathbb{C} \times \mathbb{C}^n$ *and* $(r, x) \in$

$[0,\infty[\times\mathbb{R}^n$,

$$\varphi_{(\mu,\lambda)}(r,x)=\begin{cases} C_{\alpha,n}\int\limits_0^1\int\limits_{[-\frac{\pi}{2},\frac{\pi}{2}]^n}\cos(\mu rs\psi(\theta))\exp\left(-i<\lambda,x+r\Phi(\theta)>\right)\times \\ \qquad\qquad\times(1-s^2)^{\alpha-\frac{n+1}{2}}\vartheta(\theta)ds\,d\theta \quad if\ \ \alpha>\dfrac{n-1}{2}, \\ \int\limits_{S^n}\cos(\mu r\eta)\exp(-i<\lambda,x+r\xi>)d\sigma_n(\eta,\xi) \quad if\ \ \alpha=\dfrac{n-1}{2}, \end{cases}$$

*where*

- $$C_{\alpha,n}=\frac{2\Gamma(\alpha+1)}{\pi^{(n+1)/2}\Gamma(\alpha-(n-1)/2)}, \tag{2.4}$$

- $$\begin{cases} \psi(\theta)=\prod\limits_{k=1}^n(\cos\theta_k), \\ \Phi(\theta)=(\sin\theta_1,\cos\theta_2\sin\theta_2,\ldots,\cos\theta_1\ldots\cos\theta_{n-1}\sin\theta_n), \\ \vartheta(\theta)=\prod\limits_{k=1}^n(\cos\theta_k)^{2\alpha+1-k}, \end{cases} \tag{2.5}$$

- *$S^n$ is the unit sphere in $\mathbb{R}^{n+1}$ defined by*
$$S^n=\{(\eta,\xi)\in(\mathbb{R}\times\mathbb{R}^n,\eta^2+\|\xi\|^2=1\},$$

- *$d\sigma_n$ is the surface measure on $S^n$ normalized to have total measure one, defined by*

$$\int\limits_{S^n}f(w)d\sigma_n(w)=\frac{\Gamma(\frac{n+1}{2})}{2\pi^{\frac{n+1}{2}}}\int\limits_0^{2\pi}\int\limits_{[0,\pi]^{n-1}}f(\sin\theta_1\ldots\sin\theta_n,\sin\theta_1\ldots\sin\theta_{n-1}\cos\theta_n,$$
$$\ldots\sin\theta_1\cos\theta_2,\cos\theta_1)\prod_{k=1}^{n-1}(\sin\theta_k)^{n-k}d\theta_1d\theta_2\ldots d\theta_n. \tag{2.6}$$

*Proof.* We recall that the modified Bessel function $j_\alpha$ has the Mehler integral representation (we refer to [7,9])

$$j_\alpha(s)=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}\int\limits_{-1}^1(1-t^2)^{\alpha-1/2}\exp(-ist)dt=$$
$$=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}\int\limits_{-\pi/2}^{\pi/2}\exp(-is\sin\theta)(\cos\theta)^{2\alpha}d\theta. \tag{2.7}$$

On the other hand from the following expansions of the function $j_\alpha$

$$j_\alpha(s)=\frac{2^\alpha\Gamma(\alpha+1)J_\alpha(s)}{s^\alpha}=\Gamma(\alpha+1)\sum_{k=1}^{\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\left(\frac{s}{2}\right)^{2k},$$

where $J_\alpha$ is the Bessel function of first kind and order $\alpha$, we deduce that for all $(\mu,\nu)\in\mathbb{C}\times\mathbb{C}$

$$j_\alpha\big(r\sqrt{\mu^2+\nu^2}\big)=\Gamma(\alpha+1)\sum_{k=1}^{\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\left(\frac{r\mu}{2}\right)^{2k}j_{\alpha+k}(r\nu).$$

Therefore, from relation (2.7) we get

$$j_\alpha\big(r\sqrt{\mu^2+\nu^2}\big)=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_{-1}^{1}j_{\alpha-1/2}\big(r\mu\sqrt{1-t^2}\big)\times$$
$$\times\exp(-ir\nu t)(1-t^2)^{\alpha-1/2}dt=$$
$$=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_{-1}^{1}j_{\alpha-1/2}(r\mu\cos\theta)\exp(-ir\nu\sin\theta)(\cos\theta)^{2\alpha}d\theta. \qquad (2.8)$$

By using a gain relation (2.7) we obtain

$$j_\alpha(r\sqrt{\mu^2+\nu^2})=\frac{\Gamma(\alpha+1)}{\pi\Gamma(\alpha)}\int_{-1}^{1}\int_{-\pi/2}^{\pi/2}\cos(\mu rs\cos\theta)\times$$
$$\times\exp(-ir\nu\sin\theta)(\cos\theta)^{2\alpha}(1-s^2)^{\alpha-1}d\theta ds.$$

So, to complete the proof it suffices to see that for all $(\mu,\lambda)\in\mathbb{C}\times\mathbb{C}^n$

$$j_\alpha\Big(r\,\sqrt{\mu^2+\lambda_1^2+\lambda_2^2\cdots+\lambda_n^2}\Big)=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}$$
$$\times\int_{-\pi/2}^{\pi/2}j_{\alpha-1/2}\Big(r\cos\theta_1\,\sqrt{\mu^2+\lambda_2^2+\cdots+\lambda_n^2}\Big)\exp(-ir\lambda_1\sin\theta_1)(\cos\theta_1)^{2\alpha}d\theta_1. \quad \square$$

**Definition 2.3.** The generalized spherical average operator $\mathcal{R}_{\alpha,n}$ associated with $\varphi_{\mu,\lambda}$ is the mapping defined on $\mathcal{C}_*(\mathbb{R}^{n+1})$ (the space of continuous functions on $\mathbb{R}^{n+1}$ even with respect to the first variable) by the following

$$\mathcal{R}_{\alpha,n}(f)(r,x)=\begin{cases}C_{\alpha,n}r^{-2\alpha}\displaystyle\int_{B_r^+(0,x)}f(u,v)(r^2-u^2-\|v-x\|^2)^{\alpha-\frac{n+1}{2}}du\,dv & \text{if } \alpha>\dfrac{n-1}{2},\\ \displaystyle\int_{S^n}f(r\eta,x+r\xi)d\sigma_n(\eta,\xi) & \text{if } \alpha=\dfrac{n-1}{2},\end{cases}$$

where

- $$B_r^+(0,x) = \{(u,v) \in \mathbb{R} \times \mathbb{R}^n, u > 0; u^2 + \|v-x\|^2 < r^2\} \tag{2.9}$$

and

- $C_{\alpha,n}$ is the positive constant given by relation (2.4).

The operator $\mathcal{R}_{\alpha,n}$ arise in connection with the solution $\varphi_{\mu,\lambda}$ of the system (2.2). To see this connection, we make the change of variables

$$(s,\theta) \longrightarrow (u,v) = (rs\psi(\theta), x + r\Phi(\theta)),$$

where $\psi, \Phi$ are the functions given by (2.5). The Jacobian determinant is

$$\left|\partial(u,v)/\partial(s,\theta)\right| = r^{n+1} \prod_{k=1}^{n} (\cos\theta_k)^{n+2-k},$$

and the map

$$(s,\theta) \longrightarrow (u,v),$$

takes the open set $(0,1) \times (-\pi/2, \pi/2)^n$ one-to-one and onto the open half ball $B_r^+(0,x)$ given by (2.9). So it follows

$$\mathcal{R}_{(\alpha,n)}(f)(r,x) = \begin{cases} C_{\alpha,n} \int\limits_0^1 \int\limits_{[-\frac{\pi}{2},\frac{\pi}{2}]^n} f(rs\psi(\theta), x + r\Phi(\theta)) \times \\ \qquad \times (1-s^2)^{\alpha - \frac{n+1}{2}} \vartheta(\theta) ds\, d\theta \text{ if } \alpha > \dfrac{n-1}{2}, \\ \int\limits_{S^n} f(r\eta, x + r\xi) d\sigma_n(\eta,\xi) \text{ if } \alpha = \dfrac{n-1}{2}. \end{cases} \tag{2.10}$$

Then we have

$$\varphi_{\mu,\lambda}(r,x) = \mathcal{R}_{\alpha,n}\big(\cos(\mu.)\exp(-i < \lambda, . >)\big)(r,x), \quad \alpha \geq (n-1)/2. \tag{2.11}$$

*Remark* 2.4. 1. By relation (2.11) it is obvious to see that the transform $\mathcal{R}_{\alpha,n}$ is continuous and injective from $\xi_*(\mathbb{R}^{n+1})$ (the space of infinitely differentiable function on $\mathbb{R}^{n+1}$ even with respect to the first variable) into itself.

2. For $\alpha = \frac{n-1}{2}$, it is clear from its form that the operator $\mathcal{R}_{\alpha,n}$ is the average of $f$ over the right semi-sphere of radius r, centred at $(0,x)$.

For $\alpha > \frac{n-1}{2}$, an easy calculation gives

$$\frac{2\Gamma(\alpha+1)r^{-2\alpha}}{\pi^{n+1/2}\Gamma(\alpha-(n-1)/2)} \int\limits_{B_r^+} \Big(r^2 - u^2 - \sum_{k=1}^{n}(v_k - x_k)^2\Big)^{\alpha-(n+1)/2} du\, dv = 1.$$

Showing that $\mathcal{R}_{\alpha,n}$ is an averaging operator. Indeed, $\mathcal{R}_{\alpha,n}(f)(r,x)$ is the weighted averaging of $f$ over $B_r^+(0,x)$, where the weight $\big(r^2-u^2-\sum_{k=1}^{n}(v_k - x_k)^2\big)^{\alpha-(n+1)/2}$ is a power of the distance to the boundary of the ball.

**Proposition 2.5.** *For $f \in \mathcal{C}_*(\mathbb{R}^{n+1})$, $f$ bounded and $g \in \mathcal{S}_*(\mathbb{R}^{n+1})$ (the space of infinitely differentiable function on $\mathbb{R}^{n+1}$, rapidly decreasing together with all their derivatives, even with respect to the first variable),*

$$\int_{\mathbb{R}^n}\int_0^\infty \mathcal{R}_{\alpha,n}(f)(r,x)g(r,x)r^{2\alpha+1}dr\,dx = \int_{\mathbb{R}^n}\int_0^\infty f(r,x)\,{}^t\mathcal{R}_{\alpha,n}(g)(r,x)dr\,dx,$$

*where ${}^t\mathcal{R}_{\alpha,n}$ is the dual operator defined by*

$$ {}^t\mathcal{R}_{\alpha,n}(g)(r,x) = \begin{cases} C_{\alpha,n}\displaystyle\int_r^\infty \int_{B^+_{\sqrt{u^2-r^2}}(0,0)} g(u,v+x)\big(u^2-r^2-\|v\|^2\big)^{\alpha-\frac{n+1}{2}}u\,du\,dv \\ \qquad\qquad\qquad \text{if} \quad \alpha > \dfrac{n-1}{2}, \\ k_{\alpha,n}\displaystyle\int_r^\infty\int_{S^{n-1}} g(u,x+\sqrt{u^2-r^2})(u^2-r^2)^{\frac{n-2}{2}}u\,du\,d\sigma_{n-1}(v) \\ \qquad\qquad\qquad \text{if} \quad \alpha = \dfrac{n-1}{2}, \end{cases}$$

*where*
*$C_{\alpha,n}$ is the constant given by relation (2.4) and $k_{\alpha,n} = \frac{2\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\Gamma(\frac{n}{2})}$.*

*Proof.* The result follows by using Fubini's theorem and adequate change of variables. □

In the sequel of this paper we use the expression (2.10) for technical reasons.

## 3. Determining the Generalized Spherical Average Norm

In this section we will show that the generalized spherical average operator is bounded on the Lebesgue space and we will determine its norm.

**Theorem 3.1.** *If $\alpha \geq (n-1)/2$, then $\mathcal{R}_{\alpha,n}$ is a bounded operator on $L^\infty_\beta$ with the operator norm equal to 1. That is, the inequality $\|\mathcal{R}_{\alpha,n}(f)\|_{\infty,\beta} \leq \|f\|_{\infty,\beta}$ holds for all bounded $f$ and when $C < 1$ the inequality $\|\mathcal{R}_{\alpha,n}(f)\|_{\infty,\beta} \leq C\|f\|_{\infty,\beta}$ fails to hold for some bounded $f$.*

*Proof.* Let $f$ be a bounded function and view $\|f\|_{\infty,\beta}$ as a constant function. Certainly, $f \leq \|f\|_{\infty,\beta}$ almost every where. Since the transform $\mathcal{R}_{\alpha,n}$ is an average operator, for each $(r,x)$ we have

$$\big|\mathcal{R}_{\alpha,n}(f)(r,x)\big| \leq \mathcal{R}_{\alpha,n}(\|f\|_{\infty,\beta})(r,x) = \|f\|_{\infty,\beta}.$$

Taking the essential supremum over all $(r,x)$ proves that

$$\big\|\mathcal{R}_{\alpha,n}(f)\big\|_{\infty,\beta} \leq \|f\|_{\infty,\beta}$$

and shows that the operator norm of $\mathcal{R}_{\alpha,n}$ is at most 1. To complete the proof it sufficient to take $f$ to be a non zero constant. $\square$

The case $p = 1$ is also treated separately both for technical reasons and because the operator norm is achieved for all non negative function $f$.

**Theorem 3.2.** *If $\alpha \geq \frac{n-1}{2}$, and $\beta < 0$, then $\mathcal{R}_{\alpha,n}$ is a bounded operator on $L^1_\beta$ with operator norm equal to*

$$\frac{\Gamma(\alpha+1)\Gamma(-\frac{\beta}{2})}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2}-\frac{\beta}{2})}.$$

*If $\beta \geq 0$, then $\mathcal{R}_{\alpha,n}$ is not a bounded operator on $L^1_\beta$.*

*Proof.* Suppose $f \in L^1_\beta$. Then

$$\begin{aligned}\left\|\mathcal{R}_{\frac{n-1}{2},n}(f)\right\|_{1,\beta} &= \int_{\mathbb{R}^n}\int_0^\infty |\mathcal{R}_{\frac{n-1}{2},n}(f)(r,x)| r^\beta dr\, dx \leq \\ &\leq \int_{\mathbb{R}^n}\int_0^\infty\int_{S^n} |f(r|\eta|, x+r\xi)| d\sigma_n(\eta,\xi) r^\beta dr\, dx.\end{aligned}$$

Interchanging the order of integration and making the change of variable $t = r|\eta|,\ \ y = x + r\xi$. yields

$$\begin{aligned}\left\|\mathcal{R}_{\frac{n-1}{2},n}(f)\right\|_{1,\beta} &\leq \int_{S^n} |\eta|^{-(\beta+1)} \int_{\mathbb{R}^n}\int_0^\infty |f(t,y)| t^\beta dt\, dy\, d\sigma_n(\eta,\xi) \leq \\ &\leq \|f\|_{1,\beta} \int_{S^n} |\eta|^{-(\beta+1)}\, d\sigma_n(\eta,\xi). \qquad (3.1)\end{aligned}$$

On the other hand from relation (2.6) we deduce

$$\begin{aligned}\int_{S^n} |\eta|^{-(\beta+1)} d\sigma_n(\eta,\xi) &= \frac{\Gamma(\frac{n+1}{2})}{2\pi^{\frac{n+1}{2}}}\left(\prod_{k=1}^{n}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta_k)^{n-k-\beta-1} d\theta_k\right) = \\ &= \frac{\Gamma(\frac{n+1}{2})\Gamma(-\frac{\beta}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n-\beta}{2})},\end{aligned}$$

provided that $\beta < 0$. The last integral diverges when $\beta \geq 0$. It follows

$$\left\|\mathcal{R}_{\frac{n-1}{2},n}(f)\right\|_{1,\beta} \leq \frac{\Gamma(\frac{n+1}{2})\Gamma(-\frac{\beta}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n-\beta}{2})}\|f\|_{1,\beta}$$

provided that $\beta < 0$.

Since this inequality reduces to equality when $f$ is non negative, the constant is best possible. In particular when $\beta \geq 0$ the best constant is infinite so the spherical average $\mathcal{R}_{\frac{n-1}{2},n}$ is not a bounded operator on $L^1_\beta$. This completes the proof in the case $\alpha = \frac{n-1}{2}$. When $\alpha > \frac{n-1}{2}$ we proceed similarly,

$$\|\mathcal{R}_{\alpha,n}(f)\|_{1,\beta} = \int\limits_{\mathbb{R}^n}\int\limits_0^\infty |\mathcal{R}_{\alpha,n}(f)(r,x)| r^\beta dr\, dx \leq$$

$$\leq C_{\alpha,n}\int\limits_{\mathbb{R}^n}\int\limits_0^\infty\int\limits_0^1\int\limits_{[-\pi/2,\pi/2]^n} \big|f(rs\psi(\theta), x + r\phi(\theta))\big| \times$$

$$\times\, \vartheta(\theta)(1-s^2)^{\alpha-(n+1)/2} d\theta\, ds\, r^\beta dr\, dx,$$

where the constant $C_{\alpha,n}$, the functions $\psi, \phi$ and $\vartheta$ are given by relation (2.4) respectively (2.5).

Interchanging again and making the change of variable

$$(r,x) \longrightarrow (t,y) = (rs\psi(\theta), x + r\phi(\theta))$$

yields

$$\big\|\mathcal{R}_{\alpha,n}(f)\big\|_{1,\beta} \leq C_{\alpha,n}\|f\|_{1,\beta}\Bigg(\int\limits_0^1 s^{-\beta-1}(1-s^2)^{\alpha-(n+1)/2} ds\Bigg)\times$$

$$\times \prod_{k=1}^n \Bigg(\int\limits_{-\pi/2}^{\pi/2} (\cos\theta_k)^{2\alpha-\beta-k} d\theta_k\Bigg) \leq C_{\alpha,n}\|f\|_{1,\beta}\times$$

$$\times\, \frac{1}{2}\frac{\Gamma(-\beta/2)\Gamma(\alpha-(n-1)/2)}{\Gamma(\alpha+1/2-(\beta+n)/2)} \times \prod_{k=1}^n \frac{\Gamma(\alpha+1/2-(\beta+k)/2)\Gamma(1/2)}{\Gamma(\alpha+1/2-(\beta+k-1)/2)},$$

provided $\beta < 0$. Once we cancel out the repeated terms on the top and the bottom we get

$$\|\mathcal{R}_{\alpha,n}(f)\|_{1,\beta} \leq \frac{\Gamma(\alpha+1)\Gamma(-\frac{\beta}{2})}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2}-\frac{\beta}{2})}\|f\|_{1,\beta}.$$

The "ds" integral above diverges when $\beta \geq 0$.

Since this inequality reduces to equality when $f$ is non negative, the constant is best possible. In particular when $\beta \geq 0$ the best constant is infinite so the spherical average $\mathcal{R}_{\alpha,n}$ is not a bounded operator on $L^1_\beta$. This completes the proof. □

In the following we give the values of $\beta$ for which the generalized spherical average is a bounded transform on $L^p_\beta$ when $1 < p < \infty$. We begin by looking at the case $\alpha = \frac{n-1}{2}$.

**Theorem 3.3.** *Suppose* $1 < p < \infty$. *Then* $\mathcal{R}_{\frac{n-1}{2},n}$ *is a bounded operator on* $L^p_\beta$ *if and only if* $\beta < p-1$. *Moreover, if* $\beta < p-1$ *then the operator norm is*

$$\frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2}-\frac{\beta+1}{2p})}{\Gamma(\frac{1}{2})\Gamma(\frac{n+1}{2}-\frac{\beta+1}{2p})}.$$

*Proof.* Suppose that $\beta > p-1$ and define the function $f$ by setting $f(t,x) = \frac{1}{t}$ when $(t,x) \in (0,1) \times \Omega_{(n,2)}$ and $f(t,x) = 0$ otherwise. Where $\Omega_{(n,2)}$ is a ball of $\mathbb{R}^n$ at center $(0,0)$ and radius 2. This means that $\Omega_{(n,2)} = \{x \in \mathbb{R}^n / \; \|x\| = \Big(\sum_{k=1}^n x_k^2\Big)^{\frac{1}{2}} < 2\,\}$. Then we have

$$\|f\|^p_{p,\beta} = \int\limits_{\Omega_{(n,2)}} \int\limits_0^1 t^{\beta-p} dt dx = \frac{2^{n+1}\pi^{\frac{n}{2}}}{(\beta-p+1)n\Gamma(\frac{n}{2})} < \infty.$$

So $f \in L^p_\beta$. On the other hand from identity (2.6) we deduce that if $0 < r < 1$ and $\|x\| < 1$ then

$$\mathcal{R}_{\frac{n-1}{2},n}(f)(r,x) = \int\limits_{S^n} f(r\eta, x + r\xi) d\sigma_n(\eta,\xi) = \int\limits_{S^n} \frac{1}{r|\eta|} d\sigma_n(\eta,\xi) = \infty.$$

So $\mathcal{R}_{\frac{n-1}{2},n}(f) \notin L^p_\beta$. Thus $\mathcal{R}_{\frac{n-1}{2},n}$ is not a map from $L^p_\beta$ to $L^p_\beta$.

When $\beta = p-1$, let $f(t,x) = \frac{1}{t(1-\log t)}$ when $(t,x) \in (0,1) \times \Omega_{(n,2)}$ and $f(t,x) = 0$ otherwise. We have

$$\|f\|^p_{p,\beta} = \int\limits_{\Omega_{(n,2)}} \int\limits_0^1 \frac{1}{t(1-\log t)^p} dt\, dx = \frac{2^{n+1}\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})} \int\limits_0^1 \frac{1}{t(1-\log t)^p} dt < \infty$$

so $f \in L^p_\beta$. But, if $0 < r < 1$ and $\|x\| < 1$ and by relation (2.6) then we have

$$\mathcal{R}_{\frac{n-1}{2},n}(f)(r,x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \prod_{k=1}^{n-1} \Bigg( \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta_k)^{n-k-1} d\theta_k \Bigg) \times$$

$$\times \Bigg( \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta_n}{\cos r\theta_n (1 - \log(r \prod_{k=1}^n \cos\theta_k)} \Bigg).$$

This integral is infinite. That is $\mathcal{R}_{\frac{n-1}{2},n}(f) \notin L^p_\beta$.

Now, suppose that $\beta < p-1$ and fix $f \in L^p_\beta$. Let $\gamma$ be a real constant to be determined later. Then by Hölder inequality we have

$$\left|\mathcal{R}_{\frac{n-1}{2},n}(f)(r,x)\right|^p \leq \Big(\int\limits_{S^n} |f(r|\eta|, x+r\xi)|^p |\eta|^{p\gamma} d\sigma_n(\eta,\xi)\Big)\times$$

$$\times\Big(\int\limits_{S^n} |\eta|^{-p'\gamma} d\sigma_n(\eta,\xi)\Big)^{\frac{p}{p'}} = C_1^{\frac{p}{p'}} \int\limits_{S^n} \left|f(r|\eta|, x+r\xi)\right|^p \eta^{p\gamma} d\sigma_n(\eta,\xi),$$

where

$$C_1 = \int\limits_{S^n} |\eta|^{-\gamma p'} d\sigma_n(\eta,\xi). \tag{3.2}$$

Using this estimate we get

$$\left\|\mathcal{R}_{\frac{n-1}{2},n}(f)\right\|^p_{p,\beta} \leq C_1^{\frac{p}{p'}} \int\limits_{\mathbb{R}^n} \int\limits_0^\infty \int\limits_{S^n} \left|f(r\eta, x+r\xi)\right|^p |\eta|^{p\gamma} d\sigma_n(\eta,\xi) r^\beta dr\, dx.$$

Interchanging the order of integration and making the change of variable $t = r|\eta|$ and $y = x + r\xi$ yields

$$\left\|\mathcal{R}_{\frac{n-1}{2},n}(f)\right\|^p_{p,\beta} \leq C_1^{\frac{p}{p'}} \int\limits_{S^n} \eta^{p\gamma-\beta-1} \Big(\int\limits_{\mathbb{R}^n} \int\limits_0^\infty |f(t,y)|^p t^\beta dt dx\Big) d\sigma_n(\eta,\xi) =$$

$$= C_1^{\frac{p}{p'}} C_2 \|f\|^p_{p,\beta},$$

where

$$C_2 = \int\limits_{S^n} |\eta|^{p\gamma-\beta-1} d\sigma_n(\eta,\xi). \tag{3.3}$$

If there there exists a $\gamma$ that makes both the integrals (3.2) and (3.3) finite, then the spherical average $\mathcal{R}_{\frac{n-1}{2},n}$ is a bounded operator on $L^p_\beta$.

Studying the integral $C_1$

$$C_1 = \int\limits_{S^n} \eta^{-\gamma p'} d\sigma_n(\eta,\xi) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \Big(\int\limits_0^{2\pi} \frac{1}{(|sin\theta_n|)^{p'\gamma}} d\theta_n\Big)\times$$

$$\times\Big(\prod_{k=1}^{n-1} \int\limits_0^\pi (\sin\theta_k)^{n-k-p'\gamma} d\theta_k\Big).$$

This integral to be finite requires that $-p'\gamma > -1$, furthermore

$$C_1 = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \prod_{k=1}^{n} \int_{-\pi/2}^{\pi/2} (\cos\theta_k)^{(n-k-p'\gamma+1)-1} d\theta_k =$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \prod_{1}^{n} \left( \frac{\Gamma\left(\frac{n-k-p'\gamma+1}{2}\right)\Gamma(1/2)}{\Gamma\left(\frac{n-k-p'\gamma+2}{2}\right)} \right).$$

Studying the integral $C_2$. A similar calculus as the above gives that the integral $C_2$ to be finite requires that $p\gamma - \beta - 1 > -1$. Furthermore

$$C_2 = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \prod_{1}^{n} \left( \frac{\Gamma\left(\frac{n-k+p\gamma-\beta}{2}\right)\Gamma(1/2)}{\Gamma\left(\frac{n-k+p\gamma-\beta+1}{2}\right)} \right).$$

The above conditions for the constant $\gamma$ requirement that $\gamma \in (\frac{\beta}{p}, \frac{1}{p'})$, which is non-empty interval because we have assumed that $\beta < p - 1$.
To obtain a specific upper bound to the operator norm let $\gamma = \beta + 1/(pp')$. The upper bound obtained is

$$C_1^{\frac{1}{p'}} C_2^{\frac{1}{p}} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(1/2)} \prod_{1}^{n} \left( \frac{\Gamma\left(\frac{n-k+1}{2} - \frac{\beta+1}{2p}\right)}{\Gamma\left(\frac{n-k+2}{2} - \frac{\beta+1}{2p}\right)} \right) = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2} - \frac{\beta+1}{2p}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2} - \frac{\beta+1}{2p}\right)}.$$

To get a lower bound for the spherical average operator norm, we fix $\kappa$ a positive and odd real and $a > 1$ and fix $f$ by setting $f(t,y) = t^{\frac{\kappa-\beta-1}{p}}$ when $(t,y) \in (0,1) \times \Omega_{(n,a)}$ and $f(t,y) = 0$ otherwise. The norm of $f$ in $L^p_\beta$gis

$$\|f\|_{p,\beta} = \left( \int_{\Omega(n,a)} \int_0^1 \left(t^{\frac{\kappa-\beta-1}{p}}\right)^p t^\beta dt dy \right)^{1/p} = \left( \int_{\Omega(n,a)} \left( \int_0^1 t^{\kappa-1} dt \right) dy \right)^{1/p} =$$

$$= \left( \frac{2\pi^{n/2} a^n}{n\Gamma(n/2)\kappa} \right)^{1/p}.$$

On the other hand, if $0 < r < 1$ and $\|x\| < a-1$, then for $\omega = (\eta, \xi) \in S^n$, we have $|r\eta| < |\eta| < 1$ and $\|x + r\xi\| < \|x\| + |r|\ \|\xi\| < a - 1 + 1 = a$. So

$$\mathcal{R}_{\frac{n-1}{2},n}(f)(r,x) = \int\limits_{S^n} f(r\eta, x + r\xi) d\sigma_n(\eta,\xi) = r^{\frac{\kappa-\beta-1}{p}} \int\limits_{S^n} t^{\frac{\kappa-\beta-1}{p}} d\sigma_n(t,\xi) =$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} r^{\frac{\kappa-\beta-1}{p}} \prod_{k=1}^{n} \left( \int\limits_{-\pi/2}^{\pi/2} (\cos\theta)^{n-k+\frac{\kappa-\beta-1}{p}} d\theta_k \right) =$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} r^{\frac{\kappa-\beta-1}{p}} \prod_{k=1}^{n} \left( \frac{\Gamma\left(\frac{n-k}{2} + \frac{\kappa-\beta-1}{2p} + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-k}{2} + \frac{\kappa-\beta-1}{2p} + 1\right)} \right) =$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} r^{\frac{\kappa-\beta-1}{p}} \prod_{k=1}^{n} \left( \frac{\Gamma\left(\frac{n-k}{2} + \frac{\kappa-\beta-1}{2p} + \frac{1}{2}\right)}{\Gamma\left(\frac{n-k}{2} + \frac{\kappa-\beta-1}{2p} + 1\right)} \right).$$

It follows that

$$\left\|\mathcal{R}_{\frac{n-1}{2},n}(f)\right\|_{p,\beta} \geq \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \prod_{k=1}^{n} \left( \frac{\Gamma\left(\frac{n-k}{2} + \frac{\kappa-\beta-1}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n-k}{2} + \frac{\kappa-\beta-1}{2} + 1\right)} \right) \times$$

$$\times \left( \int\limits_{\Omega(n,a-1)} \int\limits_{0}^{1} \left(r^{\frac{\kappa-\beta-1}{p}}\right)^p r^\beta dr dx \right)^{1/p} =$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \prod_{k=1}^{n} \left( \frac{\Gamma\left(\frac{n-k}{2} + \frac{\kappa-\beta-1}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n-k}{2} + \frac{\kappa-\beta-1}{2} + 1\right)} \right) \left( \frac{2\pi^{n/2}(a-1)^n}{n\Gamma(n/2)\kappa} \right)^{1/p}.$$

For each $\kappa$ and $a$ the ratio $\frac{\|\mathcal{R}_{\frac{n-1}{2},n}(f)\|_{p,\beta}}{\|f\|_{p,\beta}}$ is a lower bound for the spherical average operator norm. Canceling out the repeated terms of the top and the bottom and Letting $a \to \infty$ first and then $\kappa \longrightarrow 0$ gives the lower bound

$$\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2} - \frac{\beta+1}{2p}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2} - \frac{\beta+1}{2p}\right)}$$

as required. The upper and lower bounds coincide so the operator norm is determined. □

**Theorem 3.4.** *Suppose* $1 < p < \infty$, *and* $\alpha > \frac{n-1}{2}$. *Then* $\mathcal{R}_{\alpha,n}$ *is a bounded operator on* $L^p_\beta$ *if and only if* $\beta < p - 1$. *Moreover, if* $\beta < p - 1$ *then the operator norm is*

$$\frac{\Gamma(\alpha+1)\Gamma\left(\frac{1}{2} - \frac{\beta+1}{2p}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha + 1 - \frac{\beta+1}{2p}\right)}.$$

*Proof.* Suppose that $\beta > p-1$ and define the function $f$ by setting $f(t,y) = \frac{1}{t}$ when $(t,y) \in (0,1) \times \Omega(n,2)$ and $f(t,y) = 0$ otherwise,

$$\|f\|_{p,\beta}^p = \int_{\mathbb{R}^n} \int_0^\infty |f(t,y)|^p t^\beta dt\, dy = \int_{\Omega(n,2)} \left( \int_0^1 t^{\beta-p} dt \right) dy =$$

$$= \left( \int_0^1 t^{\beta-p} dt \right) \int_{\Omega(n,2)} dy < \infty$$

provided $\beta > p-1$. So $f \in L^p_\beta$. On the other hand if $0 < r < 1$, and $\|x\| < 1$, then

$$\mathcal{R}_{\alpha,n}(f)(r,x) = C_{\alpha,n} \left( \int_0^1 (1-s^2)^{\alpha - \frac{n+1}{2}} \frac{1}{s} ds \right) \prod_{k=1}^n \int_{[-\pi/2,\pi/2]^n} (\cos\theta_k)^{2\alpha-k} d\theta.$$

Since the "ds" integral diverges then $\mathcal{R}_{\alpha,n}(f) \notin L^p_\beta$.

When $\beta = p-1$ a similar argument shows that $\mathcal{R}_{\alpha,n}$ does not map $L^p_\beta$ to $L^p_\beta$. This time, $f(t,y) = \frac{1}{t(1-\log t)}$ when $(t,y) \in (0,1) \times \Omega(n,2)$ and $f(t,y) = 0$ otherwise. A similar calculus as above shows that, $\|f\|_{p,\beta}$ is finite and $\mathcal{R}_\alpha(f) \notin L^p_\beta$.

Now suppose $\beta < p-1$ and fix $f \in L^p_\beta$. Let $\gamma_k,\ 1 \le k \le n,\ \delta$ and $\epsilon$ be real constants to be determined later. Clearly,

$$|\mathcal{R}_\alpha f(r, x_1, \dots x_n)| \le \mathcal{R}_\alpha |f|(r, x_1, \dots x_n)$$

and by Hölder inequality this is no greater then

$$C_{\alpha,n} \times C_1^{\frac{1}{p'}} \int_0^1 \int_{[-\pi/2,\pi/2]^n} \Big| f(rs\cos\theta_1 \dots \cos\theta_n, x_1 + r\sin\theta_1, x_2 +$$

$$+ r\cos\theta_2 \sin\theta_2, \dots x_n + r\cos\theta_1 \dots \cos\theta_{n-1}\sin\theta_n) \Big|^p \times$$

$$\times \prod_{k=1}^n (\cos\theta_k)^{p\gamma_k} (1-s^2)^{p\epsilon} d\theta_1 \dots d\theta_n ds,$$

where

$$C_1 = \left( \int_0^1 (1-s^2)^{(\alpha - \frac{n+1}{2} - \epsilon)p'} s^{-\delta p'} ds \right) \prod_{k=1}^n \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta_k)^{(2\alpha+1-k-\gamma_k)p'} d\theta_k \right).$$

Using this estimate and by using the adequate change of variables as above we obtain

$$\|\mathcal{R}_{\alpha,n}\|_{p,\beta}^p \leq (C_{\alpha,n})^p C_1^{\frac{p}{p'}} \left( \int_0^1 (1-s^2)^{p\epsilon} s^{p\delta-\beta-1} ds \right) \times$$

$$\times \left( \prod_{k=1}^n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta_k)^{p\gamma_k-\beta-1} d\theta_k \right) \|f\|_{p,\beta}^p = (C_{\alpha,n})^p C_1^{\frac{p}{p'}} C_2 \, \|f\|_{p,\beta}^p,$$

where $C_2$ is the constant defined by

$$C_2 = \left( \int_0^1 (1-s^2)^{p\epsilon} s^{p\delta-\beta-1} ds \right) \times \left( \prod_{k=1}^n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta_k)^{p\gamma_k-\beta-1} d\theta_k \right).$$

If there exist $\gamma_k, \delta$, and $\epsilon$ that make $C_1$ and $C_2$ finite, then $\mathcal{R}_{\alpha,n}$ is a bounded operator on $L^p_\beta$. The requirements are that

$$(2\alpha+1-k-\gamma_k)p', \ -\delta p', \ (\alpha - \frac{n+1}{2} - \epsilon)p', \ \delta p - \beta - 1, \ \gamma_k p - \beta - 1, \ \epsilon p$$

all be greater then $-1$. These conditions reduce to

$$\gamma_k \in \Big(\frac{\beta}{p}, 2\alpha+1-k+\frac{1}{p'}\Big), \ \delta \in \Big(\frac{\beta}{p}, \frac{1}{p'}\Big), \epsilon \in \Big(-\frac{1}{p}, -\frac{1}{p} + \alpha - \frac{n-1}{2}\Big).$$

Since $\alpha > \frac{n-1}{2}, 1 \leq k \leq n$ and $\beta < p-1$ then all three intervals are non empty so it is possible to choose $\gamma_k, \delta$ and $\epsilon$ that makes $C_1$ and $C_2$ finite. Thus $\mathcal{R}_{\alpha,n}$ is a bounded operator on $L^p_\beta$.

To obtain a specific upper bound for the operator norm let

$$\gamma_k = \frac{\beta + (2\alpha-k)p' + 1}{pp'}, \ \delta = \frac{\beta+1}{pp'}, \ \epsilon = \frac{\alpha - (n+1)/2}{p}.$$

Then

$$C_1 = C_2 = \left( \int_0^1 (1-s^2)^{\alpha - \frac{n+1}{2}} s^{\frac{-\beta-1}{p}} ds \right) \times$$

$$\times \left( \prod_{k=1}^n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta_k)^{2\alpha+1-k-\frac{\beta+1}{p}} d\theta_k \right) =$$

$$= \frac{\Gamma\big(\alpha - \frac{n-1}{2}\big)\Gamma\big(\frac{1}{2} - \frac{\beta+1}{2p}\big)}{2\Gamma\big(\alpha + 1 - \frac{n}{2} - \frac{\beta+1}{2p}\big)} \pi^{\frac{n}{2}} \prod_{k=1}^n \frac{\Gamma\big(\alpha + 1 - \frac{k}{2} - \frac{\beta+1}{2p}\big)}{\Gamma\big(\alpha + 1 - \frac{k-1}{2} - \frac{\beta+1}{2p}\big)}.$$

Once we cancel out the repeated terms on the top and the bottom, the upper bound obtained is

$$C_{\alpha,n}C_1^{\frac{1}{p}}C_2^{\frac{1}{p'}}=\frac{\Gamma(\alpha+1)\Gamma\left(\frac{1}{2}-\frac{\beta+1}{2p}\right)}{\Gamma\left(\alpha+1-\frac{\beta+1}{2p}\right)\Gamma\left(\frac{1}{2}\right)}.$$

To get a lower bound for the operator norm we fix $\eta>0$ and $a>1$ and define $f$ by setting $f(t,y)=t^{\frac{\eta-\beta-1}{p}}$ where $(t,y)\in(0,1)\times\Omega_{(n,a)}$ and $f(t,y)=0$ otherwise. The norm of $f$ in $L^p_\beta$ is

$$\|f\|_{p,\beta}=\left(\int\limits_{\Omega(n,a)}\int\limits_0^1 t^{\eta-\beta-1}t^\beta dt\,dy\right)^{\frac{1}{p}}=\left(\frac{2\pi^{n/2}a^n}{\eta n\Gamma(n/2)}\right)^{\frac{1}{p}}.$$

On the other hand, if $0<r<1$ and $\|x\|<a-1$ then for any $s\in(0,1)$ and $\theta\in(-\pi/2,\pi/2)$ $|rs\cos\theta_1\ldots\cos\theta_n|<1$ and if $y=x+r\xi$, $\|y\|<a$. So we complete the proof in the same way as Theorem 3.3. $\square$

In is important to point out the operator norm calculated in the previous four theorems are all related. We do this in the following summary.

**Theorem 3.5.** *Suppose* $1\le p\le\infty$. *and* $\alpha\ge\frac{n-1}{2}$. *The operator* $\mathcal{R}_{\alpha,n}$ *is a bounded operator on* $L^p_\beta$ *if and only if* $\beta<p-1$. *In this case the operator norm is*

$$\frac{\Gamma(\alpha+1)\Gamma\left(\frac{1}{2}-\frac{\beta+1}{2p}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha+1-\frac{\beta+1}{2p}\right)}.$$

Earlier work suggest that $L^p_{2\alpha+1}$ is a naturel space for the spherical average operator. In the following we restrict our attention to the case $\beta=2\alpha+1$,

**Corollary 3.6.** *Suppose* $1\le p\le\infty$. *The generalized spherical average operator is a bounded map on* $L^p_{2\alpha+1}$ *if and only if* $2\alpha+2<p$. *In this case the operator norm is*

$$\frac{\Gamma(\alpha+1)\Gamma\left(\frac{1}{2}-\frac{\alpha+1}{p}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\alpha+1}{p'}\right)}.$$

*Remark* 3.7. The boundedness of the dual operator ${}^t\mathcal{R}_{\alpha,n}$ is obtained by duality.

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