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ALEXANDER-SPANIER COHOMOLOGY OF A WALLMEN COMPACTIFICATION

A. BERIDZE

Abstract. The Alexander-Spanier type cohomology groups based on finite cozero coverings are defined. It is shown that these groups describe classical Alexander-Spanier cohomology groups of Wallmen compactifications. It is proved that cohomology groups based on all proximity coverings and all finite cozero (with respect to proximity maps) coverings are nonisomorphic proximity invariants.

რეზიუმე. ნაშრომში განმარტებულია სასრულო კონულ დაფარვებზე დაფუძნებული ალექსანდერ-სპენიერის ტიპის კოჰომოლოგიის ჯგუფები. ნაჩვენებია, რომ ეს ჯგუფები აღწერს ვოლმენის კომპაქტიფიკაციის ალექსანდერ-სპენიერის კლასიკურ კოჰომოლოგიის ჯგუფებს. დამტკიცებულია, რომ ყველა კონულ (მახლობლური ასახვების მიმართ) დაფარვებზე და ყველა მახლობლურ დაფარვებზე დაფუძნებული კოჰომოლოგიის ჯგუფები არაიზომორფული ინვარიანტებია.

Introduction

The present paper is motivated by the following general problem: find the necessary and sufficient conditions under which the spaces of a given class have extensions with the given (co)homology groups.

This problem was studied by many authors ([1], [2], [3], [5], [11], [12], [13], [15]). In [2], the Alexander-Spanier cohomology theory is constructed for the category of pairs of uniform spaces and uniform maps. This theory gives an intrinsic characterization of the Alexander-Spanier classical cohomology groups of compactifications of pairs of completely regular spaces in terms of uniform structures of the given pairs of completely regular spaces.

The main aim of this paper is to study the above formulated problem when the class of spaces is that of the completely regular spaces, while the extensions are the Wallmen compactifications.

All the spaces considered in the paper are assumed to be completely regular and Hausdorff ones.

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Throughout the paper, the following notation is used.

The symbol $C(X)$, $(C^*(X))$, denotes a ring of all continuous (bounded) functions from a topological space X to the real straight line R [10]. The set of all (co)zero-sets of X is denoted by $Z(X)$ $(CZ(X))$, i.e., $Z(X)=\{F=f^{-1}(0)\,|\,f\in C^*(X)\}$ $(CZ(X)=\{U=X\backslash F\,|\,F\in Z(X)\})$ [6]. Let $Z(X,Y)$ $(CZ(X,Y))$ be the trace of the system $Z(Y)$ $(CZ(Y))$ on the space X, i.e., $Z(X,Y)=\{F\cap X\,|\,F\in Z(Y)\}$ $(CZ(X,Y)=\{U\cap X\,|\,U\in CZ(Y)\})$ [6]. For each proximity space T with proximity δ let $Z_\delta(T)$ $(CZ_\delta(T))$ denote the set of all (co)zero-sets [14].

We denote by βX the Stone-Čech compactification of the space X [18]. The symbols $\overline{H}^{\,*}(-,-;G)$, $\overline{H}^{\,*}_f(-,-;G)$ and $\check{H}^{\,*}_f(-,-;G)$ denote the Alexander-Spanier classical cohomology [16], the finitely defined Alexander-Spanier cohomology [4] and the Čech cohomology [9], respectively.

Let Top be the category of topological spaces and continuous maps. By Prox we denote the category of proximity spaces and proximity maps. Let Top^2 and Prox^2 denote the category of pairs of topological and proximity spaces, respectively. Let $Y\in\mathrm{Top}$ be a topological space and (X,A) be a pair of subspaces of Y such that $A\in Z(X,Y)$. Then we say that the pair of topological spaces with respect to Y is a given one and we denote it by $(X,A)_Y$. A continuous map $f:(X_1,A_1)_{Y_1}\longrightarrow(X_2,A_2)_{Y_2}$ of two pairs of this type is called a continuous map $f:(X_1,A_1)\longrightarrow(X_2,A_2)$ of the pairs from the category Top^2, for which there exists a continuous extension $F:Y_1\longrightarrow Y_2$ of the map f [6].

Let Top^2_r be the category whose objects are the pairs $(X,A)_Y$ with respect to $Y\in\mathrm{Top}$, and let the morphisms be their continuous maps $f:(X_1,A_1)_{Y_1}\longrightarrow(X_2,A_2)_{Y_2}$.

For each pair $(X,A)_Y\in\mathrm{Top}^2_r$ we consider the pair $(i,j):A_Y\longrightarrow X_Y\longrightarrow(X,A)_Y$ of the inclusion maps $i:A_Y\longrightarrow X_Y$ and $j:X_Y\longrightarrow(X,A)_Y$. Thus the category Top^2_r is a c–category [9].

As is known [14], a proximity space is completely regular and generates a fixed compactification. Moreover, every proximity map of proximity spaces is a continuous map of completely regular spaces which has a continuous extension to the corresponding compactification. Therefore the category Prox^2 of pairs of proximity spaces is isomorphically embedded into the category Top^2_r. Consequently, we consider the category Prox^2 as a subcategory of the category Top^2_r .

The idea of defining (co)homology groups for a pair of spaces with respect to a space by cozero coverings belongs to Professor V. Baladze.

1. The Alexander-Spanier Cozero Cohomological δ–Functor

Let X be an arbitrary topological space and G be an abelian group. We will use the cochain complex $C^*(X;G) = \{C^q(X;G), \delta\}$ constructed in [16] in the manner as follows. Let $C^q(X;G)$ be the abelian group of all functions $\varphi : X^{q+1} \longrightarrow G$ of a $q+1$-fold product X^{q+1} of the topological space X to G. Here $\delta : C^q(X;G) \longrightarrow C^{q+1}(X;G)$ is the coboundary operator defined by

$$\delta\,\varphi(x_0,\dots,x_i,\dots,x_{q+1}) = \sum_{i=0}^{q+1}(-1)^i\varphi(x_0,\dots,\widehat{x}_i,\dots,x_{q+1}),$$

where the symbol "$\widehat{x}_i$" means the omission of the coordinate x_i.

Let X be a subspace of a topological space Y. The family $\alpha = \{U_i\}_{i=1}^n$ is called the $CZ(X,Y)$–covering of the space X if $X = \overset{n}{\underset{i=1}{\cup}} U_i$ and $U_i \in CZ(X,Y)$, $i = 1,\dots,n$ [6].

Definition 1.1. A function $\varphi : X^{q+1} \longrightarrow G$ is called a $CZ(X,Y)$–locally zero function if there exists a $CZ(X,Y)$–covering $\alpha = \{U_i\}_{i=1}^n$ of the space X such that $\varphi_{|\alpha^{q+1}} = 0$, where $\alpha^{q+1} = \bigcup\limits_{i=1}^{n} U_i^{q+1}$.

For each integer $q \geq 0$, we denote by $C_Y^q(X;G)$ a subset of the group $C^q(X,G)$ of all $CZ(X,Y)$–locally zero functions.

Lemma 1.2. *For each integer $q \geq 0$, $C_Y^q(X;G)$ is a subgroup of the group $C^q(X;G)$.*

Proof. Let $\varphi_1,\varphi_2 \in C_Y^q(X;G)$. Then there exist $CZ(X,Y)$–coverings $\alpha_1 = \{U_i\}_{i=1}^n$ and $\alpha_2 = \{V_j\}_{j=1}^m$ of the space X such that $\varphi_{1|\alpha_1^{q+1}} = 0$ and $\varphi_{2|\alpha_2^{q+1}} = 0$. Let $\alpha = \alpha \wedge \alpha_2 = \{W_{ij} = U_i \cap V_j\}_{i,j=1}^{n,m}$. It is clear that α is a $CZ(X,Y)$–covering and $\alpha^{q+1} \subset \alpha_1^{q+2} \cap \alpha_2^{q+1}$. Therefore $(\varphi_1 - \varphi_2)_{|\alpha^{q+1}} = 0$, which implies that $\varphi_1 - \varphi_2 \in C_Y^q(X;G)$. □

Lemma 1.3. *For each integer $q \geq 0$, $\delta\big(C_Y^q(X;G)\big) \subset C_Y^{q+1}(X;G)$.*

Proof. Let $\varphi \in C_Y^q(X;G)$. Then there exists a $CZ(X,Y)$–covering $\alpha = \{U_i\}_{i=1}^n$ of the space X such that $\varphi_{|\alpha^{q+1}} = 0$. By the definition of a coboundary operator, we have $\delta\varphi_{|\alpha^{q+2}} = 0$ and hence $\delta\varphi \in C_Y^{q+1}(X;G)$. □

By Lemmas 1.2 and 1.3 we have that the cochain complex $C_Y^*(X;G) = \big\{C_Y^q(X;G),\delta\big\}$ is a subcomplex of the complex $C^*(X;G) = \big\{C^q(X;G),\delta\big\}$. The q–dimensional cohomology group of the quotient complex $\overline{C}{}_Y^*(X;G) = C^*(X;G)/C_Y^*(X;G)$ we denote by $\overline{H}{}_Y^q(X;G)$ and call it the Alexander-Spanier q–dimensional cozero cohomology group of the space X with respect to Y.

It is known that a continuous function $f : X_1 \longrightarrow X_2$ induces a cochain map $f^{\#} : C^*(X_2; G) \longrightarrow C^*(X_1; G)$. By definition,

$$f^{\#}(\varphi)(x_0, x_1, \dots, x_q) = \varphi\big(f(x_0), f(x_1), \dots, f(x_q)\big), \quad \forall\, \varphi \in C^q(X_2; G).$$

For each continuous map $f : X_{1Y_1} \longrightarrow X_{2Y_2}$ from the category Top^2_r the inverse image $f^{-1}(\alpha) = \big\{f^{-1}(U_i)\big\}_{i=1}^n$ of the $CZ(X_2, Y_2)$–covering $\alpha = \{U_i\}_{i=1}^n$ of the space X_2 is the $CZ(X_1, Y_1)$–covering of the space X_1. The cochain map $f^{\#} : C^*(X_2; G) \longrightarrow C^*(X_1; G)$ satisfies the condition $f^{\#}(C^q_{Y_2}(X_2; G) \subset C^q_{Y_1}(X_1; G)$ and therefore induces a cochain map $f^{\#} : \overline{C}^{\,*}_{Y_2}(X_2; G) \longrightarrow \overline{C}^{\,*}_{Y_1}(X_1; G)$ and a homomorphism $f^* : \overline{H}^{\,*}_{Y_2}(X_2; G) \longrightarrow \overline{H}^{\,*}_{Y_1}(X_1; G)$.

Lemma 1.4. *For each pair $(X, A)_Y \in \mathrm{Top}^2_r$ the inclusion map $i : A_Y \longrightarrow X_Y$ induces an epimorphism*

$$i^{\#} : \overline{C}^{\,*}_Y(X; G) \longrightarrow \overline{C}^{\,*}_Y(A; G).$$

Proof. Let $\overline{\varphi} \in \overline{C}^{\,q}_Y(A; G)$ and $\varphi \in \overline{\varphi}$. It is clear that the function $\varphi : A^{q+1} \longrightarrow G$ can be extended to the function $\psi : X^{q+1} \longrightarrow G$, where

$$\psi(x_0, x_1, \dots, x_q) = \begin{cases} \varphi(x_0, x_1, \dots, x_q) & \text{if } \ (x_0, x_1, \dots, x_q) \in A^{q+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Consider the equivalent class $\overline{\psi} \in \overline{C}^{\,q}_Y(X; G)$ of ψ and show that it does not depend on the representatives of $\overline{\varphi}$. Indeed, let $\varphi_1, \varphi_2 \in \overline{\varphi}$ be arbitrary representatives of $\overline{\varphi}$. Then there exists a $CZ(A, Y)$–covering $\alpha = \{U_i\}_{i=1}^n$ of the space A such that $\varphi_{1|\alpha^{q+1}} = \varphi_{2|\alpha^{q+1}}$. By the definition of a $CZ(A, Y)$–covering, for each element $U_i \in \alpha$ there exists a cozero set $V_i \in CZ(Y)$ such that $U_i = V_i \cap A$. Note that in this case $\widetilde{\alpha} = \{X \backslash A\} \cup \big\{\widetilde{U}_i = V_i \cap X\big\}_{i=1}^n$ is a $CZ(X, Y)$–covering of the space X such that $\psi_{1|\widetilde{\alpha}^{\,q+1}} = \psi_{2|\widetilde{\alpha}^{\,q+1}}$ and therefore $\overline{\psi}_1 = \overline{\psi}_2$. Note that $i^{\#}(\overline{\psi}) = \overline{\varphi}$ and hence $i^{\#}$ is an epimorphism. □

For each pair $(X, A)_Y \in \mathrm{Top}^2_r$, we denote the kernel of the cochain map $i^{\#} : \overline{C}^{\,*}_Y(X; G) \longrightarrow \overline{C}^{\,*}_Y(A; G)$ by the symbol $\overline{C}^{\,*}_Y(X, A; G)$. The q–dimensional homology group of the resulting complex is denoted by $\overline{H}^{\,q}_Y(X, A; G)$ and we call it the Alexander-Spanier q–dimensional cozero cohomology group of the pair (X, A) with respect to Y.

Let $f : (X_1, A_1)_{Y_1} \longrightarrow (X_2, A_2)_{Y_2}$ be a continuous map of pairs. Then the homomorphism $f^{\#} : \overline{C}^{\,*}_{Y_2}(X_2; G) \longrightarrow \overline{C}^{*}_{Y_1}(X_1; G)$ satisfies the condition $f^{\#}\big(\overline{C}^{\,*}_{Y_2}(X_2, A_2; G)\big) \subset \overline{C}^{\,*}_{Y_1}(X_1, A_1; G)$ and thus induces a homomorphism

$$f^* : \overline{H}^{\,*}_{Y_2}(X_2, A_2; G) \longrightarrow \overline{H}^{\,*}_{Y_1}(X_1, A_1; G).$$

Theorem 1.5. *$\overline{H}^{\,*}_{-}(-, -; G)$ is a δ–functor from the category Top^2_r to the category of the graded abelian group $\mathscr{A}\mathscr{B}$.*

Proof. I. It is clear that each identity map $id_{(X,A)_Y} : (X,A)_Y \longrightarrow (X,A)_Y$ induces the identity cochain map $id^{\#}_{\overline{C}^{*}_{Y}(X,A;G)} : \overline{C}^{*}_{Y}(X,A;G) \longrightarrow \overline{C}^{*}_{Y}(X,A;G)$ and hence the identity homomorphism $id^{*}_{\overline{H}^{*}_{Y}(X,A;G)} : \overline{H}^{*}_{Y}(X,A;G) \longrightarrow \overline{H}^{*}_{Y}(X,A;G)$.

II. Let $f_1 : (X_1,A_1)_{Y_1} \longrightarrow (X_2,A_2)_{Y_2}$ and $f_2 : (X_2,A_2)_{Y_2} \longrightarrow (X_3,A_3)_{Y_3}$ be continuous maps. Consider the cochain maps:

$$\begin{aligned}
f_1^{\#} &: \overline{C}^{*}_{Y_2}(X_2,A_2;G) \longrightarrow \overline{C}^{*}_{Y_1}(X_1,A_1;G),\\
f_2^{\#} &: \overline{C}^{*}_{Y_3}(X_3,A_3;G) \longrightarrow \overline{C}^{*}_{Y_2}(X_2,A_2;G),\\
(f_1 \circ f_2)^{\#} &: \overline{C}^{*}_{Y_3}(X_3,A_3;G) \longrightarrow \overline{C}^{*}_{Y_1}(X_1,A_1;G).
\end{aligned}$$

According to the definition of an induced map, for each integer $q \geq 0$, $\overline{\varphi} \in \overline{C}^{*}_{Y_3}(X_3,A_3;G)$ and $(x_0,x_1,\ldots,x_q) \in X_1^{q+1}$ we have

$$\begin{aligned}
&(f_2 \circ f_1)^{\#}(\varphi)(x_0,x_1,\ldots,x_q) =\\
&\quad = \varphi\big((f_2 \circ f_1)(x_0),(f_2 \circ f_1)(x_1),\ldots,(f_2 \circ f_1)(x_q)\big) =\\
&\quad = \varphi\big(f_2(f_1(x_0)),f_2(f_1(x_1)),\ldots,f_2(f_1(x_q))\big);\\
&(f_1^{\#} \circ f_2^{\#})(\varphi)(x_0,x_1,\ldots,x_q) = f_1^{\#}\big(f_2^{\#}(\varphi)(x_0,x_1,\ldots,x_q)\big) =\\
&\quad = f_2^{\#}(\varphi)\big(f_1(x_0),f_1(x_1),\ldots,f_1(x_q)\big) =\\
&\quad = \varphi\big(f_2(f_1(x_0)),f_2(f_1)(x_1)),\ldots,f_2(f_1)(x_q))\big).
\end{aligned}$$

Therefore $(f_2 \circ f_1)^{\#} = f_1^{\#} \circ f_2^{\#}$ and hence $(f_2 \circ f_1)^{*} = f_1^{*} \circ f_2^{*}$.

III. It is clear that each continuous map $f : (X_1,A_1)_{Y_1} \longrightarrow (X_2,A_2)_{Y_2}$ induces the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \overline{C}^{*}_{Y_2}(X_2,A_2;G) \arrow[r] \arrow[d, "f^{\#}"] & \overline{C}^{*}_{Y_2}(X_2;G) \arrow[r] \arrow[d, "f^{\#}"] & \overline{C}^{*}_{Y_2}(A_2;G) \arrow[r] \arrow[d, "f^{\#}_{|A_1}"] & 0\\
0 \arrow[r] & \overline{C}^{*}_{Y_1}(X_1,A_1;G) \arrow[r] & \overline{C}^{*}_{Y_1}(X_1;G) \arrow[r] & \overline{C}^{*}_{Y_1}(A_1;G) \arrow[r] & 0\,.
\end{tikzcd}$$

Thus for each integer $q \geq 0$ there exist coboundary operators $\delta_i^q : \overline{H}^{q}_{Y_i}(A_i;G) \longrightarrow \overline{H}^{q+1}_{Y_i}(X_i,A_i;G)$, $i = 1,2$ such that the diagram

$$\begin{tikzcd}
\overline{H}^{q}_{Y_2}(A_2;G) \arrow[r, "\delta_2"] \arrow[d, "f^{*}"] & \overline{H}^{q+1}_{Y_2}(X_2,A_2;G) \arrow[d, "f^{*}"]\\
\overline{H}^{q}_{Y_1}(A_1;G) \arrow[r, "\delta_2"] & \overline{H}^{q+1}_{Y_1}(X_1,A_1;G)
\end{tikzcd}$$

is commutative. □

Note that if for a pair $(X,A)_Y$, $CZ(X,Y)=CZ(X)$, then the group $\overline{H}^{\,*}_{Y}(X,A;G)$ does not depend on Y, and in this case we use the notation $\overline{H}^{\,*}_{CZ}(X,A;G)$ instead of $\overline{H}^{\,*}_{Y}(X,A;G)$.

Let T be a proximity space. Consider T as a topological space with a topological structure induced by the proximity δ on T. Let δT be the compactification of T. It is clear that $T_{\delta T}\in \mathrm{Top}^2_r$ and $CZ(T,\delta T)=CZ_\delta(T)$. So, the group $\overline{H}^{\,*}_{\delta T}(T;G)$ depends only on the proximity space T. That is why we sometimes use the notation $\overline{H}^{\,*}_{\delta}(T;G)$ instead of $\overline{H}^{\,*}_{\delta T}(T;G)$, where the subscript δ indicates that T is considered as a proximity space.

2. Cohomology of Wallmen Compactification

A family $\mathscr{F}$ of closed subsets of a topological space X is called separating if for each closed set $H\subset X$ and each point $x\notin H$ there are disjoint sets in $\mathscr{F}$, one containing H and the other containing x. The family $\mathscr{F}$ is called a ring if it is closed under a finite union and finite intersections. A sequence $\{F_i\}_{i=1}^n$ of the sets in $\mathscr{F}$ is called a nest in $\mathscr{F}$ if there is a sequences $\{H_i\}_{i=1}^n$ such that $X\backslash H_{i+1}\subset F_{i+1}\subset X\backslash H_i\subset F_i$ for all $i=1,2,\dots$. A family of closed sets is a nest generated if for each member F of the family $\mathscr{F}$ there is a nest $\{F_i\}_{i=1}^n$ in $\mathscr{F}$ such that $F=\bigcap\limits_{i=1}^{n}F_i$ [17].

Let $\mathscr{L}(X)$ denote the family of all separating, nest generated intersection rings on the topological space X. It is known that if $\mathscr{F}$ is a separating normal ring on the space X, then the Wallmen space $w(X,\mathscr{F})$ of all ultrafilters on $\mathscr{F}$ is the Hausdorff compactification of the space X [17].

In the paper we use the following statements:

1. If $\mathscr{F}\in\mathscr{L}(X)$, then $\mathscr{F}$ is precisely the trace on X of all zero-sets in the Wallmen compactification $w(X,\mathscr{F})$ [17].
2. If $\mathscr{F}\in\mathscr{L}(Y)$ and $X\subset Y$, then $\{F\cap X\mid F\in\mathscr{F}\}\in\mathscr{L}(X)$ [17].
3. For each topological space X, $Z(X)\in\mathscr{L}(X)$ [17].

Let X be a topological space and $\mathscr{A}$ be a family of subspaces. The trace of the family $\mathscr{A}$ on the subspace A of the space X is denoted by $\mathscr{A}_A$.

Let $\mathscr{F}\in\mathscr{L}(X)$ and $\mathscr{A}$ be a filter on $\mathscr{F}_A$. Denote by $\mathscr{A}^X$ a maximal subfamily of the family $\mathscr{F}$ such that the trace of $\mathscr{A}^X$ on the subspace A coincides with the family $\mathscr{A}$.

Lemma 2.1. *Let X be a complete regular space, $\mathscr{F}\in\mathscr{L}(X)$, $\mathscr{A}$ be a filter on the family $\mathscr{F}$ and $A\in\mathscr{A}$, then $\mathscr{A}_A$ is a filter on $\mathscr{F}_A$.*

Proof. 1. The subspace A is an element of the filter $\mathscr{A}$, therefore $\varnothing\notin\mathscr{A}_A$.

2. Let $A_1,A_2\in\mathscr{A}_A$, then there exist $A'_1,A'_2\in\mathscr{A}$ such that $A_1=A'_1\cap A$, $A_2=A'_2\cap A$. The family $\mathscr{A}$ is a filter and hence $A'_1\cap A'_2\in\mathscr{A}$. Therefore

$A_1 \cap A_2 = (A'_1 \cap A) \cap (A'_2 \cap A) = (A'_1 \cap A'_2) \cap A$, which implies that $A_1 \cap A_2 \in \mathscr{A}_A$;

3. Suppose that for an element $A_1 \in \mathscr{F}_A$ there exists $A_2 \in \mathscr{A}_A$ such that $A_2 \subset A_1$. It is clear that in this case $A_1 \in \mathscr{F}$ and $A_2 \in \mathscr{A}$. We know that the family $\mathscr{A}$ is a filter and hence $A_1 \in \mathscr{A}$. On the other hand, $A_1 = A_1 \cap A$ and therefore $A_1 \in \mathscr{A}_A$. □

Lemma 2.2. *If X is a complete regular space, $\mathscr{F} \in \mathscr{L}(X)$, $A \in \mathscr{F}$, and $\mathscr{A}$ is a filter on the family $\mathscr{F}_A$, then $\mathscr{A}^X$ is a filter on the family $\mathscr{F}$.*

Proof. 1. We know that $\varnothing \notin \mathscr{A}$ and therefore $\varnothing \notin \mathscr{A}^X$.

2. For each elements $A_1, A_2 \in \mathscr{A}^X$ there exist elements $A'_1, A'_2 \in \mathscr{A}$ such that $A'_1 = A_1 \cap A$ and $A'_2 = A_2 \cap A$. The family $\mathscr{A}$ is a filter and hence $A'_1 \cap A'_2 \in \mathscr{A}$. Note that $(A_1 \cap A_2) \cap A = (A_1 \cap A) \cap (A_2 \cap A) = A'_1 \cap A'_2 \in \mathscr{A}$ and therefore $A_1 \cap A_2 \in \mathscr{A}^X$.

3. Suppose that for an element $A_1 \in \mathscr{F}$ there exists $A_2 \in \mathscr{A}^X$ such that $A_2 \subset A_1$. Let us show that $A_1 \in \mathscr{A}^X$. Indeed, for the element A_2 there exists $A'_2 \in \mathscr{A}$ such that $A'_2 = A_2 \cap A$. It is clear that $A'_2 \subset A_1 \cap A$ and hence $A_1 \cap A \in \mathscr{A}$. Therefore $A_1 \in \mathscr{A}^X$. □

Lemma 2.3. *If X is a complete regular space, $\mathscr{F} \in \mathscr{L}(X)$, $\mathscr{A}$ is an ultrafilter on the family $\mathscr{F}$ and $A \in \mathscr{A}$, then $\mathscr{A}_A$ is an ultrafilter on the family $\mathscr{F}_A$.*

Proof. By Lemma 2.1, $\mathscr{A}_A$ is a filter. Our goal is to show that $\mathscr{A}_A$ is an ultrafilter. Indeed, let $\mathscr{A}'$ be a filter on the family $\mathscr{F}_A$ such that $\mathscr{A}_A \subset \mathscr{A}'$. Then $\mathscr{A} \subset \mathscr{A}'^X$. On the other hand, $\mathscr{A}$ is an ultrafilter and since by Lemma 2.2 $\mathscr{A}'^X$ is a filter, we have $\mathscr{A} = \mathscr{A}'^X$, from which it follows that $\mathscr{A}_A = \mathscr{A}'$. □

Lemma 2.4. *If X is a complete regular space, $\mathscr{F} \in \mathscr{L}(X)$, $A \in \mathscr{F}$, $\mathscr{A}$ is an ultrafilter on the family $\mathscr{F}_A$, then $\mathscr{A}^X$ is an ultrafilter on the family $\mathscr{F}$.*

Proof. By Lemma 2.2, $\mathscr{A}^X$ is a filter. Let $\mathscr{A}'$ be a filter on the family $\mathscr{F}$ such that $\mathscr{A} \subset \mathscr{A}'$. Then $\mathscr{A} \subset \mathscr{A}'_A$. On the other hand, $\mathscr{A}$ is an ultrafilter and since by Lemma 2.1 $\mathscr{A}'_A$ ia a filter, we have $\mathscr{A} = \mathscr{A}'_A$ and hence $\mathscr{A}^X = \mathscr{A}'$. □

Let X be a complete regular space, $\mathscr{F} \in \mathscr{L}(X)$, and A be a subspace of the space X. Denote by $\overline{A}^w$ the closure of the subspace A in the space $w(\mathscr{F})$. Let $O_w A = w(\mathscr{F}) \backslash \overline{X \backslash A}^w$. If $A \in \mathscr{F}$ and $U = X \backslash A$, then the sets $\{\mathscr{A} \in w(\mathscr{F}) \,|\, A \in \mathscr{A}\}$ and $\{\mathscr{A} \in w(\mathscr{F}) | X \backslash U \notin \mathscr{A}\}$ are denoted by A^* and U_*, respectively. Note that if $\mathscr{B}(X) = \{U = X \backslash A \,|\, A \in \mathscr{F}\}$, then $\mathscr{B}^*(X) = \{U_* \,|\, U \in \mathscr{B}(X)\}$ is the basis of the space $w(\mathscr{F})$.

Lemma 2.5. *If X is a complete regular space, $\mathscr{F} \in \mathscr{L}(X)$, $A \in \mathscr{F}$, then $w(\mathscr{F}_A)$ is homeomorphic to the closure $\overline{A}^{\,w}$ of the subspace A in the space $w(\mathscr{F})$.*

Proof. Let a function $f : w(\mathscr{F}_A) \longrightarrow w(\mathscr{F})$ be given by the formula $f(\mathscr{A}) = \mathscr{A}^X$, $\forall \mathscr{A} \in w(\mathscr{F}_A)$. Let us show that f is an injunction. Indeed, let $\mathscr{A}_1 \neq \mathscr{A}_2$, then there exist elements $\mathscr{A}_1, \mathscr{A}_2 \in \mathscr{F}_A$ such that $A_1 \in \mathscr{A}_1$, $A_2 \in \mathscr{A}_2$, and $A_1 \cap A_2 = \varnothing$. It is easy to show that $A_1 \in \mathscr{A}_1^X$, $A_2 \in \mathscr{A}_2^X$ and that is why $\mathscr{A}_1^X \neq \mathscr{A}_2^X$. Thus we obtain $f(\mathscr{A}_1) \neq f(\mathscr{A}_2)$.

Let us show that $\overline{A}^{\,w} = f(w(\mathscr{F}_A))$:

1) $\mathscr{A} \in \overline{\mathscr{A}}^{\,w} \Rightarrow A \in \mathscr{A}$, $\mathscr{A} \in w(\mathscr{F}) \Rightarrow A \in \mathscr{A}_A$, $\mathscr{A}_A \in w(\mathscr{F}_A) \Rightarrow (\mathscr{A}_A)^X = \mathscr{A} \Rightarrow f(\mathscr{A}_A) = \mathscr{A} \Rightarrow \mathscr{A} \in f(w(\mathscr{F}_A))$;

2) $\mathscr{A}^X \in f(w(\mathscr{A}_A)) \Rightarrow \exists\, \mathscr{A} \in w(\mathscr{F}_A)$, $f(\mathscr{A}) = \mathscr{A}^X \Rightarrow A \in \mathscr{A}^X$ (because $A \in \mathscr{F}$ and $A \in \mathscr{A}) \Rightarrow \mathscr{A}^X \in \overline{A}^{\,w}$.

Thus it remains to show that the function $f : w(\mathscr{F}_A) \longrightarrow w(\mathscr{F})$ is an open one. Indeed, let $\mathscr{B}^*(A) = \{U_* \,|\, A\backslash U \in \mathscr{F}_A\}$ be the basis of the space $w(\mathscr{F}_A)$ and let us show that $f(U_*)$ is an open subspace of the space $\overline{A}^{\,w}$. We actually have $f(U_*) = U'_* \cap \overline{A}^{\,w}$, where $U' = X\backslash(A\backslash U)$. Indeed:

1) $f(\mathscr{A}) = \mathscr{A}^X \in f(U_*) \Rightarrow \mathscr{A} \in U_* \Rightarrow \mathscr{A} \in \{\mathscr{A} \in w(\mathscr{F}_A) \,|\, A\backslash U \notin \mathscr{A}\}$ $\Rightarrow A\backslash U \notin \mathscr{A} \Rightarrow A\backslash U \notin \mathscr{A}^X$ (because $A \in \mathscr{F}$, $A\backslash U \in \mathscr{F}_A \Rightarrow A\backslash U \in \mathscr{F}$) $\Rightarrow \mathscr{A}^X \in \{\mathscr{A} \in w(\mathscr{F}) \,|\, A\backslash U \notin \mathscr{A}\} \Rightarrow \mathscr{A}^X \in \{\mathscr{A} \in w(\mathscr{F}) \,|\, X\backslash U' \notin \mathscr{A}\} \Rightarrow \mathscr{A}^X \in U'_* \Rightarrow \mathscr{A}^X \in U'_* \cap \overline{A}^{\,w}$.

2) $\mathscr{A}' \in U'_* \cap \overline{A}^{\,w} \Rightarrow \exists\, \mathscr{A} \in w(\mathscr{F}_A)$ (because $\overline{A}^{\,w} = f(w(\mathscr{F}_A))$), $f(\mathscr{A}) = \mathscr{A}^X = \mathscr{A}' \in U'_* \cap \overline{A}^{\,w} \Rightarrow \mathscr{A}^X \in \{\mathscr{A} \in w(\mathscr{F}) \,|\, X\backslash U' \notin \mathscr{A}\} \Rightarrow X\backslash U' \notin \mathscr{A}^X$ $\Rightarrow A\backslash U \notin \mathscr{A} \Rightarrow \mathscr{A} \in \{\mathscr{A} \in w(\mathscr{F}_A) \,|\, A\backslash U \notin \mathscr{A}\} \Rightarrow \mathscr{A} \in U_* \Rightarrow f(\mathscr{A}) = \mathscr{A}^X = \mathscr{A}' \in f(U_*)$. $\square$

Lemma 2.6. *Let X be a complete regular space, $\mathscr{F} \in \mathscr{L}(X)$ and $A_i \in \mathscr{F}$, $i = 1, 2, \ldots, n$, then $\overline{\bigcap\limits_{i=1}^{n} A_i}^{\,w} = \bigcap\limits_{i=1}^{n} \overline{A_i}^{\,w}$.*

Proof. It is clear that $\overline{\bigcap\limits_{i=1}^{n} A_i}^{\,w} \subset \bigcap\limits_{i=1}^{n} \overline{A_i}^{\,w}$. Let us show that $\bigcap\limits_{i=1}^{n} \overline{A_i}^{\,w} \subset \overline{\bigcap\limits_{i=1}^{n} A_i}^{\,w}$. Indeed, let $\mathscr{A} \in \bigcap\limits_{i=1}^{n} \overline{A_i}^{\,w}$, then $A_i \in \mathscr{A}$ for all $i = 1, 2, \ldots, n$. Since $\mathscr{A}$ is an ultrafilter, we have $\bigcap\limits_{i=1}^{n} A_i \in \mathscr{A}$, which means that $\mathscr{A} \in \overline{\bigcap\limits_{i=1}^{n} A_i}^{\,w}$. $\square$

Lemma 2.6 and the equation $CZ(X, w(\mathscr{F})) = \{U = X\backslash A \,|\, A \in \mathscr{F}\}$ give rise to

Corollary 2.7. *For any elements $U_1, U_2 \in CZ(X, w(\mathscr{F}))$ we have $O_w(U_1 \cup U_2) = O_w U_1 \cup O_w U_2$.*

Let X be a complete regular space, $\mathscr{F} \in \mathscr{L}(X)$ and $A \in \mathscr{F}$. Suppose $(w(\mathscr{F}), w(\mathscr{F}_A))$ is the compactificacion of the pair (X, A) and $i : (X, A)_{w(\mathscr{F})} \longrightarrow (w(\mathscr{F}), w(\mathscr{F}_A))_{w(\mathscr{F})}$ is the inclusion map. Thus we obtain the map

$$i^{-1} : \mathrm{Cov}^{CZ}_{w(\mathscr{F})}(w(\mathscr{F}), w(\mathscr{F}_A)) \longrightarrow \mathrm{Cov}^{CZ}_{w(\mathscr{F})}(X, A).$$

Lemma 2.8. *Let X be a complete regular space, $\mathscr{F} \in \mathscr{L}(X)$ and $A \in \mathscr{F}$, then the image of the map $i^{-1} : \mathrm{Cov}^{CZ}_{w(\mathscr{F})}(w(\mathscr{F}), w(\mathscr{F}_A)) \longrightarrow \mathrm{Cov}^{CZ}_{w(\mathscr{F})}(X, A)$ is a cofinite subspace of the direct set $\mathrm{Cov}^{CZ}_{w(\mathscr{F})}(X, A)$.*

Proof. Suppose $(\alpha, \alpha') \in \mathrm{Cov}^{CZ}_{w(\mathscr{F})}(X, A)$ is a $CZ(X, w(\mathscr{F}))$–covering. Let $\beta = \{O_w U_\alpha \,|\, U_\alpha \in \alpha\}$ and $\beta' = \{O_w U_{\alpha'} \,|\, U'_\alpha \in \alpha'\}$. By Corollary 2.7, (β, β') is a finite covering of the pair $(w(\mathscr{F}), w(\mathscr{F}_A))$. Since $(w(\mathscr{F}), w(\mathscr{F}_A))$ is a pair of compact spaces, there exists a $CZ(w(\mathscr{F}), w(\mathscr{F}))$–covering $(\mu, \mu') \in \mathrm{Cov}^{CZ}_{w(\mathscr{F})}(w(\mathscr{F}), w(\mathscr{F}_A))$ such that $(\mu, \mu') \geq (\beta, \beta')$ and therefore $i^{-1}(\mu, \mu') \geq (\alpha, \alpha')$. $\square$

Let (α, α') be a covering of the pair (X, A). Denote by $(X(\alpha), A(\alpha'))$ the pair of simplicial complexes defined as follows: $X(\alpha) = \{s = \{s_0, s_1, \ldots, s_n\} \subset X \,|\, n \in N,\ \exists\ U_\alpha \in \alpha,\ s \subset U_\alpha\}$, $A(\alpha') = \{s \in X(\alpha) \,|\, \exists\ U_\alpha \in \alpha',\ s \subset U_\alpha \cap A\}$. Let $C(\alpha, \alpha'; G)$ be the cochain complex of the pair $(X(\alpha), A(\alpha'))$. For each pair $(X, A)_Y \in \mathrm{Top}^2_r$ consider the direct set $\mathrm{Cov}^{CZ}_Y(X, A)$ and the corresponding limit complex

$$\varinjlim_{(\alpha,\alpha') \in \mathrm{Cov}^{CZ}_Y(X,A)} C(\alpha, \alpha'; G).$$

Lemma 2.9. *For each pair $(X, A)_Y \in \mathrm{Top}^2_r$ there is an isomorphism*

$$\widetilde{\tau} : \overline{C}\,^q_Y(X, A; G) \approx \varinjlim_{(\alpha,\alpha') \in \mathrm{Cov}^{CZ}_Y(X,A)} C^q(\alpha, \alpha'; G).$$

Proof. Let $C^q_Y(X, A; G)$ be a subgroup of the group $C^q(X; G)$ of all $\varphi : X^{q+1} \to G$ functions such that $\varphi_{|A^{q+1}} : A^{q+1} \to G$ is a $CZ(A; Y)$–locally zero function. It is clear that $C^*_Y(X, A; G) = \{C^*_Y(X, A; G), \delta\}$ is a subcomplex of the complex $C^*(X; G)$ and there is an isomorphism $\overline{C}\,^*_Y(X, A; G) \approx C^*_Y(X, A; G)/C^*_Y(X; G)$.

For each element $\varphi \in C^q_Y(X, A; G)$ there exists a $CZ(A, Y)$–covering $\alpha'' = \{U_{\alpha'_1}, U_{\alpha'_2}, \ldots, U_{\alpha'_n}\}$ of the subspace A such that $\varphi_{|(\alpha'')^{q+1}} = 0$. The covering α'' is a $CZ(A, Y)$–covering, therefore for each $U''_{\alpha_i} \in \alpha''$ there exists a cozero subspace $U'_{\alpha_i} \in CZ(X, Y)$ such that $U'_{\alpha_i} \cap A = U''_{\alpha_i}$. Let $\alpha' = \{U'_{\alpha_i}\}$ and $\alpha = \alpha' \cup \{X \backslash A\}$, then it is clear that $(\alpha, \alpha') \in \mathrm{Cov}^{CZ}_Y(X, A)$ and therefore $\varphi_\alpha = \varphi_{|\alpha^{q+1}} \in C^q(\alpha, \alpha'; G)$. Suppose that the map

$$\tau : C^q_Y(X, A; G) \longrightarrow \varinjlim_{(\alpha,\alpha') \in \mathrm{Cov}^{CZ}_Y(X,A)} C^q(\alpha, \alpha'; G)$$

is defined by the formula

$$\tau(\varphi) = \pi_\alpha(\varphi_\alpha), \ \ \forall \ \ \varphi \in C^q_Y(X, A; G),$$

where $\pi_\alpha : C^q(\alpha, \alpha'; G) \longrightarrow \varinjlim_{(\alpha,\alpha')\in \mathrm{Cov}^{CZ}_Y(X,A)} C^q(\alpha, \alpha'; G)$ is a natural projection.

Now we are to show that the map τ induces the isomorphism

$$\widetilde{\tau} : \overline{C}\,^q_Y(X, A; G) \approx \varinjlim_{(\alpha,\alpha')\in \mathrm{Cov}^{CZ}_Y(X,A)} C^q(\alpha, \alpha'; G).$$

To verify this fact it suffices to show that τ is a surjection and $\operatorname{Ker}\tau = C^q_Y(X; G)$. Indeed, for each element $\overline{\varphi} \in \varinjlim_{(\alpha,\alpha')\in \mathrm{Cov}^{CZ}_Y(X,A)} C^q(\alpha, \alpha'; G)$ there exists $(a, a') \in \mathrm{Cov}^{CZ}_Y(X, A)$ such that $\varphi_\alpha \in C^q(\alpha, \alpha'; G)$ and $\pi_\alpha(\varphi_\alpha) = \overline{\varphi}$. Define the map $\varphi : X^{q+1} \longrightarrow G$ by the formula

$$\varphi(x_0, x_1, \dots, x_n) = \begin{cases} \varphi_\alpha(x_0, x_1, \dots, x_n), \ \forall \, x_0, x_1, \dots, x_n \in U_\alpha, \ U\alpha \in \alpha, \\ 0, \qquad \text{otherwise.} \end{cases}$$

Then $\varphi_{|(\alpha')^{q+1}\cap A^{q+1}} = 0$, $\varphi_\alpha = \varphi_{|\alpha^{q+1}}$. Thus $\varphi \in C^{CZ}_Y(X, A; G)$ and $\tau(\varphi) = \overline{\varphi}$.

It remains to show that $\operatorname{Ker}\tau = C^q_Y(X; G)$. Let $\varphi \in C^q_Y(X; G)$, then, by the definition, we have $\tau(\varphi) = 0$ and therefore $\varphi \in \operatorname{Ker}\tau$. On the other hand, if $\varphi \in \operatorname{Ker}\tau$, then there exists $(\alpha, \alpha') \in \mathrm{Cov}^{CZ}_Y(X, A)$ such that $\tau(\varphi) = \pi_\alpha(\varphi_\alpha) = 0$. In this case there exists $(\beta, \beta') \in \mathrm{Cov}^{CZ}_Y(X, A)$ such that $(\alpha, \alpha') < (\beta, \beta')$ and $\pi_{\alpha\beta}(\varphi_\alpha) = \varphi_{|\beta^{q+1}} = 0$. Therefore $\varphi \in C^q_Y(X; G)$. □

Theorem 2.10. *Let X be a complete regular space, $\mathscr{F} \in \mathscr{L}(X)$ and $A \in \mathscr{F}$, then*

$$\overline{H}\,^q_{w(\mathscr{F})}(X, A; G) \approx \overline{H}\,^q\big(w(\mathscr{F}), w(\mathscr{F}_A); G\big).$$

Proof. Let $(\alpha, \alpha') \in \mathrm{Cov}^{CZ}_{w(\mathscr{F})}(X, A)$. Consider the pairs (X, A) and (α, α') as the pairs of sets. Define the pair of relations (R, R') in the following manner:

$(x, U_\alpha) \in R$ if and only if $x \in X$, $U_\alpha \in \alpha$ and $x \in U_\alpha$.

$(x, U_\alpha) \in R'$ if and only if $x \in A$, $U_\alpha \in \alpha'$ and $x \in U_\alpha$.

Following [7], the pair (R, R') defines the pairs of the simplicial complexes $(X(\alpha), A(\alpha'))$ and $(X_\alpha, A_{\alpha'})$ and these complexes have isomorphic cohomology groups, where $(X_\alpha, A_{\alpha'})$ is the nerve of the covering (α, α'). On the other hand, if $(\beta, \beta') \in \mathrm{Cov}^{CZ}_{w(\mathscr{F})}(w(\mathscr{F}), w(\mathscr{F}_A))$ and $(\alpha, \alpha') = i^{-1}(\beta, \beta') \in \mathrm{Cov}^{CZ}_{w(\mathscr{F})}(X, A)$, then $(w(\mathscr{F})_\beta, w(\mathscr{F}_A)_{\beta'}) \approx (X_\alpha, A_{\alpha'})$. By virtue of the above facts and Lemmas 2.8 and 2.9, we obtain

$$\overline{H}\,^q\big(w(\mathscr{F}), w(\mathscr{F}_A); G\big) \approx \varinjlim_{(\beta,\beta')\in \mathrm{Cov}^{CZ}_{w(\mathscr{F})}(w(\mathscr{F}),w(\mathscr{F}_A))} H^q(C^*(\beta, \beta'; G)) \approx$$

$$\approx \varinjlim_{(\beta,\beta')\in \mathrm{Cov}^{CZ}_{w(\mathscr{F})}(w(\mathscr{F}),w(\mathscr{F}_A))} H^q(C^*(w(\mathscr{F})_\beta, w(\mathscr{F}_A)_{\beta'};G)) \approx$$
$$\approx \varinjlim_{(\alpha,\alpha')\in \mathrm{Cov}^{CZ}_{w(\mathscr{F})}(X,A)} H^q(C^*(X_\alpha, A_{\alpha'};G)) \approx$$
$$\approx \varinjlim_{(\alpha,\alpha')\in \mathrm{Cov}^{CZ}_{w(\mathscr{F})}(X,A)} H^q(C^*(\alpha,\alpha';G)) \approx \overline{H}\,^q_{w(\mathscr{F})}(X,A;G). \qquad \square$$

Corollary 2.11. *Let A be a completely closed subspace of a completely regular space X, then*

$$\overline{H}\,^*_{CZ}(X,A;G) \approx \overline{H}\,^*(\beta X, \beta A;G).$$

Corollary 2.12. *Let A be a closed subspace of a normal space X, then*

$$\overline{H}\,^*_{CZ}(X,A;G) \approx \overline{H}\,^*_f(X,A;G) \approx \overline{H}\,^*(\beta X, \beta A;G).$$

3. Application of the Cozero Cohomology

Let T be a proximity space whose corresponding proximity is δ. It is clear that the proximity δ precisely defines the extension (compactification) δT. If we consider T as a completely regular space induced by δ, then δT is the compactification of the topological space T. Let $\mathrm{Cov}^\delta(T)$ be the direct set of all proximity coverings of the proximity space T and let $\mathrm{Cov}^\delta_{CZ}(T)$ be the direct set of all finite cozero (with respect to proximity maps) coverings of the proximity space. There arise the following questions:

1. How can the proximity space T be described by the direct sets $\mathrm{Cov}^\delta(T)$ and $\mathrm{Cov}^\delta_{CZ}(T)$?

2. Do the direct sets $\mathrm{Cov}^\delta(T)$ and $\mathrm{Cov}^\delta_{CZ}(T)$ generate the same cohomology or other proximity invariants?

Below we will prove the theorem that provides answers to the above questions.

Let $\overline{h}\,^*_\delta(-;G)$ be an Alexander-Spanier type cohomology defined by the direct set $\mathrm{Cov}^\delta(T)$ [2]. It is clear that then we obtain an isomorphism $\overline{h}\,^*_\delta(T;G) \approx \overline{H}\,^*(\delta T;G)$. Note that an Alexander-Spaniers type cohomology defined by the direct set $\mathrm{Cov}^\delta_{CZ}(T)$ is precisely the $\overline{H}\,^*_{\delta T}(T;G)$, where T is considered as a topological space and δT as its compactification.

Theorem 3.1. *For a proximity space T the groups $\overline{h}\,^*_\delta(T;G)$ and $\overline{H}\,^*_{\delta T}(T;G)$ are not isomorphic in general.*

Proof. To prove the theorem it suffices to find at least one proximity space T such that $\overline{h}\,^*_\delta(T;G) \neq \overline{H}\,^*_{\delta T}(T;G)$.

Let R be the real line. Consider R with the corresponding maximal β and minimal α proximity. It is clear that the proximity β gives the

Stone-Čech compactification and the proximity α gives a one-point compactification. According to [8], the group $\overline{H}^{\,1}(\beta R; Z) \neq Z$ and therefore $\overline{H}^{\,1}(\beta R; Z) \neq \overline{H}^{\,1}(\alpha R; Z)$. Thus we obtain that $\overline{h}^{\,1}_{\beta}(R; Z) \neq \overline{h}^{\,1}_{\alpha}(R; Z)$. Since the topological space R is a Lindeloff space, by virtue of [17] we have $CZ(R, \beta R) = CZ(R, \alpha R)$ and therefore $\mathrm{Cov}^{\beta}_{CZ}(R) = \mathrm{Cov}^{\alpha}_{CZ}(R)$, which implies $\overline{H}^{\,*}_{\beta R}(R; Z) \approx \overline{H}^{\,*}_{\alpha R}(R; Z)$. On the other hand, $\overline{H}^{\,*}_{CZ}(R; Z) = \overline{H}^{\,*}_{\beta R}(R; Z) \approx \overline{H}^{\,*}(\beta R; Z)$ and thus

$$\overline{H}^{\,*}_{\alpha R}(R; Z) \approx \overline{H}^{\,*}_{\beta R}(R; Z) \approx \overline{h}^{\,*}_{\beta}(R; G) \neq \overline{h}^{\,*}_{\alpha}(R; G). \qquad \square$$

Corollary 3.2. *For a proximity space T the direct sets $\mathrm{Cov}^{\delta}(T)$ and $\mathrm{Cov}^{\delta}_{CZ}(T)$ does not generate the same cohomology invariants in general.*

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Author's address:

Department of Mathematics of
Shota Rustavely State University
35, Ninoshvili St., Batumi 6010
Georgia