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ON THE SEPARABLE DOUBLE HAAR WAVELET

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Abstract. A property of double Haar wavelet coefficients of composite functions is established.

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1. Introduction

One of the classes of orthonormal systems of functions well adapted to image analysis is the class of "separable" wavelets which are formed in the following way (cf. [4, p. 108])

Let φ be a scaling function of some multiresolution analysis (in $L^2(\mathbb{R})$) and ψ be the corresponding wavelet. If we define functions h^1, h^2 and h^3 as follows

$$\begin{aligned} h^1(x,y) &= \varphi(x)\psi(y), \\ h^2(x,y) &= \psi(x)\varphi(y), \quad x,y \in \mathbb{R}, \\ h^1(x,y) &= \psi(x)\psi(y), \end{aligned} \tag{1}$$

then the family

$$h^{i,j}_{k,l}(x,y) = 2^j h^i(2^j x - k, 2^j y - l), \quad i = 1,2,3, \quad j,k \in \mathbb{Z}, \tag{2}$$

is an orthonormnal basis in $L^2(\mathbb{R}^2)$,

If the functions φ and ψ have compact supports then the supports of functions from the family (2) are well localized which causes good applicability of such system to image analysis.

The Haar wavelet (see definition in Section 2) has the shortest support among all orthonormal on $\mathbb{R}$ wavelets. Here we consider the system (2) formed by the Haar scaling function φ and Haar wavelet ψ and investigate

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Fourier expansions of functions $f \in L^2(\mathbb{R}^2)$ with respect to the system

$$\sum_{j,k,l=-\infty}^{\infty} \sum_{i=1}^{3} \langle f, h_{k,l}^{i,j} \rangle \, h_{k,l}^{i,j} \,, \tag{3}$$

where

$$\langle f, h_{k,l}^{i,j} \rangle = \int_{\mathbb{R}^2} f(x,y) h_{k,l}^{i,j}(x,y)\, dx\, dy, \quad i = 1,2,3, \quad j,k,l \in \mathbb{Z}.$$

The series (3) converges in L^2-norm to f and

$$\sum_{j,k,l=-\infty}^{\infty} \sum_{i=1}^{3} \left| \langle f, h_{k,l}^{i,j} \rangle \right|^2 = \|f\|_{L^2}^2$$

because of completeness of the system.

For $0 < q < 2$ convergence of the series

$$\sum_{j,k,l=-\infty}^{\infty} \sum_{i=1}^{3} \left| \langle f, h_{k,l}^{i,j} \rangle \right|^q \tag{4}$$

means that large coefficients are not too many and the system is "well adapted" to the function f. The sum in (4) is an example of a so called "information cost function" (see [3, p. 400]).

We represent an image as a function from $L^2(\mathbb{R}^2)$.

In many cases images obtained by various devices are distorted in the space variable. The problem we investigate gives, in particular, the answer to the following question: If the above ("separable" Haar) basis is well adapted to a function will it be well adapted to the deformated function as well?

The results of this paper are formulated in Theorem 2.3 and Theorem 2.4 at the end of Section 2. These results were announced in [1].

One-dimensional distortion in time of signals is considered in [2].

2. Definitions and Results

The Haar scaling function and the Haal wavelet ψ arc defined on the real line $\mathbb{R}$ as follows (cf. [4, p. 73])

$$\varphi(x) = \begin{cases} 1, & x \in [0,1), \\ 0, & x \in \mathbb{R} \setminus [0,1), \end{cases} \tag{5}$$

$$\psi(x) = \begin{cases} 1, & x \in \left[0, \frac{1}{2}\right), \\ -1, & x \in \left[\frac{1}{2}, 1\right), \\ 0, & x \in \mathbb{R} \setminus [0,1). \end{cases} \tag{6}$$

For these functions (1) and (2) give an orthonormal basis in $L^2(\mathbb{R}^2)$.

Let

$$
\begin{aligned}
\Delta^{1,j}_{k,l} &= \left[k\cdot 2^{-j}, (2k+1)2^{-j-1}\right) \times \left[l\cdot 2^{-j}, (2l+1)2^{-j-1}\right),\\
\Delta^{2,j}_{k,l} &= \left[(2k+1)2^{-j-1}, (k+1)2^{-j}\right) \times \left[l\cdot 2^{-j}, (2l+1)2^{-j-1}\right),\\
\Delta^{3,j}_{k,l} &= \left[(2k+1)2^{-j-1}, (k+1)2^{-j}\right) \times \left[(2l+1)2^{-j-1}, (l+1)2^{-j}\right),\\
\Delta^{4,j}_{k,l} &= \left[k\cdot 2^{-j}, (2k+1)2^{-j-1}\right) \times \left[(2l+1)2^{-j-1}, (l+1)2^{-j}\right),\\
\Delta^{j}_{k,l} &= \bigcup_{i=1}^{4} \Delta^{i,j}_{k,l}, \quad j,k,l\in\mathbb{Z},
\end{aligned} \tag{7}
$$

where "$\times$" stands for the cartesian product of the intervals. We say that j is the *rank* of the square $\Delta^j_{k,l}$.

For $j,k,l\in\mathbb{Z}$ we have (see (1), (2), (5), (6), (7))

$$
h^{1,j}_{k,l}(x,y) = \begin{cases} 2^j, & (x,y)\in \Delta^{1,j}_{k,l}\cup\Delta^{2,j}_{k,l},\\ -2^j, & (x,y)\in \Delta^{3,j}_{k,l}\cup\Delta^{4,j}_{k,l},\\ 0, & (x,y)\in\mathbb{R}^2\setminus\Delta^j_{k,l}, \end{cases} \tag{8}
$$

$$
h^{2,j}_{k,l}(x,y) = \begin{cases} 2^j, & (x,y)\in \Delta^{1,j}_{k,l}\cup\Delta^{4,j}_{k,l},\\ -2^j, & (x,y)\in \Delta^{2,j}_{k,l}\cup\Delta^{3,j}_{k,l},\\ 0, & (x,y)\in\mathbb{R}^2\setminus\Delta^j_{k,l}, \end{cases} \tag{9}
$$

$$
h^{3,j}_{k,l}(x,y) = \begin{cases} 2^j, & (x,y)\in \Delta^{1,j}_{k,l}\cup\Delta^{3,j}_{k,l},\\ -2^j, & (x,y)\in \Delta^{2,j}_{k,l}\cup\Delta^{4,j}_{k,l},\\ 0, & (x,y)\in\mathbb{R}^2\setminus\Delta^j_{k,l}. \end{cases} \tag{10}
$$

It follows that

$$
\int_{\Delta^j_{k,l}} h^{i,j}_{k,l}(x,y)\,dx\,dy = 0, \quad i=1,2,3, \quad j,k,l\in\mathbb{Z}. \tag{11}
$$

For a function $\in L^2(\mathbb{R}^2)$ we introduce the wavelet coefficients

$$
\langle\, f, h^{i,j}_{k,l}\,\rangle = \int_{\mathbb{R}^2} f(x,y)h^{i,j}_{k,l}(x,y)\,dx\,dy, \quad i=1,2,3, \quad j,k,l\in\mathbb{Z}.
$$

The Fourier expansion of f with respect to the system has the form

$$
\sum_{j,k,l=-\infty}^{\infty} \sum_{i=1}^{3} \langle\, f, h^{i,j}_{k,l}\,\rangle\; h^{i,j}_{k,l}.
$$

For $q > 0$ let A_q be the class of all functions $f \in L^2(\mathbb{R}^2)$ such that

$$\|f\|_{A_q} := \left\{ \sum_{j,k,l=-\infty}^{\infty} \sum_{i=1}^{3} \left| \langle f, h_{k,l}^{i,j} \rangle \right|^q \right\}^{\frac{1}{q}} < \infty.$$

We denote by $d(x;y)$ Euclidian distance between points $x, y \in \mathbb{R}^2$.
For a set $B \subset \mathbb{R}^2$ we introduce

$$\operatorname{diam} B = \sup_{x,y \in B} d(x;y).$$

For a measurable set $B \subset \mathbb{R}^2$ we denote by $\mu(B)$ the Lebesque measure of B.

By $\Lambda(\gamma)$ we denote the length of a rectifiable curve $\gamma \subset \mathbb{R}^2$.

By $\mathrm{Lip}_D\, 1$, where $D > 0$, we denote the class of functions $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the condition $d(\tau(x); \tau(y)) \le D d(x;y)$ for all $x, y \in \mathbb{R}^2$, and by $\mathrm{Lip}\, 1$ – the class $\bigcup_{D>0} \mathrm{Lip}_D\, 1$.

We shall use the following lemma

Lemma 2.1. *If $\tau \in \mathrm{Lip}_D\, 1$, $D > 0$, then for every measurable set $B \subset \mathbb{R}^2$ the image $\tau(B)$ is also measurable and $\mu(\tau(B)) \le D^2 \mu(B)$.*

A *homeomorphism* of $\mathbb{R}^2$ is a one-to-one continuous mapping of $\mathbb{R}^2$ onto itself.

The fact that $f \circ \tau \in L^2(\mathbb{R}^2)$ if $f \in L^2(\mathbb{R}^2)$ and τ is a homeomorphism of $\mathbb{R}^2$ with the inverse function $\tau^{-1} \in \mathrm{Lip}\, 1$ follows from the following lemma, which can be proved using Lemma 1 and the definition of Lebesgue integral.

Lemma 2.2. *If τ is a homeomorphism of the plane $\mathbb{R}^2$ with $\tau^{-1} \in \mathrm{Lip}_D\, 1$, then*

$$\|f \circ \tau\|_{L^p} \le D^{\frac{2}{p}} \|f\|_{L^p}, \quad 1 \le p < \infty.$$

We now formulate our results

Theorem 2.3. *Let τ be a homeomorphism of the plane $\mathbb{R}^2$ such that $\tau^{-1} \in \mathrm{Lip}_D\, 1$. Then, for every function $f \in A_1$ and every $q > 1$, the composite function $f \circ \tau$ belongs to A_q and*

$$\|f \circ \tau\|_{A_q} \le C(D;q) \|f\|_{A_1},$$

where

$$C(D;q) = \left(\frac{3(5D)^q}{2^q - 1} + \frac{9(8D)^q}{2^{q-1} - 1} \right)^{\frac{1}{q}}.$$

Theorem 2.4. *There exists a function $f \in A_1$ and a homeomorphism τ of the plane $\mathbb{R}^2$ with $\tau^{-1} \in \mathrm{Lip}\, 1$ such that $f \circ \tau \notin A_1$.*

3. Proofs

3.1. Proof of Lemma 2.1. It is easy to prove the inequality if B is a circle.

It is easy also to prove that τ maps any set of measure zero onto a set of measure zero.

An arbitrary open set B can be presented as a disjoint union of a set of measure zero and an union of a family of pairwise disjoint open circles. Therefore, the lemma is true for open sets also.

An arbitrary measurable set B can be approximated (in measure) by an open set $C \supset B$. It is easy to prove that the τ image of difference $C \setminus B$ will be also of little measure if the measure of $C \setminus B$ is little (because $C \setminus B$ can be covered by an union of circles of little total sum of measures). Therefore, it is easy to see that the lemma is true for arbitrary measurable sets.

The lemma is proved.

3.2. Proof of Theorem 2.3. Let τ be any homeomorphism of $\mathbb{R}^2$ with $\tau^{-1} \in \mathrm{Lip}_D\, 1$, $D > 0$, and let $f \in A_1$. First we suppose

$$\|f\|_{A_1} = 1. \tag{12}$$

We have

$$f = \sum_{j,k,l=-\infty}^{\infty} \sum_{i=1}^{3} \langle\, f, h_{k,l}^{i,j} \,\rangle\, h_{k,l}^{i,j} \tag{13}$$

in the sense of convergence in L^2-norm.

It follows from (13) and Lemma 2.2 that

$$f \circ \tau = \sum_{j,k,l=-\infty}^{\infty} \sum_{i=1}^{3} \langle\, f, h_{k,l}^{i,j} \,\rangle\, h_{k,l}^{i,j} \circ \tau \tag{14}$$

in the same sense. For the coefficients of $f \circ \tau$, we shall have (see (14))

$$\langle\, f \circ \tau, h_{p,r}^{m,n} \,\rangle = \sum_{j,k,l=-\infty}^{\infty} \sum_{i=1}^{3} \langle\, f, h_{k,l}^{i,j} \,\rangle \cdot \langle\, h_{k,l}^{i,j} \circ \tau, h_{p,r}^{m,n} \,\rangle.$$

From this representation and (12) bv Jensen's inequality

$$\left|\langle\, f \circ \tau, h_{p,r}^{m,n} \,\rangle\right|^q = \sum_{j,k,l=-\infty}^{\infty} \sum_{i=1}^{3} \left|\langle\, f, h_{k,l}^{i,j} \,\rangle\right| \cdot \left|\langle\, h_{k,l}^{i,j} \circ \tau, h_{p,r}^{m,n} \,\rangle\right|^q,$$
$$m = 1, 2, 3, \quad n, p, r \in \mathbb{Z}.$$

Therefore

$$\|f \circ \tau\|_{A_q}^q = \sum_{j,k,l=-\infty}^{\infty} \sum_{i=1}^{3} \left|\langle\, f, h_{k,l}^{i,j} \,\rangle\right| \cdot \left\|h_{k,l}^{i,j} \circ \tau\right\|_{A_q}^q . \tag{15}$$

Let $j,k,l\in\mathbb{Z}$ and $1\le i\le 3$. We estimate $\|h^{i,j}_{k,l}\circ\tau\|_{A_q}$.

We denote by $\Phi^j_{k,l}$ the boundary of $\tau^{-1}(\Delta^j_{k.l})$. It is equal to the τ^{-1}-image of the boundary of $\Delta^j_{k,l}$. By $E^j_{k,l}$ and $F^j_{k,l}$ we denote the intervals

$$\left\{(2k+1)2^{-j-1}\right\}\times\left[l\cdot 2^{-j},(l+1)2^{-j}\right)$$

and

$$\left[k\cdot 2^{-j},(k+1)2^{-j-1}\right)\times\left\{(2l+1)2^{-j-1}\right\},$$

crossing $\Delta^j_{k,l}$ in the middle.

Let

$$\Gamma^j_{k,l}=\Phi^j_{k,l}\bigcup\tau^{-1}(E^j_{k,l})\bigcup\tau^{-1}(F^j_{k,l}). \tag{16}$$

It is easy to prove that if a mapping ω of the plane $\mathbb{R}^2$ into itself is in $\mathrm{Lip}_D\,1$ and γ is a rectifiable curve then the image $\omega(\gamma)$ also is rectifiable curve and $\Lambda(\omega(\gamma))\le D\Lambda(\gamma)$. Therefore, $\Phi^j_{k,l}$, $\tau^{-1}(E^j_{k,l})$, $\tau^{-1}(F^j_{k,l})$ are rectifiable and

$$\Lambda(\tau^{-1}(E^j_{k,l}))\le D\,2^{-j},\quad \Lambda(\tau^{-1}(F^j_{k,l}))\le D\,2^{-j}, \tag{17}$$

$$\Lambda(\Phi^j_{k,l})\le D\,2^{-j+2}. \tag{18}$$

Let $s=s(j,k,l)$ be an integer such that

$$2^{-s-1}\le\Lambda(\Phi^j_{k,l})<2^{-s}. \tag{19}$$

By (18) and (19) we have

$$2^{-s}\le D\,2^{-j+3}. \tag{20}$$

It follows from (19) that

$$\operatorname{diam}\Phi^j_{k,l}\le 2^{-s-1}. \tag{21}$$

But $\operatorname{diam}\Phi^j_{k,l}(\Delta^j_{k,l})=\operatorname{diam}\tau^{-1}(\Delta^j_{k,l})$. Because of geometrical argument and (21) this gives

$$\mu(\tau^{-1}(\Delta^j_{k,l}))\le 1,25\cdot 2^{-2s-2}. \tag{22}$$

First we consider the case $n\le s$. Because of (21) the number of all squares $\Delta^n_{p,r}$ which intersect $\tau^{-1}(\Delta^j_{k,l})$ are at most 4: We can choose pairs of integers $p_u=p_u(j,k,l)$, $r_u=r_u(j,k,l)$ such that

$$\tau^{-1}(\Delta^j_{k,l})\subset\bigcup_{u=1}^{4}\Delta^n_{p_u,r_u}.$$

Therefore (see (20), (22))

$$\sum_{p,r=-\infty}^{\infty}\left|\langle\, h^{i,j}_{k,l}\circ\tau, h^{m,n}_{p,r}\,\rangle\right|=\sum_{u=1}^{4}\left|\langle\, h^{i,j}_{k,l}\circ\tau, h^{m,n}_{p_u,r_u}\,\rangle\right|^q$$

$$\leq 2^{(n+j)q} \sum_{u=1}^{4} \left(\mu\left(\Delta^n_{p_u,r_u} \bigcap \tau^{-1}(\Delta^j_{k,l})\right)\right)^q \leq 2^{(n+j)q} \left(\mu(\tau^{-1}(\Delta^j_{k,l}))\right)^q$$
$$\leq 2^{(n+j)q} \cdot (1,25)^q \cdot 2^{q(-2s-2)} \leq (1,25)^q \cdot 2^{-2q}(8 \cdot D \cdot 2^{n-s})^q$$
$$= (2,5D)^q \frac{2^{nq}}{2^{sq}}, \quad 1 \leq m \leq 3.$$

Hence
$$\sum_{n=-\infty}^{s} \sum_{p,r=-\infty}^{\infty} \sum_{m=1}^{3} \left| \langle\, h^{i,j}_{k,l} \circ \tau, h^{m,n}_{p,r} \,\rangle \right| \leq \frac{3(5D)^q}{2^q - 1}\,. \tag{23}$$

Now we consider the case $n > s$. If a support of a function $h^{m,n}_{p,r}$, $1 \leq m \leq 3$, $n, p, r \in \mathbb{Z}$, does not intersect the set $\Gamma^j_{k,l}$, then the corresponding wavelet coefficient is equal to zero because of (11), since $h^{i,j}_{k,l} \circ \tau$ is constant on the support of $h^{m,n}_{p,r}$. So, it remains to estimate the coefficients corresponding to those squares of rank n which intersect $\Gamma^j_{k.l}$. It is easy to check that each of them is less than 2^{-j-n} (see (7)–(10)).

It is easy to prove that if K is a rectifiable curve, then the number of all dyadic squares of rank v, $v \in \mathbb{Z}$, intersecting the curve is at most $4 \cdot \Lambda(K) 2^v$. Hence, the number of all dyadic squares of rank n, intersecting $\Gamma^j_{k.l}$ is at most $24 \cdot D \cdot 2^{n-j}$ (see (16), (17), (18)).

Thus (see, also, (20))

$$\sum_{n=s+1}^{\infty} \sum_{p,r=-\infty}^{\infty} \sum_{m=1}^{3} \left| \langle\, h^{i,j}_{k,l} \circ \tau, h^{m,n}_{p,r} \,\rangle \right|^q$$
$$\leq 3 \cdot 24 \cdot D \cdot \sum_{n=s+1}^{\infty} 2^{n-j}(2^{j-n})^q = 72 \cdot D \cdot 2^{j(q-1)} \sum_{n=s+1}^{\infty} 2^{-n(q-1)}$$
$$= \frac{72 \cdot D \cdot 2^{j(q-1)}}{2^{s(q-1)}(2^{q-1} - 1)} \leq \frac{9(8D)^q}{2^{q-1} - 1}\,. \tag{24}$$

If we denote
$$C(D;q) := \left(\frac{3(5D)^q}{2^q - 1} + \frac{9(8D)^q}{2^{q-1} - 1} \right)^{\frac{1}{q}},$$
then by (12), (15), (23) and (24) we have
$$\|f \cdot \tau\|_{A_q} \leq C(D;q). \tag{25}$$

For arbitrary $f \in A_1$ with $\|f\|_{A_1} \neq 0$ (for $f = 0$ assertion of the theorem is obvious) we consider the function $f_1 = \frac{f}{\|f\|_{A_1}}$. We have $\|f_1\|_{A_1} = 1$ and
$$\langle\, f_1 \circ \tau, h^{i,j}_{k,l} \,\rangle = \frac{\langle\, f \circ \tau, h^{i,j}_{k,l} \,\rangle}{\|f\|_{A_1}}, \quad 1 \leq i \leq 3, \quad j, k, l \in \mathbb{Z}.$$

Consequently (see (25))

$$\|f\circ\tau\|_{A_q} = \|f_1\circ\tau\|_{A_q}\cdot\|f\|_{A_1} \le C(D;q)\cdot\|f\|_{A_1}\,.$$

The theorem is proved.

3.3. Proof of Theorem 2.4. Let

$$f(x,y) = \begin{cases} 1, & x\in[0,1)^2,\\ 0, & x\in\mathbb{R}^2\setminus[0,1)^2,\end{cases}$$

and

$$\tau(x,y) = \frac{1}{\sqrt{5}}\,(2x+y,-x+2y), \quad (x,y)\in\mathbb{R}^2.$$

It is obvious, that for f we have to estimate only those coefficients supports of corresponding functions of which intersect the square $[0,1)^2$. These supports are squares $\Delta^j_{k,l}$ which contain $[0,1)^2$ or are contained in $[0,1)^2$. First are with $j<0$ and $k,l=0$, second – with $j\ge 0$ and $0\le k,l<2^j$.

It is easy to check that if $j<0$, $k,l=0$, then

$$a^j_{k,l}(f) = b^j_{k,l}(f) = c^j_{k,l}(f) = \sqrt{2^j}\,,$$

and if $j\ge 0$, $0\le k,l<2^j$, then

$$a^j_{k,l}(f) = b^j_{k,l}(f) = c^j_{k,l}(f) = 0.$$

So $f\in A_1$.

Let us now consider the composite function $f\circ\tau$:

$$f(\tau(x,y)) = \begin{cases} 1, & x,y\in\tau^{-1}([0,1)^2),\\ 0, & x,y\in\mathbb{R}^2\setminus\tau^{-1}([0,1)^2).\end{cases}$$

The set $\tau^{-1}([0,1)^2)$ is the unit square with vertexes at points $(0,0)$, $\left(\frac{2}{\sqrt5},\frac{1}{\sqrt5}\right)$, $\left(\frac{1}{\sqrt5},\frac{3}{\sqrt5}\right)$ and $\left(-\frac{1}{\sqrt5},\frac{2}{\sqrt5}\right)$.

If $j\ge 2$, $0\le l\le 2^{j-2}-1$, $k=2l$, then it can be easily calculated that $c^j_{2l,l}(f\circ\tau) = 2^{-j-3}$. Consequently

$$\sum_{j=2}^{\infty}\sum_{l=0}^{2^{j-2}-1}|c^j_{2l,l}(f\circ\tau)| = \sum_{j=2}^{\infty}2^{-5} = \infty.$$

Thus $f\circ\tau\notin A_1$.

The theorem is proved.

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