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THE CONTACT PROBLEM FOR A COMPOUND PLATE WITH AN ELASTIC INCLUSION

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Abstract. The contact problem of the theory of elasticity of a compound plate with an elastic semi-infinite inclusion of variable rigidity is considered. The problem is reduced to the system of integral differential equations. If the rigidity of am inclusion varies with the power law, we investigate the obtained equations, get exact solutions and establish behavior of unknown contact stresses at the ends of an elastic inclusion.

რეზიუმე. განხილულია დრეკადობის თეორიის საკონტაქტო ამოცანა შედგენილი ფირფიტებისათვის, რომელიც შეიცავს ცვლადი სიხისტის მქონე ნახევრადუსასრულო ჩართვას. ამოცანა მიიყვანება ინტეგრო-დიფერენციალურ განტოლებათა სისტემაზე. როდესაც ჩართვის სიხისტე იცვლება ხარისხოვანი კანონით, მიღებულია აღნიშნული განტოლებათა სისტემის ზუსტი ამონახსნი, გამოკვლეულია უცნობი საკონტროლო ძაბვების ყოფაქცევა ჩართვის ბოლოების მახლობლობაში.

Introduction

The contact problems on interaction of thin-shelled elements (stringers or inclusions) of various geometric forms with massive deformable bodies belong to one of the extensive fields of the theory of contact and mixes problems of mechanics of deformable rigid bodies. A vast number of works are devoted to the problems of tension and bending of finite or infinite plates with thin absolutely rigid elements or with those of constant rigidity (see, e.g., [1-3]). We investigated the contact problems of tension and bending of plates with inclusions of variable rigidity, for unknown contact stresses we obtained the Prandtl's integral differential equation with a variable coefficient. In the case if the coefficient of a singular operator tends to zero of any order at the ends of the integration line, the obtained equation is equivalent to the third lond singular equation. We have investigated this equation and obtained exact or approximate solutions [4-7].

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1. The basic relations

We consider a compound elastic plate, an unbounded elastic medium consisting of two half-planes ($y > 0$ and $y < 0$) with different elastic constants (E_+, μ_+ and E_-, μ_-). In the conditions of plane deformation, the plate is assumed to be strengthened on the semi-axis $(0, +\infty)$ by an inclusion of variable thickness $h_0(x)$, with modulus of elasticity $E_0(x)$ and Poisson coefficient ν_0. The inclusion is loaded by horizontal forces of intensity $\tau_0(x)$, and the plate is free from the loading. When crossing the semi-axis $(0, +\infty)$, the stress filed undergoes discontinuity, while when crossing the rest part of the ox-axis we observe no discontinuity of strains and displacements field.

According to the equations of equilibrium of the inclusion elements and Hooke's law, we have

$$\begin{aligned}\frac{du^{(1)}(x)}{dx} &= \frac{1}{E(x)}\int\limits_0^x [\tau^{(1)}(t) - \tau_0(t)]dt \\ \frac{dv^{(1)}(x)}{dx} &= 0, \quad x > 0,\end{aligned} \tag{1}$$

and the equilibrium equations of the inclusion are of the form

$$\int\limits_0^\infty [\tau^{(1)}(t) - \tau_0(t)]dt = 0, \quad \int\limits_0^\infty q^{(1)}(t)dt = 0, \tag{2}$$

where $u^{(1)}(x)$ and $v^{(1)}(x)$ are the horizontal and vertical displacements of the inclusion points; $\tau^{(1)}(x)$ and $q^{(1)}(x)$ are the jumps of tangential and normal contact strains, subjects to determination:

$$\begin{aligned}\tau^{(1)}(x) =& \tau_-^{(1)}(x) - \tau_+^{(1)}(x), \quad E(x) = \frac{h_0(x)E_0(x)}{1-\nu_0^2} \\ q^{(1)}(x) =& q_-^{(1)}(x) - q_+^{(1)}(x).\end{aligned}$$

Moreover, as is known, the derivatives of displacements of boundary points on a compound plane dependent on the outer load acting on the semi-axis, have the form

$$\begin{aligned}\frac{du^{(2)}(x)}{dx} &= -Aq^{(2)}(x) + \frac{B}{\pi}\int\limits_0^\infty \frac{\tau^{(2)}(t)dt}{t-x} \\ \frac{dv^{(2)}(x)}{dx} &= -A\tau^{(2)}(x) + \frac{B}{\pi}\int\limits_0^\infty \frac{q^{(2)}(t)dt}{t-x},\end{aligned} \tag{3}$$

where $u^{(2)}(x)$ and $v^{(2)}(x)$ are the boundary values of horizontal and vertical displacements on the semi-axis; $\tau^{(2)}(x)$, $q^{(2)}(x)$ are the jumps of the boundary values of tangential and normal strains, respectively.

$$A=\frac{a_+b_-(b_++a_-)-a_-b_+(b_++a_-)}{2c_0},\ B=\frac{a_+b_-(b_++a_-)+a_-b_+(b_++a_-)}{2c_0}$$

$$c_\pm=\frac{1-\mu_\pm}{E_\pm},\ \ d_\pm=\frac{\mu_\pm(1+\mu_\pm)}{E_\pm},\ \ a_\pm=3c_\pm-d_\pm,$$

$$b_\pm=c_\pm-d_\pm,\ \ c_0=4(c_++c_-)^2-[(c_--c_+)-(d_--d_+)]^2.$$

The conditions of contact of the inclusion and the plate have the form

$$\frac{du^{(1)}(x)}{dx}=\frac{du^{(2)}(x)}{dx},\quad \frac{dv^{(1)}(x)}{dx}=\frac{dv^{(2)}(x)}{dx} \tag{4}$$

2. Solution of the problem

By formulas (1.1-1.4), to find the unknown contact strains, we obtain the following system of singular integral differential equations:

$$\frac{B}{\pi}\int_0^\infty\frac{\psi'(t)dt}{t-x}-A\varphi(x)=\frac{\psi(x)}{E(x)}+f_1(x),$$

$$x>0 \tag{5}$$

$$\frac{B}{\pi}\int_0^\infty\frac{\varphi'(t)dt}{t-x}+A\psi'(x)=f_2(x),$$

$$\psi(x)=\int_0^x[\tau(t)-\tau_0(t)]dt,\ \ \tau(t)=\tau^{(1)}(t)=\tau^{(2)}(t),\ \ \varphi(x)=q^{(1)}(x)=q^{(2)}(x)$$

$$f_1(x)=-\frac{B}{\pi}\int_0^\infty\frac{\tau_0(t)dt}{t-x},\quad f_2(x)=-A\tau_0(x).$$

Assume that $E(x)=hx^\omega$, $x\geq 0$, $h_0=\text{const}$ and ω is an arbitrary real number (the inclusion at point $x=0$ has a qualitative cusp, while at infinity it becomes rigid). The system of equations (2.1) we consider with the following boundary conditions:

$$\psi(0)=0,\quad \psi(\infty)=0,\quad \int_0^\infty\varphi(t)dt=0.$$

The change of the variables $x=e^\xi$, $t=e^\zeta$ and the Fourier transformation of equations (2.1) result in

$$-A\Phi(s-i)+Bs\operatorname{cth}\pi s\Psi(s)=\frac{1}{h}\Psi(s+i\beta)+F_1(s),$$

$$As\Psi(s)+B\operatorname{cth}\pi s\Phi(s-i)=iF_2(s),$$

$$s = s_0 + i\varepsilon, \quad -\infty < s_0 < \infty, \quad \varepsilon > 0,$$

where $\Phi(s)$ and $\Psi(s)$ are the Fourier transformations of the functions $\varphi_0(\xi) = \varphi(e^\xi)$ and $\psi_0(\xi) = \psi(e^\xi)$, respectively, while $F_1(s)$ and $F_2(s)$ are those of the functions $f_1(e^\xi)e^\xi$ and $e^\xi f_2(e^\xi)$, $\beta = \omega - 1$.

From the last system, for the function $\Psi(s)$ we obtain the following boundary condition of Karleman type problem for a strip:

$$G(s)\Psi(s) - \Psi(s + i\beta) = F(s), \quad -\infty + i\varepsilon < s < \infty + i\varepsilon, \tag{6}$$

where $G(s) = s \operatorname{cth} \pi s(B^2 - A^2 \operatorname{th}^2 \pi s)h/B$, $F(s) = hF_1(s) - \dfrac{ihAF_2(s)}{B \operatorname{cth} \pi s}$.

Having solved the last problem, we can find the function $\Phi(z)$ by the formula

$$\Phi(z - i) = \frac{iF_2(z) - Az\Psi(z)}{B \operatorname{cth} \pi z}. \tag{7}$$

Thus we consider the problem: find the function $\Psi(z)$, holomorphic in the strip $0 < \operatorname{Im} z < \omega - 1$, vanishing at infinity, continuously extendable on the boundary of the strip and satisfying the condition (2.2).

We represent the coefficient of the problem (2.2) in the form

$$G(s) = -isG_0(s) \operatorname{sh} \frac{\pi}{2\beta}(s + i\beta) / \operatorname{sh} \frac{\pi}{2\beta} s \frac{B^2 - A^2}{B} h, \tag{8}$$

where $G_0(s) \equiv \frac{B^2 - A^2 \operatorname{th}^2 \pi s}{B^2 - A^2} \operatorname{th} \frac{\pi}{2\beta} s \operatorname{cth} \pi s$.

Since $B > A$, $B^2 - A^2 \operatorname{th}^2 \pi s > 0$, the function $G_0(s)$ is positive and $G_0(\pm\infty + i\varepsilon) = 1$. Taking into account that the index of the function $G_0(s)$ is equal to zero and $\ln[G_0(s)]$ is integrable on the entire axis, we can represent it in the form

$$G_0(s) = \frac{X_0(s)}{X_0(s + i\beta)}, \tag{9}$$

where

$$X_0(z) = \exp\left\{ \frac{1}{2i\beta} \int_{-\infty + i\varepsilon}^{\infty + i\varepsilon} \ln[G_0(s)] \quad \operatorname{cth} \pi(s - z) ds \right\}.$$

The function $X_0(z)$ is holomorphic in the strip, continuous on the border of the strip and bounded in the closed strip $0 \leq \operatorname{lm} z \leq \beta$ and at infinity. Substituting (2.4) and (2.5) into the condition (2.2) and introducing the notations

$$\Psi_1(z) = \frac{zX_0(z)\Psi(z)}{\operatorname{sh}(\pi z/2\beta)}, \quad \mu = \frac{B^2 - A^2}{B} h,$$

we obtain

$$\mu(\beta - is)\Psi_1(s) + \Psi_1(s + i\beta) = \frac{F(s)(s + i\beta)X_0(s + i\beta)}{\operatorname{sh} \pi(s + i\beta)/2\beta}. \tag{10}$$

The coefficient of the condition (2.6) is represented in the form

$$\mu(\beta - is) = \frac{X_1(s+i\beta)}{X_1(s)}, \quad s = s_0 + i\varepsilon,$$

where $X_1(z) = \exp\Big(-\frac{iz}{\beta}\ln\beta\mu\Big)\Gamma\Big(1-\frac{iz}{\beta}\Big)$. Substituting this value into (2.6), the solution of the problem takes the form [8]

$$\Psi(z) = -\frac{X(z)}{2iz\beta}\int\limits_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\frac{F(t)(t+i\beta)dt}{X(t+i\beta)\,\mathrm{sh}(\pi(t-z)/\beta)}, \tag{11}$$

where $X(z) = \frac{X_1(z)}{X_0(z)}\,\mathrm{sh}\,\frac{\pi z}{2\beta}$, $0 < \mathrm{Im}\, z < \beta$. The function $X(z)$ satisfies the condition

$$C_1|t|^{1/2} \le |X(t+i\tau)| \le C_2|t|^{3/2}, \quad 0 \le \tau \le \beta.$$

When the functions $F_1(t)$ and $F_2(t)$ vanish exponentially at infinity, the integral in (2.7) decreases exponentially, i.e. it is continuously extendable in the closed strip $0 \le \mathrm{Im}\, z \le \beta$ and vanishes exponentially at infinity.

Consequently, the solution of the problem under consideration is given by formula (2.7) and the function $\Phi(z)$ is determined by formula (2.3).

Suppose $\omega \ge 2$, $(\beta \ge 1)$. The contact strain can be calculated by the formula

$$\tau^{(1)}(x) - \tau_0(x) = \psi'(x) = \frac{x^{-1}}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} it\Psi(t)e^{it\ln x}dt =$$

$$= \frac{ix^{-1}}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}(t+i(\omega-1))\Psi(t+i(\omega-1))e^{-i(t+i(\omega-1))\ln x}dt = x^{\omega-2}\tau_1(x),$$

where $\tau_1(x)$ is the bounded function in the neighborhood of the point $x=0$. The function $\Phi(z)$ vanishes exponentially at infinity, is analytically extendable in the strip $0 \le \mathrm{Im}\, z < \omega - 1$, except, possibly, the points: $z_0 = \frac{i}{2}$, $z_1 = \frac{3i}{2}\ldots$, being the poles in the strip. Then using the Cauchy formula, we have

$$q^{(1)}(x) = x^{-1}\varphi_0'(\ln x) = x^{-1}\operatorname*{res}_{t=\frac{i}{2}}[it\Phi(t)e^{-it\ln x}] + x^{-1}\operatorname*{res}_{t=\frac{3i}{2}}[it\Phi(t)e^{-it\ln x}] +$$

$$+\frac{ix^{-1}}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}(t+i(\omega-1))\Phi(t+i(\omega-1))e^{-i(t+i(\omega-1))\ln x}dt =$$

$$= c_1x^{-1/2} + c_2x^{1/2} + x^{\omega-2}q_1(x),$$

where c_1 and c_2 are the known constants, $q_1(x)$ is the bounded function for $x \ge 0$.

For $1 < \omega < 2$, according to the condition (2.2), we can see that the function $\Psi(z)$, defined by formula (2.7), is analytically extendable in the strip $0 < \operatorname{Im} z < 1$, except the point $z = \frac{i}{2}$. Hence the tangential contact strain in the neighborhood of the point $x = 0$ has the following estimation: $\tau^{(1)}(x) - \tau_0(x) = O(x^{-1/2})$, $x \to 0$.

Consider now the case when $\omega \leq 1$. We introduce the following notation: $m = -(\omega - 1)$, and the condition (2.2) takes the form

$$G(s)\Psi(s) - \Psi(s - im) = F(s), \quad -\infty - i\delta < s < \infty - i\delta. \tag{12}$$

We formulate the problem as follows: find a function, which is holomorphic in the strip $-m < \operatorname{Im} z < m$, vanishing at infinity, bounded in the entire strip, except the points $z_k^+ = t_k^+ + i\tau_k^+$, $(k = 0, 1, 2, \dots, l)$, being zeros of the function $G(z)$ in the upper strip.

If we solve the problem: find a function, holomorphic in the strip $-m < \operatorname{Im} z < 0$, vanishing at infinity, continuously extendable on the border of the strip by the condition (2.8), then the solution of the preceding problem will be the function

$$\Psi_0(z) = \begin{cases} \Psi(z), & -m < \operatorname{Im} z < 0 \\ \dfrac{F(z) + \Psi(z - im)}{G(z)}, & 0 < \operatorname{Im} z < m. \end{cases}$$

Analogously to the above reasoning, for the function $\Psi(z)$ we have the following representation:

$$\Psi(z) = \frac{\widetilde{X}(z)}{2im} \int\limits_{-\infty - i\delta}^{+\infty - i\delta} \frac{F(t)dt}{\widetilde{X}(t)\operatorname{sh}(\pi(t - z)/m)}, \quad -m < \operatorname{Im} z < 0$$

$$\widetilde{X}(z) = \frac{\widetilde{X}_0(z)\widetilde{X}_1(z)}{z} \operatorname{sh}\frac{\pi z}{2m}, \quad \widetilde{X}_1(z) = \Gamma\Big(1 + \frac{iz}{m}\Big)\exp\Big(\frac{iz}{m}\ln \mu m\Big)$$

$$\widetilde{X}_0(z) = \exp\bigg\{\frac{1}{2im}\int\limits_{-\infty}^{\infty} \ln[\widetilde{G}_0(t)] \operatorname{cth}\pi(t - z)dt\bigg\}$$

$$\widetilde{G}_0(t) = \frac{B^2 - A^2\operatorname{th}^2\pi t}{B^2 - A^2}\operatorname{th}\frac{\pi t}{2m}\operatorname{cth}\pi t$$

The unknown contact strain: $\tau^{(1)}(x) - \tau_0(x) = c_0 x^{\tau_0^+ - 1} + \varphi_2(x)$, $c_0 = \text{const}$, $\varphi_2(x)$ is the bounded function for $x \geq 0$, $0 < \tau_0^+ < 1$.

For $\omega = 1$, the condition (2.2) yields

$$\Psi(z) = \frac{F(z)}{G(z) + 1},$$

and for the tangential strain we have $\tau^{(1)}(x) - \tau_0(x) = O(x^{\mu - 1})$, for $x \to 0$, where μ is the zero of the function $G(z) + 1$, the closest to the real axis in the upper semi-plane.

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