Proceedings of A. Razmadze
Mathematical Institute
Vol. **149** (2009), 17–33

ON PROPER OSCILLATIONS OF PRESTRESSED ORTHOTROPIC CYLINDRICAL SHELLS

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Abstract. A system of equations and a corresponding resolving equation of oscillations of prestressed orthotropic cylindrical shells of arbitrary length are obtained. The system of equations is obtained on the basis of refined relations of the theory of shells. We consider both the axi-symmetric as well as asymmetric proper oscillations of prestressed shells. Using the obtained equations, we get the already known formulas and certain new ones for finding the lower frequencies.

რეზიუმე. ნაშრომში მიღებულია რხევების განტოლებათა სისტემა და შესაბამისი ამოხსნადი განტოლება წინასწარ დაძაბული ორთოტროპული ნებისმიერი სიგრძის ცილინდრული გარსებისათვის. განტოლებათა სისტემა მიღებულია გარსთა თეორიის დაზუსტებული თანაფარდობებისაგან. განიხილება წინასწარ დაძაბული გარსების როგორც ღერძსიმეტრიული, ისე არასიმეტრიული საკუთარი რხევები. მიღებული განტოლებებიდან გამომდინარეობს როგორც ცნობილი ფორმულები, ისე ზოგიერთი ახალი ფორმულა ქვედა სიხშირის განსაზღვრისათვის.

In the present work we obtain a system of equations and a corresponding resolving equation of oscillations of prestressed orthotropic cylindrical shells of an arbitrary length, with the exclusion very short ones, when the initial stressed state cannot be considered as momentless. The system of equations is obtained on the basis of refined relations of the shell theory. We consider both the axisymmetric and asymmetric proper oscillations under the action of initial stresses, independent of the coordinates. The main attention is focused on the investigation of lower frequencies in middle length shells and in long prestressed shells. Relying on the obtained equations, we obtained the already known formulas ([1]-[8])and also new ones. The dependence of the lower frequency value on the shear modulus of an orthotropic material is found out. Simple enough formulas are obtained and the graphs for finding lower frequencies depending on the given load, geometrical sizes and elastic parameters of the shell are drawn.

2000 *Mathematics Subject Classification.* 35J60.
Key words and phrases. Shell, proper oscillation, lower frequency, orthotropy, load, equation, symmetric form.

1. The system of oscillation equations of a prestressed orthotropic cylindrical shell is obtained by virtue of equilibrium equations written with regard for lengthening and angle of revolution of linear elements of its middle surface ([9]) and also from the refined relations of the shell theory ([11]). The system in the operational form looks as

$$\overline{L}_{11} + L_{12}v = L_1 w \tag{1.1}$$

$$L_{21}u + \overline{L}_{22}v = L_2 w \tag{1.2}$$

$$L_{31}u + L_{32}v + \overline{L}_{33}w = 0 \tag{1.3}$$

$$\overline{L}_{ii} = L_{ii} - \Omega_i \frac{\partial^2}{\partial t^2} \quad (i = 1, 2)$$

$$\overline{L}_{33} = L_{33} - \Omega_2 \frac{\partial^2}{\partial t^2}, \quad \Omega_i = \frac{\rho R^2}{A_i}, \quad A_i = \frac{E_i}{1 - \nu_1\nu_2},$$

$$L_{11} = \frac{\partial^2}{\partial \xi^2} + p_1(1 + \varepsilon_*)\frac{\partial^2}{\partial \varphi^2} + (1 - \nu_1\nu_2)s_1^0 \frac{\partial^2}{\partial \xi \partial \varphi}$$

$$L_{12} = [\overline{p}_1 + (1 - \nu_1\nu_2)(t_{11}^0 - t_{21}^0)]\frac{\partial^2}{\partial \xi \partial \varphi} - (1 - \nu_1\nu_2)s_1^0 \frac{\partial^2}{\partial \xi^2}$$

$$L_1 = \varepsilon_*\Big(p_1 \frac{\partial^3}{\partial \xi \partial \varphi^2} - \frac{\partial^2}{\partial \xi^3}\Big) + \nu_2\Big[1 + \frac{1 - \nu_1\nu_2}{\nu_2}(t_{11}^0 - t_{21}^0)\Big]\frac{\partial}{\partial \xi}$$

$$L_{21} = [\overline{p}_2 + (1 - \nu_1\nu_2)t_{22}^0]\frac{\partial^2}{\partial \xi \partial \varphi}$$

$$L_{22} = \frac{\partial^2}{\partial \varphi^2} + [p_2(1 + 3\varepsilon_*) + (1 - \nu_1\nu_2)t_{12}^0]\frac{\partial^2}{\partial \xi^2} + 2(1 - \nu_1\nu_2)s_2^0 \frac{\partial^2}{\partial \xi \partial \varphi}$$

$$L_2 = \frac{\partial}{\partial \varphi} - \varepsilon_* p_3 \frac{\partial^3}{\partial \xi^2 \partial \varphi} + 2(1 - \nu_1\nu_2)s_2^0 \frac{\partial}{\partial \xi} \tag{1.4}$$

$$L_{31} = \nu_1 \frac{\partial}{\partial \xi} - \varepsilon_*\Big(\frac{p_2}{p_1}\frac{\partial^3}{\partial \xi^3} - p_2 \frac{\partial^3}{\partial \xi \partial \varphi^2}\Big)$$

$$L_{32} = \frac{\partial}{\partial \varphi} - \varepsilon_* p_3 \frac{\partial^3}{\partial \xi^2 \partial \varphi} + 2(1 - \nu_1\nu_2)s_2^0 \frac{\partial}{\partial \xi}$$

$$L_{33} = -1 - \varepsilon_* + (1 - \nu_1\nu_2)t_{22}^0 - \varepsilon_*\Big(\Delta_1 + 2\frac{\partial^2}{\partial \varphi^2}\Big) +$$

$$+ (1 - \nu_1\nu_2)\Big(t_{12}^0 \frac{\partial^2}{\partial \xi^2} + t_{22}^0 \frac{\partial^2}{\partial \varphi^2} + 2s_2^0 \frac{\partial^2}{\partial \xi \partial \varphi}\Big)$$

$$p_i = B/A_i, \quad \overline{p}_1 = p_1 + \nu_2, \quad \overline{p}_2 = p_2 + \nu_1, \quad p_3 = 3p_2 + \nu_1$$

$$t_{i1}^0 = T_i^0/E_1 h, \quad t_{i2}^0 = T_i^0/E_2 h, \quad s_i^0 = S^0/E_i h \quad (i = 1, 2) \tag{1.5}$$

$$\Delta_1 = \delta_1 \frac{\partial^4}{\partial \xi^4} + \delta_2 \frac{\partial^4}{\partial \xi^2 \partial \varphi^2} + \frac{\partial^4}{\partial \varphi^4} \tag{1.6}$$

$$\delta_1 = p_2/p_1, \quad \delta_2 = 2(2/p_2 + \nu_1), \quad \varepsilon_* = h^2/12R^2$$

where u, v, w are the dynamical components of displacement in axial, circular and normal direction; T_i^0 and S^0 are the normal and shearing forces of the initial statical state; ξ and φ are the dimensionless coordinates in the axial and angular direction; R, l and h are, respectively, radius, length and thickness of the shell; E_1, E_2, ν_1, ν_2 are the modules of elasticity and Poisson's coefficients in the axial and angular direction ($E_1\nu_2 = E_2\nu_1$); G is the shear modulus. Since the stresses $\sigma_1^0 = T_i/h$, $(i = 1, 2)$, $\tau^0 = S^0/h$ corresponding to the initial state, are the values which are in many times less than E_i, therefore t_{i1}^0, t_{i2}^0, $s_i^0 \ll 1$. It seems that these values can be neglected as compared with unity in the coefficients of the system (1.1)-(1.3), however, this is, generally speaking, invalid, because when solving the system, some principal terms may be reduced and then the corresponding secondary terms may be of importance. From equations (1.1) and (1.2) we exclude, respectively, the unknowns v and u, and from (1.3) the both unknowns. Then instead of (1.1)-(1.3) we obtain the following system of equations:

$$\overline{L}u = (L_1\overline{L}_{22} - L_2L_{12})w \tag{1.7}$$

$$\overline{L}v = (L_2\overline{L}_{11} - L_1L_{21})w \tag{1.8}$$

$$[L_{31}(L_1\overline{L}_{22} - L_2L_{12}) + L_{32}(L_2\overline{L}_{11} - L_1L_{22}) + \overline{L}_{33}\overline{L}]w = 0 \tag{1.9}$$

where

$$\overline{L} = \overline{L}_{11}\overline{L}_{22} - L_{12}L_{21}.$$

Having defined w from (1.9) and substituted in (1.7) and (1.8), we find u and v. Thus after reduction of like terms it becomes possible in the coefficients of the operators appearing before u, v, w to neglect t_{i1}^0, t_{i2}^0, s_i^0, ε_* and their products as compared with unity. Having done all this, the system (1.7)-(1.9) takes the form

$$\begin{gathered}\Delta_2 u + p_2^{-1}\Omega_2\frac{\partial^2}{\partial t^2}\Big[(1+p_2)\frac{\partial^2 u}{\partial\xi^2} + p_1(1+p_2^{-1})\frac{\partial^2 u}{\partial\varphi^2} - \Omega_1\frac{\partial^2 u}{\partial t^2}\Big] = \\ = \frac{\partial}{\partial\xi}\Big\{\nu_2\frac{\partial^2 w}{\partial\xi^2} - \frac{p_1}{p_2}\frac{\partial^2 w}{\partial\varphi^2} + \\ +\varepsilon_*\Big[\frac{p_1}{p_2}\frac{\partial^4 w}{\partial\varphi^4} - \frac{\partial^4 w}{\partial\xi^4} + p_1p_2^{-1}(4(p_2+\nu_1) - 1 + \nu_1\nu_2)\frac{\partial^4 w}{\partial\xi^2\partial\varphi^2}\Big] - \\ -p_2^{-1}\Omega_2\frac{\partial^2}{\partial t^2}\Big[\nu_2 + \varepsilon_*\Big(p_1\frac{\partial^2}{\partial\varphi^2} - \frac{\partial^2}{\partial\xi^2}\Big)\Big]w\Big\}\end{gathered} \tag{1.10}$$

$$\begin{gathered}\Delta_2 v + p_2^{-1}\Omega_2\frac{\partial^2}{\partial t^2}\Big[(1+p_2)\frac{\partial^2 v}{\partial\xi^2} + p_1(1+p_2^{-1})\frac{\partial^2 v}{\partial\varphi^2} - \Omega_1\frac{\partial^2 v}{\partial t^2}\Big] = \\ = \frac{\partial}{\partial\varphi}\Big\{(1-\nu_1\nu_2-\nu_2p_2)\frac{\partial^2 w}{\partial\xi^2} +\end{gathered}$$

$$+\frac{p_1}{p_2}\frac{\partial^2 w}{\partial\varphi^2}-2\varepsilon_*\Big[\frac{\partial^4 w}{\partial\xi^4}+p_1p_2^{-1}(2p_2+\nu_1)\frac{\partial^4}{\partial\xi^2\partial\varphi^2}\Big]-$$

$$-p_2^{-1}\Omega_1\frac{\partial^2}{\partial t^2}\Big(1-\varepsilon_*\frac{\partial^2}{\partial\xi^2}\Big)w\Big\} \tag{1.11}$$

$$\varepsilon\Big(\Delta_1\Delta_2 w+\Big\{d_1\frac{\partial^6 w}{\partial\xi^6}+d_2\frac{\partial^6 w}{\partial\xi^4\partial\varphi^2}+d_3\frac{\partial^6 w}{\partial\xi^2\partial\varphi^4}+$$

$$+2d_4\frac{\partial^6 w}{\partial\varphi^6}+d_5\frac{\partial^4 w}{\partial\xi^2\partial\varphi^2}+d_4\frac{\partial^4 w}{\partial\varphi^4}\Big\}\Big)+$$

$$+\frac{\partial^4 w}{\partial\xi^4}+d_4(t_{12}^0-t_{22}^0)\frac{\partial^2 w}{\partial\xi^2\partial\varphi^2}-\Big[t_{12}^0\frac{\partial^2}{\partial\xi^2}+t_{22}^0\Big(\frac{\partial^2}{\partial\varphi^2}+1\Big)+2_s^0\frac{\partial^2}{\partial\xi\partial\varphi}\Big]\Delta_2 w-$$

$$-2s_2^0\frac{\partial^2}{\partial\xi\partial\varphi}\Big(g_1\frac{\partial^2 w}{\partial\xi^2}+g_2\frac{\partial^2 w}{\partial\varphi^2}\Big)=\overline{\Omega}_2\frac{\partial^2}{\partial t^2}\Big\{-\Delta_2 w+p_1^{-1}(1+p_2-\nu_1\nu_2)\frac{\partial^2 w}{\partial\xi^2}+$$

$$+p_1p_2^{-1}\frac{\partial^2 w}{\partial\varphi^2}+(1+p_2)p_2^{-1}\Big(\frac{\partial^2}{\partial\xi^2}+\rho_1\frac{\partial^2}{\partial\varphi^2}\Big)\Big[\varepsilon_*\Big(\Delta_1+2\frac{\partial^4}{\partial\varphi^4}+1\Big)-$$

$$-\Big(\overline{t}_{12}^0\frac{\partial^2}{\partial\xi^2}+\overline{t}_{22}^0\frac{\partial^2}{\partial\varphi^2}+2s_2^0\frac{\partial^2}{\partial\xi\partial\varphi}\Big)\Big]w+p_2^{-1}\Omega_2\frac{\partial^2}{\partial t^2}\Big[-p_1p_2^{-1}-$$

$$-\varepsilon_*p_1p_2^{-1}\Big(\Delta_1+2\frac{\partial^4}{\partial\varphi^4}+1\Big)+(1+p_2)\Big(\frac{\partial^2}{\partial\xi^2}+$$

$$+\rho_1\frac{\partial^2}{\partial\varphi^2}\Big)+\Big(2\overline{t}_{12}^0\frac{\partial^2}{\partial\xi^2}+\overline{t}_{22}^0\frac{\partial^2}{\partial\varphi^2}+2\overline{s}_1^0\frac{\partial^2}{\partial\xi\partial\varphi}\Big)-\Omega_1\frac{\partial^2}{\partial t^2}\Big]w\Big\}. \tag{1.12}$$

$$\Delta_2=\frac{\partial^4}{\partial\xi^4}+\delta_3\frac{\partial^4}{\partial\xi\partial\varphi}+\delta_1^{-1}\frac{\partial^4}{\partial\varphi^4},\quad \frac{p_1}{p_2}=\frac{E_2}{E_1} \tag{1.13}$$

$$\delta_1=\frac{E_1}{E_2},\quad \delta_2=\frac{4G(1-\nu_1\nu_2)}{E_2},\quad \delta_3=\frac{E_2}{G}-2\nu_2$$

$$d_1=2\nu_1,\quad d_2=6-8\nu_1\nu_2+2\nu_1\Big[\frac{E_2}{G}-\frac{4G(1-\nu_1\nu_2)}{E_1}\Big]$$

$$d_3=2\Big[\frac{4G(1-\nu_1\nu_2)}{E_1}+\frac{E_2}{G}-\nu_2\Big],\quad d_4=\frac{E_2}{E_1},\quad \rho_1=d_4 \tag{1.14}$$

$$d_5=\frac{4G(1-\nu_1\nu_2)}{E_1}+\frac{E_2}{G}-2\nu_2,\ \ \varepsilon=\varepsilon_*(1-\nu_1\nu_2)^{-1},\ \ \overline{\Omega}_i=\Omega_i(1-\nu_1\nu_2)^{-1}$$

$$g_1=\frac{E_2}{G}+\frac{\nu_1E_2}{2G(1-\nu_1\nu_2)}\Big(1-\frac{E_2}{E_1}-(1+2\nu_1)\frac{E_2}{E_1}\Big)-(1+2\nu_1)\frac{E_2}{E_1},$$

$$g_2=\delta_1^{-1}=\frac{E_2}{E_1},\ \ \overline{t}_{ij}^0=(1-\nu_1\nu_2)t_{ij}^0,\ \ \overline{s}_i^0=(1-\nu_1\nu_2)s_i^0\ \ (i,j=1,2).$$

The above equations and the corresponding boundary conditions for the formulated by us problem can be satisfied by means of the expression

$$V(\xi,\varphi,t)=e^{i\omega t}V(\xi,\varphi),$$

where V are the components of the perturbed dynamical stressed-strained state, ω is a constant to be defined. Bearing in mind that a solution of the system with respect to v is defined in terms of a linear combination of the product of trigonometric functions or trigonometric and exponential functions $n\varphi$ and $\mu_i\xi$, we out the conditions

$$\Big|\frac{\partial^m V}{\partial\xi^i\partial\varphi^k}\Big| \ll \Big|\frac{\partial^m V}{\partial\xi^{i'}\partial\varphi^{k'}}\Big|, \tag{1.15}$$

where $V = u, v, w$; $m \leq 8$; $k + 2 \leq k'$ (this condition is fulfilled if in the solution of the above-mentioned type $\mu_i^2 \ll n^2$). Assuming $E_1 \sim E_2$, we can neglect several terms in the corresponding system. Thus we have

$$\frac{\partial^4 u}{\partial\varphi^4} - p_1^{-1}\Omega_2\frac{\partial^2}{\partial t^2}\Big\{p_1(1+p_2^{-1})\frac{\partial^2 u}{\partial\varphi^2} - \Omega_1\frac{\partial^2 u}{\partial t^2}\Big\} = \\ = \frac{\partial}{\partial\xi}\Big\{-\frac{\partial^2\omega}{\partial\varphi^2} + \varepsilon_*\frac{\partial^4\omega}{\partial\varphi^4} - \nu_2 p_1^{-1}\Omega_2\frac{\partial^2 w}{\partial t^2}\Big\} \tag{1.16}$$

$$\frac{\partial^4 v}{\partial\varphi^4} - p_1^{-1}\Omega_2\frac{\partial^2}{\partial t^2}\Big\{p_1(1+p_2^{-1})\frac{\partial^2 v}{\partial\varphi^2} - \Omega_1\frac{\partial^2 v}{\partial t^2}\Big\} = \\ = \frac{\partial}{\partial\varphi}\Big\{\frac{\partial^2 w}{\partial\varphi^2} + 2\varepsilon_*\frac{\partial^4 w}{\partial\xi^2\partial\varphi^2} - p_1^{-1}\Omega_2\frac{\partial^2 w}{\partial t^2}\Big\} \tag{1.17}$$

$$\varepsilon\Big(\frac{\partial^8 w}{\partial\varphi^8} + 2\frac{\partial^6 w}{\partial\varphi^6} + \frac{\partial^4 w}{\partial\varphi^4}\Big) + \frac{p_2}{p_1}\frac{\partial^4 w}{\partial\xi^4} - t_{12}^0\frac{\partial^2}{\partial\xi^2}\Big(\frac{\partial^4 w}{\partial\varphi^4} - \frac{\partial^2 w}{\partial\varphi^2}\Big) - \\ - \Big(t_{22}^0\frac{\partial^2}{\partial\varphi^2} + 2s_2^0\frac{\partial^2}{\partial\xi\partial\varphi}\Big)\Big(\frac{\partial^4 w}{\partial\varphi^4} + \\ + \frac{\partial^2 w}{\partial\varphi^2}\Big) = \overline{\Omega}_2\Big\{-\frac{\partial^4 w}{\partial\varphi^4} + \frac{\partial^2 w}{\partial\varphi^2} + p_2^{-1}(1+p_2)\varepsilon_*\frac{\partial^2}{\partial\varphi^2}\Big(\frac{\partial^4 w}{\partial\varphi^4} + \\ + 2\frac{\partial^2 w}{\partial\varphi^2} + w\Big) + p_2^{-1}\Omega_2\frac{\partial^2}{\partial t^2}\Big[-w + (1+p_2)\frac{\partial^2 w}{\partial\varphi^2} - \\ - \varepsilon_*\Big(\frac{\partial^4 w}{\partial\varphi^4} + 2\frac{\partial^2 w}{\partial\varphi^2} + w\Big) - \Omega_2\frac{\partial w}{\partial t^2}\Big]\Big\}. \tag{1.18}$$

As for equation (1.18), it should be noted that if in the left-hand side of equation (1.18) the terms appearing in the first and in the last brackets cancel each other, i.e., if the sums appearing in these brackets considerably less in absolute value than their summands, than we cannot neglect secondary terms. In this case, the passage from (1.12) to (1.18) is invalid. However, if the secondary terms appearing in the first bracket are small as compared with the fourth term in equation (1.18), then this disturbance does not cover the first bracket.

If we adopt the condition $|\partial^2 w/\partial\varphi^2| \gg |w|$ (which is fulfilled if a number n of waves in a circular direction is sufficiently large, $n^2 \gg 1$), then some terms in (1.18) can be neglected, and thus we obtain

$$\varepsilon\frac{\partial^8 w}{\partial\varphi^8}+\frac{E_1}{E_2}\frac{\partial^4 w}{\partial\xi^4}-\Big(t_{12}^0\frac{\partial^2}{\partial\xi^2}+t_{22}^0\frac{\partial^2}{\partial\varphi^2}+2s_2^0\frac{\partial^2}{\partial\xi\partial\varphi}\Big)\frac{\partial^4 w}{\partial\varphi^4}=$$
$$=\overline{\Omega}_2=\frac{\partial^2}{\partial t^2}\Big[-\frac{\partial^4 w}{\partial\varphi^4}+p_2^{-1}(1+p_2)\varepsilon_*\frac{\partial^6 w}{\partial\varphi^6}+$$
$$+p_2^{-1}\Omega_2\frac{\partial^2}{\partial t^2}\Big(-(1+p_2)\frac{\partial^2 w}{\partial\varphi^2}-\varepsilon_*\frac{\partial^4 w}{\partial\varphi^4}-\Omega_2\frac{\partial^2 w}{\partial t^2}\Big)\Big]. \qquad (1.19)$$

Consequently, in the case under consideration we obtain for u, v, w the system of equations (1.16), (1.17) and (1.19).

2. In the sequel, we will consider proper oscillations of middle length shells as well as of long shells with freely supported edges. We proceed from equations (1.10)-(1.12) and assume $s_2^0=0$. Substituting the known solution from the products of trigonometric functions (satisfying the given boundary conditions) into the resolving equation (1.12), we obtain the following cubic equation with respect to ω_{mn}^2 (in what follows, the indices m, n and y of proper frequency ω_{mn} will be omitted):

$$a_0\alpha^3-a_1\alpha^2+a_2\alpha-c=0,\quad c=a_3-a_4,\quad \alpha=\Omega_2\omega^2, \qquad (2.1)$$
$$a_0=E_2/E_1,\quad \Omega_1=\Omega_2E_2/E_1,\quad B=\lambda_m^2+\delta_1^{-1}n^2,\quad \gamma_m=m\pi R/l, \quad (2.2)$$
$$a_2=(\overline{\Delta}_2+\{p_2^{-1}(1+p_2-\nu_1\nu_2)\lambda_m^2+p_1p_2^{-1}n^2+$$
$$+(1+p_2)p_2^{-1}B[\varepsilon_*(\overline{\Delta_1}+2n^2+1)-(\overline{t}_{12}^0\lambda_m^2+t_{22}n^2)]\})p_2,$$
$$a_3=[\varepsilon(\overline{\Delta}_1\overline{\Delta}_2-\{d_1\lambda_m^6+d_2\lambda_m^4n^2+d_3\lambda_m^2n^4+d_4(2n^6-n^4)-$$
$$-d_5\lambda_m^2n^2\})+\lambda_m^4](1-\nu_1\nu_2)p_2, \qquad (2.3)$$
$$a_4=[(t_{12}^0-t_{22}^0)d_4\lambda_m^2n^2+t_{12}^0\lambda_m^2\overline{\Delta}_2+t_{22}^0(n^2-1)\overline{\Delta}_2](1-\nu_1\nu_2)p_2.$$

Equation (2.1) is of the third degree with respect to ω^2, and hence every pair m, n is associated to three values ω^2. For an isotropic cylindrical shell, it is shown in [5] and [6] that the least of the values ω^2 corresponds to oscillations which have the largest of the displacement components ω, and two large values ω^2 correspond to oscillations mostly with the displacements u or v. Note that to the lower frequency value ω_0 (unloaded shell) there corresponds the value $\Omega\omega_0^2$, the least one as compared with unity. In the case under consideration, in the presence of preliminary longitudinal compression and external pressure the lower frequency $\omega<\omega_0$. This frequency is of special interest.

Consider first proper oscillations of middle length shells, free from pre-stresses ($t_{12}^0=t_{22}^0=s_2^0=0$). For orthotropic shells, it is assumed that elastic parameters do not significantly differ from each other ($E_1\sim E_2$). Estimating the coefficients of equation (2.1) and taking into account that $\varnothing_{,i}\,\omega^2$ ($i=1,2$) is small, we can, when defining lower frequency, neglect the terms of higher order for $\Omega_i\omega^2$. Moreover, in the expressions a_2 and a_3 we can omit the terms appearing in braces. Below, these conditions are justified for certain class of orthotropic shells.

Taking the above-said into consideration, to find the lower frequency we obtain

$$\Omega_2\omega^2 = \varepsilon\overline{\Delta}_1 + \lambda_m^4/\overline{\Delta}_2 \tag{2.4}$$
$$\overline{\Delta}_1 = \delta_1\lambda_m^4\widetilde{\Delta}_1, \quad \widetilde{\Delta}_1 = 1 + \delta_1^{-1}\delta_2\gamma + \delta_1^{-1}\gamma^2$$
$$\overline{\Delta}_2 = \lambda_m^4\widetilde{\Delta}_2, \quad \widetilde{\Delta}_2 = 1 + \delta_3\gamma + \delta_1^{-1}\gamma^2, \quad \gamma = n^2/\lambda_m^2.$$

Introduce the notation $\theta = \widetilde{\Delta}_2$. Then

$$\overline{\Delta}_1 = \delta_1\lambda_m^4[\theta + (a-b)\gamma], \quad \overline{\Delta}_2 = \lambda_m^4\theta, \quad a = \delta_1^{-1}\delta_2, \quad b = \delta_3. \tag{2.5}$$

As a result, equality (2.4) takes the form

$$\Omega_2\omega^2 = \varepsilon\delta_1\lambda_m^4[\theta + (a-b)\gamma] + \theta^{-1}. \tag{2.6}$$

The least value $\omega^2(\theta > 0)$ depending on θ is defined from the condition $(\omega^2)'_\theta = 0$, whence

$$\theta_* = (\varepsilon\delta_1)^{-1/2}/\lambda_m^2. \tag{2.7}$$

Thus we have

$$\Omega_2\omega^2 = 2(\varepsilon\delta_1)^{1/2}\lambda_m^2\Big[1 + \frac{1}{2}(\varepsilon\delta_1)^{1/2}(a-b)n^2\Big].$$

It is not difficult to see that the least value ω depending on m is realized for $m = 1$. Hence

$$\Omega_2\omega^2 = 2(\varepsilon\delta_1)^{1/2}\lambda_1^2\Big[1 + \frac{1}{2}(\varepsilon\delta_1)^{1/2}(a-b)n^2\Big], \tag{2.8}$$

where n is defined by virtue of equality (2.7) for $\lambda_m = \lambda_1$ which in an extended form represents the equality

$$\delta_1^{-1}\lambda_1^{-2}n^4 + \delta_3 n^2 - [(\varepsilon\delta_1)^{-1/2} - \lambda_1] = 0.$$

The positive root of the equation is of the form

$$n_0^2 = \frac{1}{2}\delta_1\lambda_1^2\{\sqrt{4\delta_1^{-1}\lambda_1^{-2}[(\varepsilon\delta_1)^{-1/2} - \lambda_1^2] + \delta_3^2} - \delta_3\}. \tag{2.9}$$

Introduce the notation

$$\begin{gathered} K(a,b,n) = \frac{1}{2}\varepsilon^{1/2}\alpha_1 n^2, \\ \alpha_1 = \delta_1^{1/2}(a-b) = \frac{4G(1-\nu_1\nu_2)}{\sqrt{E_1E_2}} + 4\sqrt{\nu_1\nu_2} - \frac{\sqrt{E_1E_2}}{G}, \\ E_1 = \gamma_1 E, \quad E_2 = \gamma_2 E, \quad \Omega\omega_*^2 = 2\varepsilon^{1/2}\lambda_1^2, \quad \Omega = \rho R^2/E \end{gathered} \tag{2.10}$$

where γ_1, γ_2-the parameters of orthotropy; E-module of elasticity isotropic shell's; ω_* is the lower frequency for an isotropic shell of middle length [3]. Then (2.8) takes the form

$$\omega^2/\omega_*^2 = (\gamma_1\gamma_2)^{1/2}[1 + K(a,b,n)]. \tag{2.11}$$

Introduce the parameter

$$\rho_1 = G/G_0, \quad G_0 = \sqrt{E_1E_2}/2(1+\sqrt{\nu_1\nu_2}). \tag{2.12}$$

Then

$$\alpha_1 = 2\rho_1(1-\sqrt{\nu_1\nu_2}) - 2\rho_1^{-1}(1+\sqrt{\nu_1\nu_2}) + 4\sqrt{\nu_1\nu_2}. \tag{2.13}$$

As a result, we find that for $G = G_0$, $(\rho_1 = 1)$ $\alpha_1 = 0$; for $G > G_0$, $(\rho_1 > 1)$ $\alpha_1 > 0$ and for $G < G_0$, $(\rho_1 < 1)$ $\alpha_1 < 0$.

For an isotropic shell, when $E_1 = E_2 = E(\gamma_1 = \gamma_2 = 1)$, $G = G_0 = E/2(1+\nu)$ we obtain $\alpha_1 = 0$. Consequently, $K = 0$, and formula (2.11) transforms into the well-known formula $\omega^2/\omega_*^2 = 1$ [3].

For $E_1 \neq E_2$ and the condition $\rho_1 = 1$ $(\alpha_1 = 0)$ which corresponds to $G = G_0 = \sqrt{E_1E_2}/2(1+\sqrt{\nu_1\nu_2})$, we obtain $K = 0$, and the formula $\omega^2/\omega_*^2 = (\gamma_1\gamma_2)^{1/2}$ holds.

For $E_1 \neq E_2$ and the condition $\rho_1 > 1$ $(\alpha_1 > 0)$ which corresponds to $G > G_0$, we obtain $K > 0$, and the formula $\omega^2/\omega_*^2 = (\gamma_1\gamma_2)^{1/2}[1+K(a,b,n)]$ holds.

For $E_1 \neq E_2$ and $\rho_1 < 1$ $(\alpha_1 < 0)$ which corresponds to $G < G_0$, $K < 0$ and hence $\omega^2/\omega_*^2 = (\gamma_1\gamma_2)^{1/2}$ $[1 - |K(a,b,n)|]$, where $K(a,b,n)$ is defined by formula (2.10).

Find out under what conditions the terms appearing in the braces in the expressions for a_2 and a_3 can be neglected. By D_1 and D_2 we denote a set of these terms:

$$D_1 = d_1\lambda_m^6 + d_2\lambda_m^4 n^2 + d_3\lambda_m^2 n^4 + d_4(2n^6 - n^4) - d_5\lambda_m^2 n^2 =$$
$$= \lambda_m^4 m^2\Big(d_1\frac{\lambda_m^2}{n^2} + d_2 + d_3\frac{n^2}{\lambda_m^2} + d_4\frac{n^2(2n^2-1)}{\lambda_m^4} - d_5\frac{1}{\lambda_m^2}\Big), \quad \varepsilon_* = h^2/12R^2$$
$$D_2 = \lambda_m^2\Big(\delta_1^{-1}\frac{n^2}{\lambda_m^2} + \Big(1 + \frac{E_2}{G}\Big) +$$
$$+\varepsilon_*\Big[\Big(1 + \frac{E_2}{G}(1-\nu_1\nu_2)\Big(1 + \delta_1^{-1}\frac{n^2}{\lambda_m^2}\Big)\Big)\Big(\overline{\Delta}_1 - 2n^2 + 1\Big)\Big]\Big).$$

Find the conditions under which $\overline{\Delta}_1\overline{\Delta}_2 \gg D_1$, $\overline{\Delta}_2 \gg D_2$, $\theta = \theta_* = (\varepsilon\delta_1)^{-1/2}\lambda_m^{-2}$. In the expression D_2 we can neglect the third term as compared with the sum of the first two terms. Taking into account that

$$\overline{\delta}_2 = \lambda_m^4\widetilde{\Delta}_2 = \lambda_m^4\theta_*, \quad \overline{\Delta}_1 = \lambda_m^4\widetilde{\delta}_1 = \lambda_m^4\delta_1[\theta_* + (a-b)n^2\lambda_m^{-2}], \quad \alpha_1 = \delta_1^{1/2}(a-b)$$
$$\overline{\Delta}_1\overline{\Delta}_2 = \delta_1\lambda_m^8\theta_*[\theta_* + (a-b)n^2\lambda_m^{-2}] =$$

$$= \varepsilon^{-1}\lambda_m^4[1+(a-b)(\varepsilon\delta_1)^{1/2}n^2] = \varepsilon^{-1}\lambda_m^4(1+\varepsilon^{1/2}\alpha_1 n^2)$$

the above inequalities are reduced to the conditions

$$\varepsilon^{-1}(1+\varepsilon^{1/2}\alpha_1 n^2) \gg n^2\Big(d_1\frac{\lambda^2 m}{n^2} + d_2 + d_3\frac{n^2}{\lambda_m^2} + d_4\frac{n^2(2n^2-1)}{\lambda_m^4} - d_5\frac{1}{\lambda_m^2}\Big)$$

$$\varepsilon^{-1/2}\Big(\frac{E_1}{E_2}\Big)^{1/2} \gg \frac{n^2}{\gamma_m^2} + \Big(\frac{E_1}{E_2} + \frac{E_1}{G}\Big).$$

In particular, we can show that the above conditions are fulfilled for thin shells of middle length ($\lambda_1 \sim 1$), $m \lesssim n$, $n^2 \sim \varepsilon^{-1/4}$, $E_1 \sim E_2$, $0,3G_0 < G \le G_0$ ($\alpha_1 < 0$) and $G_0 < G \le 3G_0$ ($\alpha_1 > 0$).

Consider the numerical examples. Consider first the case $G < G_0$,

$$E_1 = E, \quad E_2 = 2E, \quad \nu_1 = 0,2, \quad \nu_2 = 0,4 \quad (G_0 = 0,551E)$$

$$G = 0,25E, \quad \varepsilon^{-1} = 10^4\cdot 11,04, \quad \lambda_1 = \pi/2.$$

By virtue of formula (2.9), we obtain $n_0 = 5$. Moreover, $K(a,b,n) = -0,146$, $\omega^2/\omega_*^2 = (\gamma_1\gamma_2)^{1/2}0,854$. For the same parameters, just as in the above case and for $G = 1,2E_0$ ($G > G_0$), we obtain $n_0 = 5$, $K(a,b,n) = 0,115$, $\omega^2/\omega_*^2 = 1,115(\gamma_1\gamma_2)^{1/2}$.

Next, consider the case $E_1 = 2E$, $E_2 = E$, $\nu_1 = 0,2$, $\nu_2 = 0,1$ ($G_0 = 0,636E$), $G = 0,25E$ $G < G_0$, $\varepsilon^{-1} = 10^4\cdot 11,76$, $\lambda_1 = \frac{\pi}{2}$. We obtain $n_0 = 6$, $K(a,b,n) = -0,238$, $\omega^2/\omega_*^2 = 0,762(\gamma_1\gamma_2)^{1/2}$. For the same parameters as above and for $G = 1,2E$, we obtain $n_0 = 6$, $K(a,b,n) = 0,147$, $\omega^2/\omega_*^2 = 1,147(\gamma_1\gamma_2)^{1/2}$.

Thus we find that for orthotropic shells of middle length for $G < G_0$ the lower frequency decreases as G decreases and we define it by the formula $\omega^2/\omega_*^2 = (\gamma_1\gamma_2)^{1/2}(1-|K|)$, whereas for $G > G_0$ it increases and we define it by the formula $\omega^2/\omega_*^2 = (\gamma_1\gamma_2)^{1/2}(1+K)$. For $G = G_0$, the lower frequency of the orthotropic shell is defined by the formula $\omega^2/\omega_*^2 = (\gamma_1\gamma_2)^{1/2}$.

Consider now long shells. First, we investigate axisymmetric (longitudinal-transverse) oscillations of a preliminarily compressed shell ($t_{12}^0 \ne 0$, $t_{22}^0 = 0$, $s_2^0 = 0$, $v = 0$). Thus the system (1.1)-(1.3) takes the form

$$\overline{L}_{11}u = L_1 w, \quad L_{31}u + \overline{L}_{33}w = 0 \tag{2.14}$$

$$L_{11} = \frac{\partial^2}{\partial\xi^2}, \quad L_1 = \nu_2\frac{\partial}{\partial\xi} + \varepsilon_*\frac{\partial^2}{\partial\xi^2} + \overline{t}_{11}^0\frac{\partial}{\partial\xi}$$

$$L_{31} = \nu_1\frac{\partial}{\partial\xi} - \varepsilon_*\frac{E_1}{E_2}\frac{\partial^3}{\partial\xi^3}, \quad L_{33} = -1 - \varepsilon_*\Big(\frac{E_1}{E_2}\frac{\partial^4}{\partial\xi^4} + 1\Big) + \overline{t}_{12}^0\frac{\partial^2}{\partial\xi^2}.$$

The shell edges are assumed to be freely supported. Substituting the known solution $u = A_m\cos\lambda_m\xi\cos\omega_m t$, $w = C_m\sin\lambda_m\xi\cos\omega_m t$ ($\lambda_m = m\pi R/l$) into the system (2.14), we obtain the system of homogeneous algebraic equations with respect to the coefficients A_m, C_m. To obtain a non-trivial solution, it is necessary that the determinant of the system is equal to zero. Multiplying out the determinant, we obtain the quadratic equation with respect to $\Omega_1\omega_m^2$ (in the sequel, the index m will be omitted):

$$\Omega_1^2\omega^4 - \Big[\frac{E_2}{E_1} + \lambda_m^2 + \varepsilon_*\Big(\lambda_m^4 + \frac{E_2}{E_1}\Big) + \bar{t}_{11}^0\lambda_m^2\Big]\Omega_1\omega^2 +$$
$$+\Big[(1-\nu_1\nu_2)\frac{E_2}{E_1}\lambda_m^2 + \varepsilon_*\lambda_m^2\Big(\lambda_m^4 + \frac{E_2}{E_1}\Big) - (\nu_2 - \lambda_m^2)\lambda_m^2 t_{11}^0\Big] = 0. \quad (2.15)$$

Equation (2,15) is of the second degree with respect to ω^2. Hence every m is associated with two values ω^2,

$$\Omega_1\omega^2 = \frac{1}{2}\Big[\frac{E_2}{E_1} + \lambda_m^2 + \varepsilon_*\Big(\lambda_m^4 + \frac{E_2}{E_1}\Big) + t_{11}^0\lambda_m^2\Big] \pm$$
$$\pm\Big[\Big(\frac{E_2}{E_1} + \lambda_m^2 + \varepsilon_*\Big(\lambda_m^4\frac{E_2}{E_1}\Big) + \bar{t}_{11}^0\lambda_m^0\Big)^2 -$$
$$-4\Big((1-\nu_1\nu_2)\frac{E_2}{E_1}\lambda_m^2 + \varepsilon_*\lambda_m^2\Big(\lambda_m^4 + \frac{E_2}{E_1}\Big) - (\nu_2 - \lambda m^2)\lambda_m^2\bar{t}_{11}^0\Big)^{1/2}\Big]. \quad (2.16)$$

Of special interest are the lower frequencies. Taking into account that for long shells $\lambda_m^2 \ll 1$ (if m is small), $\bar{t}_{11}^0 \ll 1$, and if $E_2 \gtrsim E_1$, then the expression (2.16) for the least root takes the form

$$\Omega_1\omega^2 = \frac{1}{2}\frac{E_2}{E_1}\Big[1 - \sqrt{1 - 4(1-\nu_1\nu_2)\frac{E_1}{E_2}\lambda_m^2}\Big] \approx (1-\nu_1\nu_2)\lambda_m^2, \qquad (2.17)$$

and in (2.17) we neglect the values of second order smallness. Thus it is not difficult to see that the lower frequency is realized for $m = 1$. Consequently, the lower frequency is defined by means of the formula

$$\Omega_1\omega^2 = (1-\nu_1\nu_2)\lambda_1^2.$$

This corresponds go to the above-made assumption that m is not too large and $\Omega_2\omega^2$ is a small value (since λ_1^2 is small).

Thus for thin shells in the case of axi2.34 symmetric oscillations for $E_2 \gtrsim E_1$, the lower frequency does not depend on preliminary compression to within small values. For an isotropic case, the expression (2.17) differs by the value of the coefficient at λ_m^2 from the formula given in [7], but coincides with the coefficient given in [3] for the case under consideration.

If E_1 is greater than E_2, we take all the terms appearing in (2.16) into consideration.

Next, we investigate the case $n = 1$. As is known, in this case there take place oscillations which correspond to the lateral swelling, when cross-section remains circular and shifts only in its own plane. Since $\lambda_m^2 \ll 1$, $t_{2i}^0 \ll 1$ (because their critical values t_{i1}^*, $t_{2i}^* \ll 1$), neglecting in (2.9) small terms, we btain

$$a_1 = 2\frac{E_2}{E_1} + \frac{G}{E_1}(1-\nu_1\nu_2), \quad a_2 = \Big[2\frac{E_2}{E_1} + \Big(1+\frac{E_2}{G}\Big)\lambda_m^2\Big]p_2$$
$$a_3 = (1-\nu_1\nu_2)\lambda_m^4 p_2, \quad a_4 = (1-\nu_1\nu_2)(2t_{11}^0 - t_{21}^0)\lambda_m^2 p_2. \tag{2.18}$$

Taking into account the coefficients in (2.18), as well as the fact that for lower frequency the value $\Omega_2\omega^2$ is small as compared to unity, we can neglect in (2.1) the terms of higher order for $\Omega_2\omega^2$ and define the least root of that equation. As a result, we have

$$\Omega_1\omega^2 = \lambda_m^2\Big[\lambda_m^2 + (2t_{11}^0 - t_{21}^0)\Big]\Big/\Big[1+\frac{1}{2}\Big(\frac{E_1}{E_2}+\frac{E_1}{G}\Big)\lambda_m^2\Big].$$

Thus we find that ω is realized for $m = 1$, i.e., when under the oscillation there takes place only one half-wave in the axial direction. Consequently,

$$\omega^2 = \frac{E_1\rho}{2R^2}\lambda_1^2[\lambda_1^2 + (2t_{11}^0 - t_{21}^2)]\Big/\Big[1+\frac{1}{2}\Big(\frac{E_1}{E_1}+\frac{E_1}{G}\Big)\lambda_1^2\Big]. \tag{2.19}$$

For the axial compression $t_{11}^0 < 0$, and for the tension $t_{11}^0 > 0$. For the external pressure $t_{21}^0 < 0$, and for the internal pressure $t_{21}^0 > 0$. In case of only one axial compression ($t_{11}^0 = -t^0$, $t_{21}^0 = 0$, $E_2 \gtrsim E_1$), we can neglect the second term in the denominator of the expression (2.19) and obtain

$$\omega^2 = \frac{E_1\rho}{2R^2}\lambda_1^2(\lambda_1^2 - 2t^0).$$

In this case, the shell oscillates like a precompressed support, however, taking tangential forces of inertia into account, we obtain coefficient 2 in the denominator, i.e., tangential forces of inertia in this case affect considerably the lower frequency.

Consider now the case $n \geq 2$. We consider the oscillations free from cross-section displacements in their own plane and take starlike shape.

Taking into account that for long shells

$$\lambda_m^2 \ll n^2, \tag{2.20}$$

we obtain

$$a_1 = \frac{E_2}{E_1} + \Big[1+\frac{G}{E_2}(1-\nu_1\nu_2)\Big]\frac{E_2}{E_1}n^2 + \varepsilon_*\frac{E_2}{E_1}(n^2-1)^2$$
$$a_2 = \Big[\frac{E_2}{E_1}n^2(n^2+1) + \varepsilon_*\frac{1+p_2}{p_2}\frac{E_2}{E_1}n^2(n^2-1)^2\Big]p_2$$
$$a_3 = \Big[\varepsilon\frac{E_2}{E_1}n^4(n^2-1)^2 + \lambda_m^4\Big](1-\nu_1\nu_2)p_2$$
$$a_4 = \Big[t_{12}^0\frac{E_2}{E_1}\lambda_m^2 n^2(n^2+1) + t_{22}^0\frac{E_2}{E_1}n^4(n^2-1)\Big](1-\nu_1\nu_2)p_2. \tag{2.21}$$

Since $a_2 > a_1 > 1$, $\varepsilon_* n^2 \gg 1$ for the least root of equation (2.1) we get

$$\omega^2 = \frac{E_2\rho}{R^2}\times$$
$$\times\frac{\varepsilon n^4(n^2-1)^2 + E_1E_2^{-1}\lambda_m^4 + t_{12}^0\lambda_m^2 n^2(n^2+1) + t_{22}^0 n^4(n^2-1)^2}{n^2(n^2+1)}. \quad (2.22)$$

This relation shows that for long shells we have to take into consideration both the tangential forces of inertia and the tension of the middle surface; $\min\omega$, depending on λ_m^2, can be easily obtained from (2.22), for

$$\lambda_m^2 = \frac{1}{2}t_{11}^0 n^2(n^2+1), \quad t_{11}^0 = \frac{E_1}{E_2}t_{12}^0 \quad (2.23)$$

that is, λ_m^2 is the linear function of t_{11}^0, and as $T_1^0 \to 0$, we obtain $\lambda_m = 0$. However, for $\lambda_m = 0$ we arrive at a trivial case. For a non-trivial solution, the least possible value for $\lambda_m = m\lambda_1$ is $\lambda_1 = \pi R/l$. Incidentally, the value $\lambda_m = \lambda_1$ realizes the least frequency value for an unloaded shell, $\omega(T_1^0 = 0) = \omega_0$ [6].

In the sequel, under long shells we will mean those for which

$$0 < \lambda_1^2 \le 12\varepsilon^{1/2}(E_2/E_1)^{1/2}. \quad (2.24)$$

Introduce the dimensionless parameters

$$k_1 = -t_1/t_{1*}, \quad k_2 - t_2/t_{2*}, \quad t_1 = T_1^0/Eh, \quad t_2 = T_2^0/Eh$$
$$E_1 = \gamma_1 E, \quad E_2 = \gamma_2 E, \quad t_{1*} = (\gamma_1\gamma_2)^{1/2}\frac{6}{5}\varepsilon^{1/2}, \quad t_{2*} = 3\gamma_2\varepsilon. \quad (2.25)$$

Thus the prestressed state will be characterized in the sequel by the parameters k_1, k_2 for the compression $t_1 < 0$ and, consequently, $k_1 = |t_1|/t_{1*} > 0$. Note that $k_1 < 1$, since $|t_1| < t_{1*}$, t_{1*} is an absolute value of the critical load for the problem of stability [12]. For the external pressure, $t_2 < 0$, and hence, $k_2 > 0$; for the internal pressure $t_2 > 0$, $k_2 < 0$ (here we assume that $k_2 < 1$).

Taking into account (2.25), the expressions (2.22) and (2.23) take the form

$$\frac{\rho R^2}{E}\omega^2 = \frac{\gamma_2\varepsilon n^4(n^2-1)^2 + \gamma_1\lambda_m^4}{n^2(n^2+1)} - k_1 t_{1*}\lambda_m^2 - k_2 t_{2*}\frac{n^2(n^2-1)}{n^2+1} \quad (2.26)$$

$$\lambda_m^2 = k_1\frac{t_{1*}}{2\gamma_1}n^2(n^2+1). \quad (2.27)$$

It is not difficult to see that for small k_1,

$$k_1\frac{t_{1*}}{2\gamma_1}n^2(n^2+1) < \lambda_1^2, \quad (2.28)$$

therefore for such values of loading t_1 $\min\omega$, depending on λ_m, is realized not by formula (2.27), but for $\lambda_m = \lambda_1$. Then

$$\frac{\rho R^2}{E}\omega^2 = \frac{\gamma_2\varepsilon n^4(n^2-1)^2+\gamma_1\lambda_1^4}{n^2(n^2+1)} - k_1 t_{1*}\lambda_1^2 - k_2 t_{2*}\frac{n^2(n^2-1)}{n^2+1}. \qquad (2.29)$$

Here of we find that the least value ω is obtained for $n = 2$, independently of k_1 and k_2 $(k_1, k_2 < 1)$. Since $t_{1*} = \frac{6}{5}(\gamma_1\gamma_2)^{1/2}\varepsilon^{1/2}$, inequality (2.28) and the relation (2.29) take the form

$$12(\gamma_2/\gamma_1)^{1/2}k_1\varepsilon^{1/2} < \lambda_1,$$
$$\omega^2 = \frac{E}{\rho R^2}\frac{36}{5}\gamma_2\varepsilon\Big\{1-\Big(\frac{\gamma_1}{\gamma_2}\Big)^{1/2}\frac{\lambda_1\varepsilon^{-1/2}}{6}\Big[k_1-\Big(\frac{\gamma_1}{\gamma_2}\Big)^{1/2}\frac{\lambda_1\varepsilon^{-1/2}}{24}\Big]-k_2\Big\}. \qquad (2.30)$$

For an arbitrary value $k_1 = k_{1*}$, inequality (2.28) transforms into the equality

$$k_{1*}\frac{t_{1*}}{2\gamma_1}n^2(n^2+1) = \lambda_1^2. \qquad (2.31)$$

In this case, just as in the previous one, ω adopts the least value for $n = 2$, and from the relation (2.31) we obtain

$$k_{1*}12\Big(\frac{\gamma_2}{\gamma_1}\Big)^{1/2}\varepsilon^{1/2} = \lambda_1^2, \quad k_{1*} = \frac{1}{12}\Big(\frac{\gamma_1}{\gamma_2}\Big)^{1/2}\varepsilon^{-1/2}\lambda_1^2. \qquad (2.32)$$

It is not difficult to see that as l increases, the value k_{1*} decreases and when $l \to \infty$ the value $k_{1*} \to 0$. Thus when $k_1 = k_{1*}$ on the basis of formulas (2.27) and (2.32) we find that $\min\omega$ is realized for $\lambda_m^2 = \lambda_1^2 = 12(\gamma_2/\gamma_1)^{1/2}\varepsilon^{1/2}k_{1*}$.

Taking into account the expression for the coefficient k_{1*}, according to equality (2.32), we can represent the relation (2.30) as follows:

$$\omega^2/\omega_\infty^2 = 1 - 2k_{1*}\Big(k_1 - \frac{1}{2}k_{1*}\Big) - k_2, \quad \omega_\infty^2 = \frac{\gamma_2 E}{\rho R^2}\frac{36}{5}\varepsilon. \qquad (2.33)$$

Next, as k_1 $(k_1 < 1)$ increases, there occurs the case for which

$$k_1\frac{t_{1*}}{2\gamma_1}n^2(n^2+1) > \gamma_1^2 \quad (k_1 > k_{1*}) \qquad (2.34)$$

and hence formula (2.27) holds for the least value ω. Substituting (2.27) into (2.26), we obtain

$$\omega^2 = \frac{\gamma_2 E}{\rho R^2}n^2(n^2+1)\Big[\frac{\varepsilon(n^2-1)^2}{(n^2+1)^2} - \frac{k_1^2}{\gamma_1\gamma_2}\Big(\frac{t_{1*}}{2}\Big)^2 - \frac{k_2 t_{2*}(n^2-1)}{\gamma_2(n^2+1)^2}\Big].$$

This implies that $\min\omega$ holds for $n = 2$. Thus we have

$$\omega^2/\omega_\infty^2 = 1 - k_1^2 - k_2. \qquad (2.35)$$

For $\omega = 0$, $k_2 = 0$, for an isotropic shell from formula (2.35) follows the well-known Sautwell formula for a critical load $t_{1*} = \frac{6}{5}\varepsilon^{1/2}$ [13]. From the relation (2.27) for $k_1 = 1$ we obtain the critical number of longitudinal waves $\lambda_{m*}^2 = 12\varepsilon^{1/2}$.

It should be noted that the relation (2.24) follows directly from the condition $0 < k_{1*} \leq 1$.

Thus we find that if the value of the preliminary compression k_1 such that $0 < k_1 < k_{1*}$, then the lower frequency is defined by formula (2.33), whereas for the compression lying in the interval $k_{1*} < k_1 \leq (1-k_2)^{1/2}$, the lower frequency is defined by formula (2.35). For $k_1 = k_{1*}$, frequency values defined by formulas (2.33) and (2.35) coincide. This, for example, occurs in shells with $\lambda_1^2 = 4\varepsilon^{1/2}(E_2/E_1)^{1/2}$, $k_{1*} = 0,333$. If $\lambda_1^2 = \varepsilon^{1/2}(E_2/E_1)^{1/2}$, we have $k_{1*} = 0,083$. In this case the frequency value ω defined by formulas (2.33) and (2.35) in the interval $0 < k_1 \leq 0,083$ differ by a small value Δ which can be neglected. The maximal divergence for $k_1 = 0$ $\Delta(k_1 = 0) = k_{1*}^2 = 0,0067$ is less than $0,7\%$. An analogous phenomenon takes place when $\lambda_1^2 < \varepsilon^{1/2}(E_2/E_1)^{1/2}$, since $k_{1*} < 0,083$, and consequently, for sufficiently long shells

$$0 < \lambda_1^2 \leq \varepsilon^{1/2}(E_2/E_1)^{1/2}$$

for an arbitrary value of preliminary compression from the interval $0 \leq k_1 \leq (1-k_2)^{1/2}$, the lower frequency ω is defined by formula (2.35).

For $\varepsilon^{1/2}(E_2/E_1)^{1/2} < \lambda_1^2 \leq 12\varepsilon^{1/2}(E_2/E_1)^{1/2}$, we have two intervals for the values k_1, and for each of them formulas (2.33) and (2.35), as is said above, work respectively.

Consider now the action of a preliminary axial tension $t_1 > 0$ ($k_1 < 0$) together with a normal pressure t_2 on the lower frequency.

In this case, just as in the previous one, for the least root of equation (2.8) we obtain

$$\frac{\rho R^2}{E}\omega^2 = \frac{\gamma_2 \varepsilon n^4(n^2-1)^2 + \gamma_1 \lambda_m^4}{n^2(n^2+1)} + |k_1| t_{1*}\lambda_m^2 - k_2 t_{2*}\frac{n^2(n^2-1)}{n^2+1}. \quad (2.36)$$

It is not difficult to notice that unlike of the case of compression, $\min \omega$, depending on λ_m^2, is realized for $\lambda_m = \lambda_1$ independently of k_1, k_2, ($|k_1|, k_2 < 1$) and n. Moreover, it is not difficult to find that the least value ω holds for $n = 2$, since $\omega'_{n^2} > 0$ for $n > 2$.

As a result, we have that the lower frequency value is defined by the formula

$$\frac{\omega^2}{\omega_\infty^2} = 1 + \Big(\frac{\gamma_1}{\gamma_2}\Big)^{1/2}\frac{\lambda_1 \varepsilon^{-1/2}}{6}\Big[|k_1| + \Big(\frac{\gamma_1}{\gamma_2}\Big)^{1/2}\frac{\lambda_1\varepsilon^{-1/2}}{24}\Big] - k_2. \quad (2.37)$$

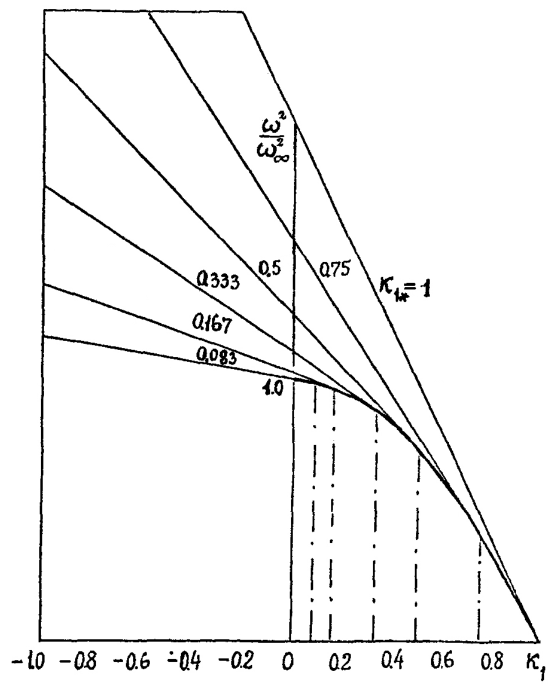

Fig. 1

Taking into account (2.32) for the coefficient k_{1*}, we can represent equality (2.37) in the form

$$\omega^2/\omega_\infty^2 = 1 + 2k_{1*}\Big(|k_1| + \frac{1}{2}k_{1*}\Big) - k_2.$$

Fig. 1 shows the dependence of the lower frequency (for $k_2 = 0$) on the preliminary stress k_1 when the parameter k_{1*}, defined by (2.32), varies. Note that the dimensionless frequency ω^2/ω_∞^2 is given on the ordinate axis, and the dimensionless stress $k_1 = -t_1/t_{1*}$ is given on the abscissa axis. In case of compression $k_1 > 0$, for tension $k_1 < 0$.

Fig. 2 shows the dependence of lower frequency ($k_2 = 0,2; 0; -0,2$) on the preliminary compression k_1, when the parameter k_{1*} varies.

Thus in the present work we have obtained simple formulas for finding lower frequency of long orthotropic shells depending on the given preliminary stresses, geometrical sized of the shell and elastic orthotropy parameters.

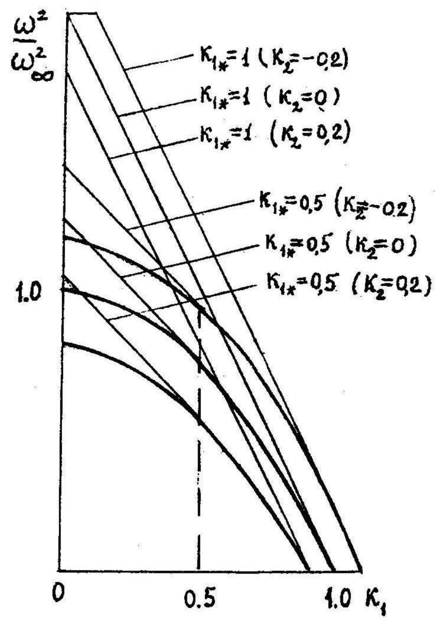

Fig. 2

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(Received 20.01.2009)

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