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# EXISTENCE THEOREMS OF THE BOUNDARY VALUE PROBLEMS OF THE ELASTIC MIXTURE THEORY

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Abstract. The purpose of this paper is to study the boundary value problems of the elastic mixture theory for the equation transversally isotropic body. The existence and uniqueness of solutions are proved.

**რეზიუმე**. ნაშრომის მიზანია დრეკად ნარევთა თეორიის სასაზღვრო ამოცანების შესწავლა ტრანსვერსალურად იზოტროპული სხეულის განტოლებისათვის. დამტკიცებულია ამონახსნების არსებობა და ერთადერთობა.

## Introduction

In the present paper we consider a three-dimensional version of the theory of elastic mixture for a transversally isotropic body. The idea to apply singular integral equations to the boundary value problems (BVPs) is due to M. Basheleishvili and D. Natroshvily [1]. In this paper we intend to extend this result to BVPs of elastic mixture for a transversally isotropic elastic body.

## 1. Some Previous Results

We use the following notation. Let $E_3$ be a 3-dimensional Euclidean space, $D^+ \in E_3$ be a finite domain bounded by the surface $S$. Let $S$ be a Lyapunov surface and $D^- = E_3 \cap \bar{D^+}$. In the theory of elastic mixtures, the displacement vector is usually denoted by $U = U^T(u', u'')$, where $u'(u_1', u_2', u_3')$ and $u^{''}(u_1'', u_2'', u_3'')$ are the partial displacement vectors. Everywhere below by $T$ we denote transposition.

**Basic Equations of Elastic Mixture**. The basic equations of statics of a transversally isotropic mixture can be written in the form [2]

$$C(\partial x)U = \begin{pmatrix} C^{(1)}(\partial x) & C^{(3)}(\partial x) \\ C^{(3)}(\partial x) & C^{(2)}(\partial x) \end{pmatrix} U = 0, \tag{1}$$

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where

$$C^{(j)}=\big(C^{(j)}_{pq}(\partial x)\big)_{3x3},\ \ C^{(j)}_{pq}=C^{(j)}_{qp},\ \ p,q=1,2,3,\ \ j=1,2,3,$$

$$C^{(j)}_{11}(\partial x)=c^{(j)}_{11}\frac{\partial^2}{\partial x_1^2}+c^{(j)}_{66}\frac{\partial^2}{\partial x_2^2}+c^{(j)}_{44}\frac{\partial^2}{\partial x_3^2},$$

$$C^{(j)}_{12}(\partial x)=\big(c^{(j)}_{11}-c^{(j)}_{66}\big)\frac{\partial^2}{\partial x_1\partial x_2},$$

$$C^{(j)}_{k3}(\partial x)=(c^{(j)}_{13}+c^{(j)}_{44})\frac{\partial^2}{\partial x_k\partial x_3},\quad k=1,2,$$

$$C^{(j)}_{22}(\partial x)=c^{(j)}_{66}\frac{\partial^2}{\partial x_1^2}+c^{(j)}_{11}\frac{\partial 2}{\partial x_2^2}+c^{(j)}_{44}\frac{\partial^2}{\partial x_3^2},$$

$$C^{(j)}_{33}(\partial x)=c^{(j)}_{44}\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)+c^{(j)}_{33}\frac{\partial^2}{\partial x_3^2},$$

where $c^{(j)}_{pq}$ are the constants, characterizing the physical properties of the mixture and satisfying certain inequalities which follow from the positive definiteness of potential energy.

We introduce the following definitions:

**Definition 1.** A vector-function $U(x)$ defined in the domain $D^+(D^-)$ is said to be regular if it has integrable continuous second derivatives in $D^+(D^-)$, and the function $U(x)$ itself and its first derivatives are continuously extendable at every boundary point of the domain $D^+(D^-)$. Note that for the infinite domain $D^-$ the vector $U(x)$ additionally satisfies at infinity the following conditions:

$$U(x)=O(|x|^{-1}),\frac{\partial U_k}{\partial x_j}=O(|x|^{-2}),|x|^{-2}=x_1^2+x_2^2+x_3^2,$$
$$j=1,2,3;\quad k=1,2,\dots,6.$$

**The Stress Vector**. Throughout this paper, $n(x)=n(n_1,n_{2,}n_{3)})$ denotes the exterior to $D^+$ unit normal vector at the point $x\in S$. Denoting the generalized stress vector by $P(\partial x,n)U$, we have [2]

$$P(\partial x,n)U=\begin{pmatrix} P^{(1)}(\partial x,n) & P^{(3)}(\partial x,n)\\ P^{(3)}(\partial x,n) & P^{(2)}(\partial x,n)\end{pmatrix}U, \tag{2}$$

where $P^{(j)}(\partial x,n)=\|P^{(j)}_{pq}\|_{3x3},j=1,2,3,$ is the generalized stress tensor with the elements

$$P^{(j)}_{11}=c^{(j)}_{11}n_1\frac{\partial}{\partial x_1}+c^{(j)}_{66}n_2\frac{\partial}{\partial x_2}+c^{(j)}_{44}n_3\frac{\partial}{\partial x_3},$$

$$P^{(j)}_{12}=\alpha^{(j)}_0n_1\frac{\partial}{\partial x_2}+\gamma^{(j)}_0n_2\frac{\partial}{\partial x_1},$$

$$P_{13}^{(j)} = \beta_0^{(j)} n_1 \frac{\partial}{\partial x_3} + \delta_0^{(j)} n_3 \frac{\partial}{\partial x_1}, \quad P_{21}^{(j)} = \alpha_0^{(j)} n_2 \frac{\partial}{\partial x_1} + \gamma_0^{(j)} n_1 \frac{\partial}{\partial x_2},$$

$$P_{22}^{(j)} = c_{66}^{(j)} n_1 \frac{\partial}{\partial x_1} + c_{11}^{(j)} n_2 \frac{\partial}{\partial x_2} + c_{44}^{(j)} n_3 \frac{\partial}{\partial x_3},$$

$$P_{23}^{(j)} = \beta_0^{(j)} n_2 \frac{\partial}{\partial x_3} + \delta_0^{(j)} n_3 \frac{\partial}{\partial x_2},$$

$$P_{31}^{(j)} = \beta_0^{(j)} n_3 \frac{\partial}{\partial x_1} + \delta_0^{(j)} n_1 \frac{\partial}{\partial x_3}, \quad P_{32}^{(j)} = \beta_0^{(j)} n_3 \frac{\partial}{\partial x_2} + \delta_0^{(j)} n_2 \frac{\partial}{\partial x_3},$$

$$P_{33}^{(j)} = c_{44}^{(j)} \Big( n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2} \Big) + c_{33}^{(j)} n_3 \frac{\partial}{\partial x_3},$$

$$\alpha_0^{(j)} + \gamma_0^{(j)} = c_{11}^{(j)} - c_{66}^{(j)}, \delta_0^{(j)} + \beta_0^{(j)} = c_{13}^{(j)} + c_{44}^{(j)}.$$

When $\beta_0^{(j)} = c_{13}^{(j)}, \gamma_0^{(j)} = c_{66}^{(j)}, \alpha_0^{(j)} = c_{11}^{(j)} - 2c_{66}^{(j)}, \delta_0^{(j)} = c_{44}^{(j)}, j = 1, 2, 3$, we have the stress vector. The stress operator we denote by $T(\partial x, n)$.

For equation (1) we pose the following BVPs. Find in the domain $D^+(D^-)$ a regular solution $U(x)$ of equation (1), satisfying on the boundary $S$ one of the following boundary conditions:

**Problem 1.** $U^{\pm} = f(z), z \in S$;

**Problem 2.** $(TU)^{\pm} = F(z), z \in S$,

where $(.)^{\pm}$ denote the limiting values from $D^{\pm}$, $f(z)$, $F(z)$ is a given function on $S$, $Tu$ is the stress vector.

**The Uniqueness Theorems**. In this subsection we investigate the question on the uniqueness of solutions of the above-mentioned problems.

Let the first BVP has in the domain $D^+$ two regular solutions $U^{(1)}$ and $U^{(2)}$. We write $u = U^{(1)} - U^{(2)}$. Evidently, the vector $u$ satisfies (1) and the boundary condition $u^+ = 0$ on $S$. Note that if $u$ is a regular solution of equation (1), we have the following Green's formula:

$$\begin{aligned} \iiint_{D^+} E(u,u) d\sigma &= \iint_S u^+ (Tu)^+ ds, \\ \iiint_{D^-} E(u,u) d\sigma &= -\iint_S u^- (Tu)^- ds, \end{aligned} \tag{3}$$

where $E(u, u)$ is a potential energy [2]:

$$E(u,u) = E^{(1)}(u,u) + E^{(2)}(u,u) + 2E^{(3)}(u,u),$$

$$E^{(1)}(u,u) = c_{66}^{(1)} e_{12}^{(')2} + c_{44}^{(1)} (e_{13}^{(')2} + e_{23}^{(')2}) + c_{33}^{(1)} e_{33}^{(')2} +$$

$$
\begin{aligned}
&+c_{11}^{(1)}(e_{11}^{(')2}+e_{22}^{(')2})+2c_{13}^{(1)}(e_{22}^{'}e_{33}^{'}+e_{11}^{'}e_{33}^{'})+\\
&+2(c_{11}^{(1)}-2c_{66}^{(1)})e_{11}^{'}e_{22}^{'},\\
E^{(2)}(u,u)&=c_{66}^{(2)}e_{12}^{('')2}+c_{44}^{(2)}(e_{13}^{('')2}+e_{23}^{('')2})+c_{33}^{(2)}e_{33}^{('')2}+\\
&+c_{11}^{(2)}(e_{11}^{('')2}+e_{22}^{('')2})+2c_{13}^{(2)}(e_{22}^{''}e_{33}^{''}+e_{11}^{''}e_{33}^{''})+ \qquad (4)\\
&+2(c_{11}^{(2)}-2c_{66}^{(2)})e_{11}^{''}e_{22}^{''},\\
E^{(3)}(u,u)&=c_{33}^{(3)}e_{33}^{'}e_{33}^{''}+c_{11}^{(3)}(e_{11}^{'}e_{11}^{''}+e_{22}^{'}e_{22}^{''})+\\
&+c_{44}^{(3)}(e_{13}^{'}e_{13}^{''}+e_{23}^{'}e_{23}^{''})+c_{66}^{(3)}e_{12}^{'}e_{12}^{''}+\\
&+c_{13}^{(3)}[e_{33}^{''}(e_{11}^{'}+e_{22}^{'})+e_{33}^{'}(e_{11}^{''}+e_{22}^{''})]+\\
&+(c_{11}^{(3)}-2c_{66}^{(3)})(e_{11}^{''}e_{22}^{'}+e_{22}^{''}e_{11}^{'}),
\end{aligned}
$$

$e'_{ij}$ and $e''_{ij}$ are the components of the strain tensor.

Using (3) and taking into account the fact that the potential energy is positive definite, we conclude that $u=\text{const}$, $x\in D^+$. Since $u^+=0$, we have $u=0, x\in D^+$. Thus the first BVP has in the domain $D^+$ at most one regular solution. The vectors $u^{'}$ and $u^{''}$ in the domain $D^-$ must satisfy the condition at infinity. In this case the regular vector $u=u^{(1)}-u^{(2)}$ satisfies (1) and hence formula (3) is valid and $u(x)=\text{const}$, $x\in D^-$, but $u^-=0$, which implies that $u=0$, $x\in D^-$. Thus the first BVP has at most one regular solution in the domain $D^-$.

Let $(TU)^+=0$ on $S$. Then applying (3), we have

$$u^{'}=a+[b,x],\ \ u^{''}=c+[d,x],\ \ x\in D^+,$$

where $a,b,c,d$ are arbitrary real constant vectors.

The vector $u$ in the domain $D^-$ must satisfy the condition at infinity. In this case, from (3) we obtain $u=0$. Therefore we can formulate the final results.

**Theorem 1.** *A regular solution of the BVP* $(2)^+$ *is not unique in the domain* $D^+$. *Two regular solutions may differ by an additive vector of rigid displacement.*

**Theorem 2.** *Problems* $(1)^{\pm}$ *and* $(2)^-$ *have at most one regular solution.*

**Matrix of Fundamental Solutions**. The matrix of fundamental solutions of equation (1) can be represented as follows:

$$\Gamma(x-y)=\sum_{k=1}^{6}\|\Gamma_{pq}^{(k)}\|_{6x6},$$

where

$$
\begin{aligned}
&\Gamma^{(k)}_{pq} = \Gamma^{(k)}_{qp}, \ \ \Gamma^{(k)}_{pq} = \delta_{pq}\tfrac{A^{(k)}_{11}}{r_k} + A^{(k)}_{12}\frac{\partial^2\Phi_k}{\partial x_p \partial x_q}, \ \ p=1,2; \ \ q=1,2;\\
&\Gamma^{(k)}_{p3} = A^{(k)}_{13}\frac{\partial^2\Phi_k}{\partial x_p \partial x_3}, \ \ \Gamma^{(k)}_{p4} = \delta_{p1}\frac{A^{(k)}_{14}}{r_k} + A^{(k)}_{42}\frac{\partial^2\Phi_k}{\partial x_p \partial x_1}, \ \ p=1,2;\\
&\Gamma^{(k)}_{15} = A^{(k)}_{42}\frac{\partial^2\Phi_k}{\partial x_1 \partial x_2}, \ \ \Gamma^{(k)}_{44} = \frac{A^{(k)}_{44}}{r_k} + A^{(k)}_{45}\frac{\partial^2\Phi_k}{\partial x_1^2},\\
&\Gamma^{(k)}_{p6} = A^{(k)}_{13}\frac{\partial^2\Phi_k}{\partial x_p \partial x_3}, \ \ p=1,2, \ \ \Gamma^{(k)}_{34} = A^{(k)}_{34}\frac{\partial^2\Phi_k}{\partial x_1 \partial x_3},\\
&\Gamma^{(k)}_{35} = A^{(k)}_{34}\frac{\partial^2\Phi_k}{\partial x_2 \partial x_3}, \ \ \Gamma^{(k)}_{45} = A^{(k)}_{45}\frac{\partial^2\Phi_k}{\partial x_1 \partial x_2},\\
&\Gamma^{(k)}_{46} = A^{(k)}_{46}\frac{\partial^2\Phi_k}{\partial x_1 \partial x_3}, \ \ \Gamma^{(k)}_{56} = A^{(k)}_{46}\frac{\partial^2\Phi_k}{\partial x_2 \partial x_3}, \ \ \Gamma^{(k)}_{33} = \frac{A^{(k)}_{33}}{r_k},\\
&\Gamma^{(k)}_{36} = \frac{A^{(k)}_{36}}{r_k}, \ \ \Gamma^{(k)}_{66} = \frac{A^{(k)}_{66}}{r_k}, \ \ \Gamma^{(k)}_{55} = \frac{A^{(k)}_{44}}{r_k} + A^{(k)}_{45}\frac{\partial^2\Phi_k}{\partial x_2^2},\\
&\Phi_k = (x_3-y_3)\ln(x_3-y_3+r_k) - r_k,\\
&r_k^2 = a_k\big[(x_1-y_1)^2 + (x_2-y_2)^2\big] + x_3^2.
\end{aligned}
\tag{5}
$$

The coefficients $A^{(k)}_{pq}$ have the form

$$
\begin{aligned}
A^{(k)}_{11} &= \frac{(-1)^k b_0 (c^{(2)}_{44} - c^{(2)}_{66} a_k)}{r_0(a_1-a_2)},\\
A^{(k)}_{14} &= -\frac{(-1)^k b_0 (c^{(3)}_{44} - c^{(3)}_{66} a_k)}{r_0(a_1-a_2)}, \ \ k=1,2,\\
A^{(k)}_{12} &= \frac{A^{(k)}_{11}}{a_k}, \ \ A^{(k)}_{24} = \frac{A^{(k)}_{14}}{a_k}, \ \ A^{(k)}_{45} = \frac{A^{(k)}_{44}}{a_k},\\
A^{(k)}_{44} &= \frac{(-1)^k b_0 (c^{(1)}_{44} - c^{(1)}_{66} a_k)}{r_0(a_1-a_2)}, \ \ k=1,2,\\
A^{(k)}_{12} &= \frac{\delta_k}{a_k}[-q_3 c^{(2)}_{44} + a_k t_{12} - a_k^2 t_{11} + c^{(2)}_{11} q_4 a_k^3],\\
A^{(k)}_{42} &= \frac{\delta_k}{a_k}[q_3 c^{(3)}_{44} + a_k t_{13} - a_k^2 t_{22} - c^{(3)}_{11} q_4 a_k^3],\\
A^{(k)}_{45} &= \frac{\delta_k}{a_k}[-q_3 c^{(1)}_{44} + a_k t_{23} - a_k^2 t_{33} + c^{(1)}_{11} q_4 a_k^3],\\
A^{((k))}_{33} &= \delta_k[q_4 c^{(2)}_{33} - a_k t_{42} + a_k^2 t_{44} - c^{(2)}_{44} q_1 a_k^3],\\
A^{(k)}_{36} &= \delta_k[-q_4 c^{(3)}_{33} - a_k t_{62} + a_k^2 t_{66} + c^{(3)}_{44} q_1 a_k^3],\\
A^{(k)}_{66} &= \delta_k[q_4 c^{(1)}_{33} - a_k t_{52} + a_k^2 t_{55} - c^{(1)}_{44} q_1 a_k^3],\\
A^{(k)}_{13} &= \delta_k[v_{13} - v_{11} a_k + v_{12} a_k^2],
\end{aligned}
$$

$$A_{16}^{(k)}=\delta_k[w_{13}-w_{12}a_k+v_{11}a_k^2],\ \ k=3,\dots,6,\ \ \sum_{k=1}^{6}A_{12}^{(k)}=0,$$
$$A_{34}^{(k)}=\delta_k[v_{23}-v_{21}a_k+v_{22}a_k^2],$$
$$A_{46}^{(k)}=\delta_k[w_{34}-w_{14}a_k+w_{24}a_k^2],\ \ k=3,\dots,6,\ \ \sum_{k=1}^{6}A_{13}^{(k)}=0,$$
$$\sum_{k=1}^{6}A_{45}^{(k)}=0,\ \ \sum_{k=1}^{6}A_{34}^{(k)}=0,\ \ \sum_{k=1}^{6}A_{16}^{(k)}=0,$$
$$\sum_{k=1}^{6}A_{46}^{(k)}=0,\ \ \delta_k=d_k(a_1-a_k)(a_2-a_k),$$

$a_k$, $k=1,\dots,6$ are the positive roots of the characteristic equation

$$(r_0a^2-c_0a+q_{4)}(b_0a^4-b_1a^3+b_2^3a^2-b_3a+b_4)=0.$$
$$r_0=c_{66}^{(1)}c_{66}^{(2)}-c_{66}^{(3)2},\ \ c_0=c_{66}^{(1)}c_{44}^{(2)}+c_{66}^{(2)}c_{44}^{(1)}-2c_{66}^{(3)}c_{44}^{(3)}.$$

The coefficients $d_k, b_k, v_{ij},\dots,t_{ij}$ are given in [3].

It is evident that all elements of $\Gamma(x)$ have the singularity

$$|\Gamma_{pq}^{(k)}(x)|\le\frac{\text{const}}{|x|}\,.$$

**Singular Matrix of Solutions**. Applying the operator $P(\partial x,n)$ to the matrix $\Gamma(x-y)$, we construct the so-called singular matrix of solutions. Let us consider the matrix $[P(\partial y,n)\Gamma]^*$ which is obtained from $P(\partial x,n)\Gamma(x-y)=\sum\limits_1^6(M_{pq}^{(k)})_{6x6}$ by transposition of the columns, rows and the variables $x$ and $y$. It is not difficult to prove that every column of the matrix $[P(\partial y,n)\Gamma]^*$ is a solution of the system (1) with respect to the point $x$, if $x\neq y$ and all elements $M_{pq}^{(k)}$ have singularity of type $|x|^{-2}$.

The elements $M_{pq}^{(k)}$ are given as follows:

$$\begin{aligned}
M_{11}^{(k)}&=R_{11}^{(k)}\frac{\partial_k}{\partial n}\frac{1}{r_k}+R_{12}^{(k)}\frac{\partial}{\partial s_2}\frac{\partial^2\Phi_k}{\partial x_1x_3}-q_{12}^{(k)}\frac{\partial}{\partial s_3}\frac{\partial^2\Phi_k}{\partial x_1x_2},\\
M_{12}^{(k)}&=-q_{11}^{(k)}\frac{\partial}{\partial s_3}\frac{1}{r_k}-q_{12}^{(k)}\frac{\partial}{\partial s_3}\frac{\partial^2\Phi_k}{\partial x_1^2}+R_{12}^{(k)}\frac{\partial}{\partial s_2}\frac{\partial^2\Phi_k}{\partial x_2x_3},\\
M_{13}^{(k)}&=R_{13}^{(k)}\frac{\partial}{\partial s_2}\frac{1}{r_k}-q_{13}^{(k)}\frac{\partial}{\partial s_3}\frac{\partial^2\Phi_k}{\partial x_2x_3},
\end{aligned}\tag{6}$$

$$\begin{aligned}
M_{14}^{(k)}&=R_{14}^{(k)}\frac{\partial_k}{\partial n}\frac{1}{r_k}+R_{24}^{(k)}\frac{\partial}{\partial s_2}\frac{\partial^2\Phi_k}{\partial x_1x_3}-q_{14}^{(k)}\frac{\partial}{\partial s_3}\frac{\partial^3\Phi_k}{\partial x_1x_2},\\
M_{15}^{(k)}&=-R_{15}^{(k)}\frac{\partial}{\partial s_3}\frac{1}{r_k}-q_{14}^{(k)}\frac{\partial}{\partial s_3}\frac{\partial^2\Phi_k}{\partial x_2^2}+R_{24}^{(k)}\frac{\partial}{\partial s_2}\frac{\partial^2\Phi_k}{\partial x_2x_3},
\end{aligned}$$

$$M_{16}^{(k)} = -q_{16}^{(k)} \frac{\partial}{\partial s_3} \frac{\partial^2 \Phi_k}{\partial x_2 x_3} + R_{16}^{(k)} \frac{\partial}{\partial s_2} \frac{1}{r_k},$$

$$M_{21}^{(k)} = q_{11}^{(k)} \frac{\partial}{\partial s_3} \frac{1}{r_k} + q_{12}^{(k)} \frac{\partial}{\partial s_3} \frac{\partial^2 \Phi_k}{\partial x_1^2} - R_{12}^{(k)} \frac{\partial}{\partial s_1} \frac{\partial^2 \Phi_k}{\partial x_1 x_3},$$

$$M_{22}^{(k)} = R_{11}^{(k)} \frac{\partial_k}{\partial n} \frac{1}{r_k} + R_{12}^{(k)} \frac{\partial}{\partial s_1} \frac{\partial^2 \Phi_k}{\partial x_2 x_3} + q_{12}^{(k)} \frac{\partial}{\partial s_3} \frac{\partial^2 \Phi_k}{\partial x_1 x_2},$$

$$M_{23}^{(k)} = -R_{13}^{(k)} \frac{\partial}{\partial s_3} \frac{1}{r_k} + q_{13}^{(k)} \frac{\partial}{\partial s_3} \frac{\partial^2 \Phi_k}{\partial x_1 x_3},$$

$$M_{24}^{(k)} = R_{15}^{(k)} \frac{\partial}{\partial s_3} \frac{1}{r_k} + q_{14}^{(k)} \frac{\partial}{\partial s_3} \frac{\partial^2 \Phi_k}{\partial x_1^2} - R_{24}^{(k)} \frac{\partial}{\partial s_1} \frac{\partial^2 \Phi_k}{\partial x_1 x_3},$$

$$M_{25}^{(k)} = R_{14}^{(k)} \frac{\partial_k}{\partial n} \frac{1}{r_k} - R_{24}^{(k)} \frac{\partial}{\partial s_1} \frac{\partial^2 \Phi_k}{\partial x_2 x_3} - q_{14}^{(k)} \frac{\partial}{\partial s_3} \frac{\partial^2 \Phi_k}{\partial x_1 x_2},$$

$$M_{26}^{(k)} = -R_{16}^{(k)} \frac{\partial}{\partial s_1} \frac{1}{r_k} + q_{16}^{(k)} \frac{\partial}{\partial s_3} \frac{\partial^2 \Phi_k}{\partial x_1 x_3},$$

$$M_{31}^{(k)} = -R_{31}^{(k)} \frac{\partial}{\partial s_2} \frac{1}{r_k} + R_{32}^{(k)} \frac{\partial_k}{\partial n} \frac{\partial^2 \Phi_k}{\partial x_1 x_3},$$

$$M_{32}^{(k)} = R_{31}^{(k)} \frac{\partial}{\partial s_1} \frac{1}{r_k} + R_{32}^{(k)} \frac{\partial_k}{\partial n} \frac{\partial^2 \Phi_k}{\partial x_2 x_3},$$

$$M_{33}^{(k)} = -R_{33}^{(k)} \frac{\partial}{\partial s_2} \frac{1}{r_k}, \quad M_{34}^{(k)} = -R_{34}^{(k)} \frac{\partial}{\partial s_2} \frac{1}{r_k} + R_{35}^{(k)} \frac{\partial_k}{\partial n} \frac{\partial^2 \Phi_k}{\partial x_1 x_3},$$

$$M_{35}^{(k)} = R_{34}^{(k)} \frac{\partial}{\partial s_1} \frac{1}{r_k} + R_{35}^{(k)} \frac{\partial_k}{\partial n} \frac{\partial^2 \Phi_k}{\partial x_2 x_3}, \quad M_{36}^{(k)} = -R_{36}^{(k)} \frac{\partial_k}{\partial n} \frac{1}{r_k},$$

$$M_{41}^{(k)} = -R_{41}^{(k)} \frac{\partial_k}{\partial n} \frac{1}{r_k} + R_{42}^{(k)} \frac{\partial}{\partial s_2} \frac{\partial^2 \Phi_k}{\partial x_1 x_3} - q_{41}^{(k)} \frac{\partial}{\partial s_3} \frac{\partial^2 \Phi_k}{\partial x_1 x_2},$$

$$M_{42}^{(k)} = -\mu_{42}^{(k)} \frac{\partial}{\partial s_3} \frac{1}{r_k} + q_{41}^{(k)} \frac{\partial}{\partial s_3} \frac{\partial^2 \Phi_k}{\partial x_2^2} + R_{42}^{(k)} \frac{\partial}{\partial s_2} \frac{\partial^2 \Phi_k}{\partial x_2 x_3},$$

$$M_{43}^{(k)} = R_{43}^{(k)} \frac{\partial}{\partial s_2} \frac{1}{r_k} - q_{43}^{(k)} \frac{\partial}{\partial s_3} \frac{\partial^2 \Phi_k}{\partial x_2 x_3},$$

$$M_{44}^{(k)} = \mu_{44}^{(k)} \frac{\partial_k}{\partial n} \frac{1}{r_k} + R_{45}^{(k)} \frac{\partial}{\partial s_2} \frac{\partial^2 \Phi_k}{\partial x_1 x_3} - q_{44}^{(k)} \frac{\partial}{\partial s_3} \frac{\partial^2 \Phi_k}{\partial x_1 x_2},$$

$$M_{45}^{(k)} = -\mu_{45}^{(k)} \frac{\partial}{\partial s_3} \frac{1}{r_k} - q_{44}^{(k)} \frac{\partial}{\partial s_3} \frac{\partial^2 \Phi_k}{\partial x_2^2} + R_{45}^{(k)} \frac{\partial}{\partial s_2} \frac{\partial^2 \Phi_k}{\partial x_2 x_3},$$

$$M_{46}^{(k)} = R_{46}^{(k)} \frac{\partial}{\partial s_2} \frac{1}{r_k} - q_{46}^{(k)} \frac{\partial}{\partial s_3} \frac{\partial^2 \Phi_k}{\partial x_2 x_3},$$

$$M_{51}^{(k)} = \mu_{42}^{(k)} \frac{\partial}{\partial s_3} \frac{1}{r_k} + q_{41}^{(k)} \frac{\partial}{\partial s_3} \frac{\partial^2 \Phi_k}{\partial x_1^2} - R_{42}^{(k)} \frac{\partial}{\partial s_1} \frac{\partial^2 \Phi_k}{\partial x_1 x_3},$$

$$M_{52}^{(k)} = R_{41}^{(k)} \frac{\partial_k}{\partial n} \frac{1}{r_k} - R_{42}^{(k)} \frac{\partial}{\partial s_1} \frac{\partial^2 \Phi_k}{\partial x_2 x_3} + q_{41}^{(k)} \frac{\partial}{\partial s_3} \frac{\partial^2 \Phi_k}{\partial x_1 x_2},$$

$$
\begin{aligned}
&M_{53}^{(k)}=-R_{43}^{(k)}\frac{\partial}{\partial s_1}\frac{1}{r_k}+q_{43}^{(k)}\frac{\partial}{\partial s_3}\frac{\partial^2\Phi_k}{\partial x_1x_3},\\
&M_{54}^{(k)}=\mu_{45}^{(k)}\frac{\partial}{\partial s_3}\frac{1}{r_k}+q_{44}^{(k)}\frac{\partial}{\partial s_3}\frac{\partial^2\Phi_k}{\partial x_1^2}-R_{45}^{(k)}\frac{\partial}{\partial s_1}\frac{\partial^2\Phi_k}{\partial x_1x_3},\\
&M_{55}^{(k)}=\mu_{44}^{(k)}\frac{\partial_k}{\partial n}\frac{1}{r_k}-R_{45}^{(k)}\frac{\partial}{\partial s_1}\frac{\partial^2\Phi_k}{\partial x_2x_3}+q_{44}^{(k)}\frac{\partial}{\partial s_3}\frac{\partial^2\Phi_k}{\partial x_1x_2},\\
&M_{56}^{(k)}=-R_{46}^{(k)}\frac{\partial}{\partial s_1}\frac{1}{r_k}+q_{46}^{(k)}\frac{\partial}{\partial s_3}\frac{\partial^2\Phi_k}{\partial x_1x_3},\\
&M_{61}^{(k)}=-R_{61}^{(k)}\frac{\partial}{\partial s_2}\frac{1}{r_k}+R_{62}^{(k)}\frac{\partial_k}{\partial n}\frac{\partial^2\Phi_k}{\partial x_1x_3},\\
&M_{62}^{(k)}=R_{61}^{(k)}\frac{\partial}{\partial s_1}\frac{1}{r_k}+R_{62}^{(k)}\frac{\partial_k}{\partial n}\frac{\partial^2\Phi_k}{\partial x_2x_3},\ \ M_{63}^{(k)}=R_{63}^{(k)}\frac{\partial_k}{\partial n}\frac{1}{r_k},\\
&M_{64}^{(k)}=-R_{64}^{(k)}\frac{\partial}{\partial s_2}\frac{1}{r_k}+R_{65}^{(k)}\frac{\partial_k}{\partial n}\frac{\partial^2\Phi_k}{\partial x_1x_3},\\
&M_{65}^{(k)}=R_{64}^{(k)}\frac{\partial}{\partial s_1}\frac{1}{r_k}+R_{65}^{(k)}\frac{\partial_k}{\partial n}\frac{\partial^2\Phi_k}{\partial x_2x_3},\ \ M_{66}^{(k)}=R_{66}^{(k)}\frac{\partial_k}{\partial n}\frac{1}{r_k},
\end{aligned}
$$

where

$$
\begin{aligned}
&\frac{\partial_k}{\partial n}=n_1\frac{\partial}{\partial x_1}+n_2\frac{\partial}{\partial x_2}+n_3a_3\frac{\partial}{\partial x_3},\ \frac{\partial}{\partial s_1}=n_2\frac{\partial}{\partial y_3}-n_3\frac{\partial}{\partial y_2},\\
&\frac{\partial}{\partial s_2}=n_3\frac{\partial}{\partial y_1}-n_1\frac{\partial}{\partial y_3},\ \frac{\partial}{\partial s_3}=n_1\frac{\partial}{\partial y_2}-n_2\frac{\partial}{\partial y_1},
\end{aligned}
$$

The coefficients $R_{pq}^{(k)}\dots$ are constant and they are expressed by the Hook's constants

$$
\begin{aligned}
&R_{11}^{(k)}=c_{66}^{(1)}A_{11}^{(k)}+c_{66}^{(3)}A_{41}^{(k)},\ R_{12}^{(k)}=\delta_0^{(1)}A_{13}^{(k)}+\delta_0^{(3)}A_{61}^{(k)}+c_{44}^{(1)}A_{12}^{(k)}+c_{44}^{(3)}A_{42}^{(k)},\\
&q_{12}^{(k)}=(\gamma_0^{(1)}+c_{66}^{(1)})A_{12}^{(k)}+(\gamma_0^{(3)}+c_{66}^{(3)})A_{42}^{(k)},\ \ q_{11}^{(k)}=\gamma_0^{(1)}A_{11}^{(k)}+\gamma_0^{(3)}A_{41}^{(k)},\\
&R_{13}^{(k)}=\delta_0^{(1)}A_{33}^{(k)}+\delta_0^{(3)}A_{63}^{(k)}+c_{44}^{(1)}A_{13}^{(k)}+c_{44}^{(3)}A_{43}^{(k)},\\
&q_{13}^{(k)}=(\gamma_0^{(1)}+c_{66}^{(1)})A_{13}^{(k)}+(\gamma_0^{(3)}+c_{66}^{(3)})A_{43}^{(k)},\ \ R_{14}^{(k)}=c_{66}^{(1)}A_{41}^{(k)}+c_{66}^{(3)}A_{44}^{(k)},\\
&R_{24}^{(k)}=\delta_0^{(1)}A_{43}^{(k)}+\delta_0^{(3)}A_{46}^{(k)}+c_{44}^{(1)}A_{42}^{(k)}+c_{44}^{(3)}A_{45}^{(k)},\\
&q_{14}^{(k)}=(\gamma_0^{(1)}+c_{66}^{(1)})A_{42}^{(k)}+(\gamma_0^{(3)}+c_{66}^{(3)})A_{45}^{(k)},\ \ R_{15}^{(k)}=\gamma_0^{(1)}A_{41}^{(k)}+\gamma_0^{(3)}A_{44}^{(k)},\\
&q_{16}^{(k)}=(\gamma_0^{(1)}+c_{66}^{(1)})A_{61}^{(k)}+(\gamma_0^{(3)}+c_{66}^{(3)})A_{46}^{(k)},\\
&R_{16}^{(k)}=\delta_0^{(1)}A_{63}^{(k)}+\delta_0^{(3)}A_{66}^{(k)}+c_{44}^{(1)}A_{61}^{(k)}+c_{44}^{(3)}A_{46}^{(k)},\\
&R_{31}^{(k)}=\delta_0^{(1)}A_{11}^{(k)}+\delta_0^{(3)}A_{41}^{(k)},\ R_{32}^{(k)}=\delta_0^{(1)}A_{12}^{(k)}+\delta_0^{(3)}A_{42}^{(k)}+c_{44}^{(1)}A_{13}^{(k)}+c_{44}^{(3)}A_{61}^{(k)},\\
&R_{33}^{(k)}=\delta_0^{(1)}A_{13}^{(k)}+\delta_0^{(3)}A_{43}^{(k)}+c_{44}^{(1)}A_{33}^{(k)}+c_{44}^{(3)}A_{63}^{(k)},\ R_{34}^{(k)}=\delta_0^{(1)}A_{41}^{(k)}+\delta_0^{(3)}A_{44}^{(k)},\\
&R_{35}^{(k)}=\delta_0^{(1)}A_{42}^{(k)}+\delta_0^{(3)}A_{45}^{(k)}+c_{44}^{(1)}A_{43}^{(k)}+c_{44}^{(3)}A_{46}^{(k)},
\end{aligned}
\tag{7}
$$

$R_{36}^{(k)} = \delta_0^{(1)} A_{61}^{(k)} + \delta_0^{(3)} A_{46}^{(k)} + c_{44}^{(1)} A_{63}^{(k)} + c_{44}^{(3)} A_{66}^{(k)},$
$R_{41}^{(k)} = c_{66}^{(3)} A_{11}^{(k)} + c_{66}^{(2)} A_{41}^{(k)}, \ \ R_{42}^{(k)} = \delta_0^{(3)} A_{13}^{(k)} + \delta_0^{(2)} A_{61}^{(k)} + c_{44}^{(3)} A_{12}^{(k)} + c_{44}^{(2)} A_{42}^{(k)},$
$q_{41}^{(k)} = (\gamma_0^{(3)} + c_{66}^{(3)}) A_{12}^{(k)} + (\gamma_0^{(2)} + c_{66}^{(2)}) A_{42}^{(k)},$
$q_{44}^{(k)} = (\gamma_0^{(3)} + c_{66}^{(3)}) A_{42}^{(k)} + (\gamma_0^{(2)} + c_{66}^{(2)}) A_{45}^{(k)}, \ \ \mu_{42}^{(k)} = \gamma_0^{(3)} A_{11}^{(k)} + \gamma_0^{(2)} A_{41}^{(k)},$
$q_{43}^{(k)} = (\gamma_0^{(3)} + c_{66}^{(3)}) A_{13}^{(k)} + (\gamma_0^{(2)} + c_{66}^{(2)}) A_{43}^{(k)},$
$R_{43}^{(k)} = \delta_0^{(3)} A_{33}^{(k)} + \delta_0^{(2)} A_{63}^{(k)} + c_{44}^{(3)} A_{13}^{(k)} + c_{44}^{(2)} A_{43}^{(k)}, \ \ \mu_{44}^{(k)} = c_{66}^{(3)} A_{41}^{(k)} + c_{66}^{(2)} A_{44}^{(k)},$
$R_{45}^{(k)} = \delta_0^{(3)} A_{43}^{(k)} + \delta_0^{(2)} A_{46}^{(k)} + c_{44}^{(3)} A_{42}^{(k)} + c_{44}^{(2)} A_{45}^{(k)},$
$\mu_{45}^{(k)} = \gamma_0^{(3)} A_{41}^{(k)} + \gamma_0^{(2)} A_{44}^{(k)}, \ \ R_{46}^{(k)} = \delta_0^{(3)} A_{63}^{(k)} + \delta_0^{(2)} A_{66}^{(k)} + c_{44}^{(3)} A_{61}^{(k)} + c_{44}^{(2)} A_{46}^{(k)},$
$q_{46}^{(k)} = (\gamma_0^{(3)} + c_{66}^{(3)}) A_{61}^{(k)} + (\gamma_0^{(2)} + c_{66}^{(2)}) A_{46}^{(k)},$
$R_{62}^{(k)} = \delta_0^{(3)} A_{12}^{(k)} + \delta_0^{(2)} A_{42}^{(k)} + c_{44}^{(3)} A_{13}^{(k)} + c_{44}^{(2)} A_{16}^{(k)},$
$R_{61}^{(k)} = \delta_0^{(3)} A_{11}^{(k)} + \delta_0^{(2)} A_{41}^{(k)}, \ \ R_{63}^{(k)} = \delta_0^{(3)} A_{13}^{(k)} + \delta_0^{(2)} A_{43}^{(k)} + c_{44}^{(3)} A_{33}^{(k)} + c_{44}^{(2)} A_{63}^{(k)},$
$R_{64}^{(k)} = \delta_0^{(3)} A_{41}^{(k)} + \delta_0^{(2)} A_{44}^{(k)}, \ R_{65}^{(k)} = \delta_0^{(3)} A_{42}^{(k)} + \delta_0^{(2)} A_{45}^{(k)} + c_{44}^{(3)} A_{43}^{(k)} + c_{44}^{(2)} A_{46}^{(k)},$
$R_{66}^{(k)} = \delta_0^{(3)} A_{61}^{(k)} + \delta_0^{(2)} A_{46}^{(k)} + c_{44}^{(3)} A_{63}^{(k)} + c_{44}^{(2)} A_{66}^{(k)},$

We introduce the following definitions.

**Definition 2.** The vector defined by the equality

$$v(x,h) = \frac{1}{2\pi} \iint\limits_S \Gamma(x-y) h(y) ds,$$

is called a simple-layer potential, where $h$ is a real 6-dimensional vector.

**Definition 3.** The vector defined by the equality

$$w(x,g) = \frac{1}{2\pi} \iint\limits_S [P(\partial x, n)\Gamma(x-y)]^* g(y) ds,$$

is called a double-layer potential, where $g$ is a real 6-dimensional vector.

These potentials are the solutions of the system (1) both in the domain $D^+$ and in the domain $D^-$, $v, w \in C^\infty(D^\pm)$.

The following theorems are valid and we cite them without proof.

**Theorem 3.** *If $S$ is a closed Lyapunov surface, i.e., $S \in L_1(\alpha)$ and $g \in C^{1,\beta}(S)$, $0 < \beta < \alpha \le 1$, then $w(x,g)$ is regular in $D^\pm$; there exist boundary values of the vector $w$ inside and outside of the surface $S$, and there take place the equality*

$$w^\pm(z,g) = \mp g(z) + \frac{1}{2\pi} \iint\limits_S [P(\partial x, n)\Gamma(x-y)]^* g(y) ds. \qquad (8)$$

**Theorem 4.** *If $S$ is a closed surface of the class $L_1(\alpha)$, and $h \in C^{0,\beta}(S)$, $0 < \beta < \alpha < 1$, then* (a) *the vector $v$ is regular in $D^\pm$;* (b) *there exist boundary values of the vector $P(\partial x, n)$ inside and outside of the surface $S$, and there take place the equality*

$$\big[P(\partial z, n)v(z)\big]^\pm = \pm h(z) + \frac{1}{2\pi}\iint_S P(\partial x, n)\Gamma(x-y)h(y)ds. \tag{9}$$

**Integral Equations of BVPs**. A solution of the BVP $(1)^\pm$ is sought in the form of a double-layer potential, while the solution of the BVP $(2)^\pm$ is sought in the form of a simple-layer potential. To determine the unknown real vector functions $g$ and $h$ from (8) and (9), we obtain the following singular integral second kind equations:

$$\mp g(z) + \frac{1}{2\pi}\iint_S \big[T(\partial y, n)\Gamma(y-z)\big]^* g(y)ds = f(z), \tag{10}$$

$$\pm h(z) + \frac{1}{2\pi}\iint_S T(\partial y, n)\Gamma(z-y)h(y)ds = F(z). \tag{11}$$

The integrals appearing in (10) and (11) are understood in a sense of the principal value [4].

## 2. Calculation of a Symbolic Matrix

Consider the operator

$$K^{1+} = -g(z) + \frac{1}{2\pi}\iint_S [T(\partial y, n)\Gamma(y-x)]^* g(y)ds. \tag{12}$$

We follow the results obtained in [1]. According to [1] (for details, see [1]), we have

$$\frac{\partial}{\partial s_j}\sum_{k=1}^{6} R_{pq}^{(k)}\frac{\partial^2\Phi_k}{\partial x_q\partial x_p} \approx -2\pi\iota\xi_j\left(\sum_{k=1}^{6} R_{pq}^{(k)}\frac{\partial^2\Phi_k}{\partial x_q\partial x_p}\right)_{x-y=\xi}.$$

Now we can construct a symbolic matrix for the operator $K^{1+}$. We have

$$\sigma^{1+} = \begin{pmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{12} & \sigma_{44} \end{pmatrix}, \tag{13}$$

where

$$\sigma_{kk} = \begin{pmatrix} l(1+\xi_1\xi_2\xi_3)A_k & (B_k-\xi_1^2A_k)i\xi_3 & i\xi_2C_k \\ (-B_k+\xi_2^2A_k)i\xi_3 & (1-\xi_1\xi_2\xi_3)A_k & -i\xi_1C_k \\ i\xi_2D_k & -i\xi_1D_k & 0 \end{pmatrix}, \quad k=1, \quad k=4,$$

$$\sigma_{1k} = \begin{pmatrix} \xi_1\xi_2\xi_3A_k & (B_k-\xi_1^2A_k)i\xi_3 & i\xi_2C_k \\ (-B_k+\xi_2^2A_k)i\xi_3 & -\xi_1\xi_2\xi_3A_k & -i\xi_1C_k \\ i\xi_2D_k & -i\xi_1D_k & 0 \end{pmatrix}, \quad k=3, \quad k=2,$$

The coefficients $A_k, B_k, C_k, D_k$ are written in the form

$$A_1 = \sum_1^5 m_k(R_{12}^{(k)} - q_{12}^{(k)} n_k), \quad A_2 = \sum_1^5 m_k(R_{24}^{(k)} - q_{14}^{(k)} n_k),$$
$$A_3 = \sum_1^5 m_k(R_{42}^{(k)} - q_{41}^{(k)} n_k), \quad A_4 = \sum_1^5 m_k(R_{45}^{(k)} - q_{44}^{(k)} n_k),$$
$$B_1 = \sum_1^2 \frac{q_{11}^{(k)}}{\rho_k} - \sum_1^5 q_{12}^{(k)} m_k \rho_k \rho_6, \quad B_2 = \sum_1^2 \frac{R_{15}^{(k)}}{\rho_k} - \sum_1^5 q_{14}^{(k)} m_k \rho_k \rho_6,$$
$$B_3 = \sum_1^2 \frac{\mu_{42}^{k}}{\rho_k} - \sum_1^5 q_{41}^{(k)} m_k \rho_k \rho_6, \quad B_4 = \sum_1^2 \frac{\mu_{45}^{(k)}}{\rho_k} - \sum_1^5 q_{44}^{(k)} m_k \rho_k \rho_6,$$
$$C_1 = -\sum_1^2 \frac{R_{31}^{(k)}}{\rho_k} + \sum_1^5 R_{31}^{(k)} m_k \rho_k \rho_6, \quad C_2 = -\sum_1^2 \frac{R_{34}^{(k)}}{\rho_k} + \sum_1^5 R_{35}^{(k)} m_k \rho_k \rho_6,$$
$$C_3 = -\sum_1^2 \frac{R_{61}^{(k)}}{\rho_k} + \sum_1^5 R_{62}^{(k)} m_k \rho_k \rho_6, \quad C_4 = -\sum_1^2 \frac{R_{64}^{(k)}}{\rho_k} + \sum_1^5 R_{65}^{(k)} m_k \rho_k \rho_6,$$
$$D_1 = \sum_1^6 \frac{R_{13}^{(k)}}{\rho_k} - \xi_3^2 \sum_3^5 q_{13}^{(k)} m_k, \quad D_2 = \sum_1^6 \frac{R_{16}^{(k)}}{\rho_k} - \xi_3^2 \sum_3^5 q_{16}^{(k)} m_k,$$
$$D_3 = -\sum_1^6 \frac{R_{43}^{(k)}}{\rho_k} - \xi_3^2 \sum_3^5 q_{43}^{(k)} m_k, \quad D_4 = -\sum_1^6 \frac{R_{46}^{(k)}}{\rho_k} - \xi_3^2 \sum_3^5 q_{46}^{(k)} m_k,$$
$$m_k = \frac{a_k - a_6}{\rho_k \rho_6(\rho_k + \rho_6)}, \quad n_k = \frac{a_6 \rho_k + a_k \rho_6}{\rho_6 + \rho_k}, \quad \rho_k^2 = a_k(\xi_1^2 + \xi_2^2) + \xi_3^2.$$

Since $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$, it is possible to take $\sqrt{\xi_1^2 + \xi_2^2} = \cos\theta, \xi_3 = \sin\theta$. From (13) we have

$$\begin{aligned}
\det \sigma^{1+} = {} & (1 + C_4 D_4 \cos^2\theta - k_4 B_4 \sin^2\theta)(1 + C_1 D_1 \cos^2\theta - k_1 B_1 \sin^2\theta) + \\
& + (D_2 C_3 + D_3 C_2)\cos^2\theta + (\sin^2\theta B_4 k_2 - \cos^2\theta D_2 C_4)(\cos^2\theta C_1 D_3 - \\
& - \sin^2\theta B_1 k_3) + (\sin^2\theta B_2 k_1 + \cos^2\theta D_1 C_2)(-C_3 D_4 \cos^2\theta + B_3 k_4 \sin^2\theta) + \\
& + \cos^2\theta \sin^2\theta[(k_2 D_4 - k_4 D_2)(C_3 B_4 - C_4 B_3) + B_3 A_2 - B_2 A_3 + \\
& + (D_1 k_3 - D_3 k_1)(C_2 B_1 + C_1 B_2)] + \\
& + (\sin^2\theta B_2 k_3 + \cos^2\theta D_3 C_2)(\cos^2\theta C_3 D_2 - \sin^2\theta B_3 k_2), \qquad (14)
\end{aligned}$$
$$k_j = B_j - A_j \cos^2\theta.$$

Let us now prove that if $\forall\theta \in [0, 2\pi]$ then $\det \sigma'^{+}(z, \theta) \neq 0$. For this reason, we derive some auxiliary formulas.

A general representation of a regular solution of equation (1) in the half-plane $x_1>0$ has the form

$$u(x)=\frac{1}{2\pi}\iint\limits_S\big[(T(\partial y,n)\Gamma(y-x))^*u^+-\Gamma(y-x)(Tu)^+\big]dy_2dy_3, \tag{15}$$

where $(u)^+$ and $(Tu)^+$ are the boundary values of the displacement and stress vectors on the plane $x_1=0$, $\Gamma(y-x)$ and $(T\Gamma)^*$ are given by formulas (5) and (6), for $n(1,0,0)$, $\beta_0^{(j)}=c_{13}^{(j)}$, $\gamma_0^{(j)}=c_{66}^{(j)}$, $\alpha_0^{(j)}=c_{11}^{(j)}-2c_{66}^{(j)}$, $\delta_0^{(j)}=c_{44}^{(j)}$.

Formula (15) is obtained under the restriction that $u^+$ and $(Tu)^+$ are the continuous functions and satisfy at infinity the conditions $u(x)=O(|x|^{-\alpha})$, $Tu=O(|x|^{-1-\alpha})$.

Taking into account the identities [1]

$$\begin{aligned}
&\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\frac{1}{r_k}\exp\Big(i\sum_2^3 p_jy_j\Big)dy_2dy_3=\frac{\exp(-x_1R_k)}{R_k}\exp\Big(i\sum_2^3 p_jx_j\Big),\\
&\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\int\Phi_k\exp\Big(i\sum_2^3 p_jy_j\Big)dy_2dy_3=-\frac{\exp(-x_1R_k)}{p_3^2R_k}\exp\Big(i\sum_2^3 p_jx_j\Big),
\end{aligned}\tag{16}$$

where $p_k$, $k=1,2,3$ are independent variables,

$$r_k=\sqrt{a_k[x_1^2+(x_2-y_2)^2]+(x_3-y_3)^2},$$

$$\Phi_k=(x_3-y_3)\ln(x_3-y_3+r_k)-r_k,\quad R_k=\sqrt{p_2^2+a_kp_3^2},$$

After performing the Fourier transforms and substituting $x_1=0$ from (15), we obtain

$$\rho\hat{A}\hat{(u)}^+ +\hat{B}\hat{(Tu)}^+=0, \tag{17}$$

where $\hat{u}^+$, $\hat{(Tu)}^+$, $A$ and $B$ are the Fourier transforms of $u, Tu, (P\Gamma)^*$ and $\Gamma$,

$$B=\sum_{k=1}^6\begin{pmatrix} B_{11}^{1(k)} & B_{12}^{3(k)}\\ B_{12}^{3(k)} & B_{22}^{2(k)}\end{pmatrix},$$

$$B_{ij}^{l(k)}=\begin{pmatrix}\alpha_{11}^{l(k)} & 0 & 0\\ 0 & \alpha_{22}^{l(k)} & \alpha_{23}^{lk}\\ 0 & \alpha_{23}^{l(k)} & \alpha_{33}^{l(k)}\end{pmatrix},\quad l=1,2,3;\ \ i,j=1,2,3;\ \ i=1,2,3,$$

$$\alpha_{11}^{1(k)}=\frac{A_{11}^{(k)}}{R_k}-A_{12}^{(k)}\frac{R_{(k)}}{p_3^2},\ \ \alpha_{22}^{1(k)}=\frac{A_{11}^{(k)}}{R_{(k)}}+A_{12}^{(k)}\frac{p_2^2}{p_3^2R_k},\ \ \alpha_{23}^{1(k)}=A_{13}^{(k)}\frac{p_2}{p_3R_k},$$

$$\alpha_{33}^{1(k)}=\frac{A_{33}^{(k)}}{R_k},\ \ \alpha_{11}^{3(k)}=\frac{A_{41}^{(k)}}{R_k}-A_{42}^{(k)}\frac{R_k}{p_3^2},\ \ \alpha_{22}^{3(k)}=\frac{A_{41}^{(k)}}{R_k}-A_{42}^{(k)}\frac{p_2^2}{p_3^2R_k},$$

$$\alpha_{33}^{3(k)} = \frac{A_{36}^{(k)}}{R_k}, \;\; \alpha_{23}^{3(k)} = \frac{A_{16}^{(k)p_2}}{p_3 R_k}, \;\; \alpha_{11}^{2(k)} = \frac{A_{44}^{(k)}}{R_k} - A_{45}^{(k)} \frac{R_k}{p_3^2}, \;\; \alpha_{23}^{2(k)} = \frac{A_{46}^{(k)} p_2}{p_3 R_k},$$
$$\alpha_{22}^{2(k)} = \frac{A_{44}^{(k)}}{R_k} + A_{45}^{(k)} \frac{p_2^2}{p_3^2 R_k}, \;\; \alpha_{33}^{2(k)} = \frac{A_{66}^{(k)}}{R_k}, \;\; \rho^2 = p_2^2 + p_3^2.$$

After certain simplifications, we obtain $\det A(\theta + \frac{\pi}{2}) = \det \sigma^{1+}$.

According to the uniqueness theorem, we get $\det B \neq 0$, and from (17) we obtain

$$(T\bar{u})^+ = -B^{-1} A(\bar{u})^+ \rho. \tag{18}$$

We rewrite (4) in a bilinear form with respect to the vectors $u$ and $\bar{u}$ ($\bar{u}$ is the complex conjugate of the vector $u$) and assume that

$$\begin{aligned} &u(x) = M(x_1, p_2, p_3) \exp\left( -i \sum_2^3 p_j x_j \right), \\ &M = (M_1, M_2, M_3, M_4, M_5, M_6). \end{aligned} \tag{19}$$

After calculation we obtain

$$\begin{aligned}
E^{(1)}(\bar{u}, u) = &\left(c_{11}^{(1)} - 2c_{66}^{(1)}\right) ip_2 \left( \bar{M}_2 \frac{dM_1}{dx_1} - M_2 \frac{d\bar{M}_1}{dx_1} \right) + \\
&+ c_{66}^{(1)} \left[ ip_2 \left( \bar{M}_1 \frac{dM_2}{dx_1} - M_1 \frac{d\bar{M}_2}{dx_1} \right) + p_2^2 |M_1|^2 + \left| \frac{dM_2}{dx_1} \right|^2 \right] + \\
&+ c_{11}^{(1)} \left( p_2^2 |M_2|^2 + \left| \frac{dM_1}{dx_1} \right|^2 \right) + c_{44}^{(1)} \Bigg[ p_3^2 |M_1|^2 + p_3^2 |M_2|^2 + \\
&+ p_2^2 |M_3|^2 + \left| \frac{dM_3}{dx_1} \right|^2 + ip_3 \left( \bar{M}_1 \frac{dM_3}{dx_1} - M_1 \frac{d\bar{M}_3}{dx_1} \right) \Bigg] + \\
&+ p_2 p_3 \left( \bar{M}_2 M_3 + \bar{M}_3 M_2 \right) (c_{13}^{(k)} + c_{44}^{(k)}) + \\
&+ c_{13}^{(1)} ip_3 \left( \bar{M}_3 \frac{dM_1}{dx_1} - M_3 \frac{d\bar{M}_1}{dx_1} \right) + c_{33}^{(1)} p_3^2 |M_3|^2, \\
E^{(2)}(\bar{u}, u) = &\left(c_{11}^{(2)} - 2c_{66}^{(2)}\right) ip_2 \left( \bar{M}_4 \frac{dM_5}{dx_1} - M_5 \frac{d\bar{M}_4}{dx_1} \right) + \\
&+ c_{66}^{(2)} \left[ ip_2 \left( \bar{M}_4 \frac{dM_5}{dx_1} - M_4 \frac{d\bar{M}_5}{dx_1} \right) + p_2^2 |M_4|^2 + \left| \frac{dM_5}{dx_1} \right|^2 \right] + \\
&+ c_{11}^{(2)} \left( p_2^2 |M_5|^2 + \left| \frac{dM_4}{dx_1} \right|^2 \right) + c_{44}^{(2)} \big[ p_3^2 |M_5|^2 + p_3^2 |M_4|^2 + \\
&+ p_2^2 |M_6|^2 + \left| \frac{dM_6}{dx_1} \right|^2 + ip_3 \left( \bar{M}_4 \frac{dM_6}{dx_1} - M_4 \frac{d\bar{M}_6}{dx_1} \right) +
\end{aligned}$$

$$+p_2p_3\left(\bar{M}_5M_6+\bar{M}_6M_5\right)]+c_{13}^{(2)}\left[ip_3\left(\bar{M}_6\frac{dM_4}{dx_1}-M_6\frac{d\bar{M}_4}{dx_1}\right)+\right.$$

$$\left.+\,p_2p_3\left(\bar{M}_5M_6+\bar{M}_6M_5\right)\right]+c_{33}^{(2)}p_3^2|M_6|^2,$$

$$E^{(3)}=c_{33}^{(3)}p_3^2\left(\bar{M}_3M_6+\bar{M}_6M_3\right)+c_{11}^{(3)}\left(\frac{d\bar{M}_1}{dx_1}\frac{dM_6}{dx_1}+\frac{d\bar{M}_6}{dx_1}\frac{dM_1}{dx_1}\right)+$$

$$+\,c_{44}^{(3)}\left[\left(-ip_3\bar{M}_4+\frac{d\bar{M}_6}{dx_1}\right)\left(ip_3\bar{M}_1+\frac{d\bar{M}_3}{dx_1}\right)+\right.$$

$$\left.+\left(-ip_3\bar{M}_4+\frac{d\bar{M}_6}{dx_1}\right)\left(-ip_3\bar{M}_1+\frac{d\bar{M}_3}{dx_1}\right)\right]+$$

$$+\,ip_3c_{13}^{(3)}\left[\bar{M}_6\left(\frac{dM_1}{dx_1}-ip_2M_2\right)-M_6\left(\frac{\bar{M}_1}{dx_1}+ip_2\bar{M}_2\right)+\right.$$

$$\left.+\,\bar{M}_3\left(\frac{dM_6}{dx_1}-ip_2M_5\right)-M_3\left(\frac{\bar{M}_6}{dx_1}+ip_2\bar{M}_5\right)\right]+$$

$$+\,C_{66}^{(3)}\left[\left(ip_2\bar{M}_4+\frac{d\bar{M}_5}{dx_1}\right)\left(-ip_2M_1+\frac{dM_2}{dx_1}\right)+\right.$$

$$\left.+\left(-ip_2M_4+\frac{dM_5}{dx_1}\right)\left(ip_2\bar{M}_1+\frac{\bar{M}_2}{dx_1}\right)\right]+ip_2(c_{11}^{(3)}-$$

$$-\,2c_{66}^{(3)})\left[M_5\frac{d\bar{M}_1}{dx_1}-\bar{M}_5\frac{dM_1}{dx_1}+\bar{M}_5\frac{dM_6}{dx_1}-M_5\frac{d\bar{M}_6}{dx_1}\right].$$

Let

$$u(x)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}M(x_1,p_2,p_3)\exp(-i\sum_{2}^{3}p_jx_jdp_2dp_3.$$

The vector $u(x)$ to be a solution of system (1), it is necessary and sufficient that

$$C(\partial x,-ip_2,-ip_3)M=0,x_1>0,\left[\frac{d^jM_k}{dx_1^j}\right]_{x_1=\infty}=0,\quad j=0,1,\quad k=\bar{1,6}.$$

Consider the expression $\int\limits_0^h\sum\limits_{k=1}^{6}\bar{M}_k[C(\partial x,-ip_2,-ip_3)M]_kdx_1$, where $h$ is an arbitrary positive number. After transformation we obtain

$$\int\limits_0^{\infty}E(u,\bar{u})dx_1=-(\hat{u})^+(T\hat{u})^+,\tag{20}$$

Substituting (18) into (20), we get

$$\int_0^\infty E(u,\bar{u})dx_1 = \rho\hat{(u)}^+ B^{-1}A\hat{(u)}^+, \tag{21}$$

Taking into account the fact that the energy $E(u,\bar{u})$ is positive definite, from (21) we can conclude that the matrix $B^{-1}A$ is also positive definite. In particular, from (21) we have $\det B^{-1}A \neq 0$, that gives $\det A \neq 0$, hence $\det \sigma^{1+} \neq 0$. Analogously, we obtain $\det \sigma^{1-} \neq 0$. $\det \sigma^{2\pm} \neq 0$.

Finally, we have singular normal type integral equations of second kind for which Fredholm theorems are valid.

**Problems** $(1)^+$**and** $(2)^-$. To define the vectors $g$ and $h$, from (8) and (9), using the boundary conditions, we obtain the following Fredholm integral equation of the second order

$$-g(z) + \frac{1}{2\pi}\iint_S [T(\partial y,n)\Gamma(z-x)]^* g(y)ds = f(z), \tag{22}$$

$$h(z) + \frac{1}{2\pi}\iint_S T(\partial y,n)\Gamma(z-x)h(y)ds = F(z), \tag{23}$$

where $f(z) \in C^{1,\beta}(S)$, $F(z) \in C^{0,\beta}(S)$ are the given vectors on the boundary.

Let us prove that the homogeneous equation corresponding to (23) has only a trivial solution. Assume that it has a nontrivial solution denoted by $h_0(z)$. $h_0(z) \in C^{0,\beta}(S)$. Consider the potential

$$v(x) = \frac{1}{2\pi}\iint_S \Gamma(z-x)h_0(y)ds.$$

Obviously, $C(\partial x)v(x) = 0$, $x \in D^+ \cup D^-$, $[T(\partial x,n)v]^- = 0$, $v(x) \in C^{0,\beta}(\bar{D^-})$, $v(x) = O(|x|^{-1})$, $\dfrac{\partial v}{\partial x_k} \in O(|x|^{-2})$, $k = 1,2,3$. This implies that $v(x,h_0) = 0$, $x \in D^-$, whence $[v]^- = [v]^+ = 0$, and $v(x) = 0$, $x \in D^+$. Thus $v(x)$ vanishes on the whole space and therefore $h_0 = 0$. Due to the Fredholm theorem we conclude that the nonhomogeneous equation is solvable for an arbitrary Hölder continuous vector $F(x)$.

Finally, from the solvability of equations (22) and (23) it follows that the solutions of the problems $(1)^+$ and $(2)^-$ are representable in the form of the second kind double and single-layer potentials, respectively. On the basis of the general theory, the following theorem is valid.

**Theorem 5.** *If* $S \in L_2(\alpha)$ *and* $f \in C^{1,\beta}(S)$, *then the BVP* $(1)^+$ *has a unique solution. Moreover, this solution is given in the form of the double-layer potential* $w(x,g)$, *where* $g$ *is a solution of equation* (22). *If*

*$F \in C^{1,\beta}(s)$, then the BVP $(2)^-$ has a unique solution satisfying the condition $u(x) = O(|x|^{-1})$, and $\frac{\partial u}{\partial x_k} = O(|x|^{-2})$, $k = 1, 2, 3$ in the neighborhood of a point at infinity. Moreover, this solution is given in the form of a simple-layer potential $v$, where $h$ is a solution of equation (23).*

**Problems** $(1)^-$ **and** $(2)^+$. Consider the first external BVP. Its solution is sought in the form

$$w(x,g) = \frac{1}{2\pi} \iint\limits_S [N(\partial y, n)\Gamma(z-x)]^* g(y)ds - \frac{1}{2}\Gamma(x)\alpha, \tag{24}$$

where

$$\alpha = \frac{1}{2\pi} \iint\limits_S [N(\partial y, n)\Gamma(y)]^* g(y)ds,$$

$[N(\partial y, n)\Gamma(z-x)]^*$ is the Fredholm kernel.

The origin is assumed to lie in the domain $D^+$. Taking into account the boundary behavior of the potential $w(x)$ and the boundary condition, to define the unknown vector $g$ we obtain from (24) the Fredholm integral equation of second order

$$g(z) + \frac{1}{2\pi} \iint\limits_S [N(\partial y, n)\Gamma(y-z)]^* g(y)ds - \frac{1}{2}\Gamma(z)\alpha = f(z). \tag{25}$$

Its conjugate equation is

$$h(z) + \frac{1}{2\pi} \iint\limits_S [N(\partial y, n)\Gamma(y-z) + \frac{1}{2} \iint\limits_S N(\partial z, n)\Gamma(z)\alpha h(y)ds = F(z), \tag{26}$$

Let us show that equation (25) is always solvable. For this it is sufficient that the homogeneous equation corresponding to (25) has only a trivial solution. Denote the homogeneous equation by $(26)_0$ and assume that it has a solution $h_0$ different from zero.

From $(26)_0$ we get

$$\frac{1}{2\pi} \iint\limits_S \Gamma(y)h(y)ds = 0. \tag{27}$$

and equation $(26)_0$ takes the form

$$h(z) + \frac{1}{2\pi} \iint\limits_S N(\partial y, n)\Gamma(y-z)h(y)ds = 0. \tag{28}$$

Construct now the potential

$$V(x) = \frac{1}{2\pi} \iint\limits_S \Gamma(y-x)h(y)ds,$$

here $(Nv)^+ = 0$ and $v(0) = 0$. Hence we get $v(x) = 0, x \in D^-$, since $h(x) = 0$. Thus our assumption is not valid. Equation (25) has a solution for an arbitrary right-hand side. We can notice that a solution of equation (25) exists if $S \in L_2(\alpha), f(z) \in C^{1,\beta}(S), 0 < \beta < \alpha \leq 1$. The solution of the BVP $(2)^+$ is sought in the form

$$v(x) = \frac{1}{2\pi} \iint\limits_S \Gamma(y-z)h(y)ds - \frac{1}{2}\Gamma(x)A - \frac{1}{2}\Gamma_0(x)B. \tag{29}$$

To define $h(z)$, we have the integral equation

$$h(z)+\frac{1}{2\pi} \iint\limits_S T(\partial y, n)\Gamma(y-z)h(y)ds - \frac{1}{4}T(\partial z, n)\Gamma(z)A- \\ - \frac{1}{4}T(\partial z, n)\Gamma_0(z)B = F(z), \tag{30}$$

where $A$, $B$ and $F$ are defined as follows:

$$\begin{aligned} A &= u(0) = \frac{1}{2\pi} \iint\limits_S \Gamma(y)h(y)ds, \\ B &= \begin{pmatrix} \operatorname{rot} u' \\ \operatorname{rot} u'' \end{pmatrix}_{x=0} = \frac{1}{2\pi} \iint\limits_S \Gamma_0(y)h(y)ds, \\ F(z) &= \begin{pmatrix} F' \\ F'' \end{pmatrix}, \quad F' = (F_1, F_2, F_3)^T, \quad F'' = (F_4, F_5, F_6)^T. \end{aligned} \tag{31}$$

Let us now show that the integral equation (30) is always solvable. Let $h(y) \neq 0$. From (30) we have

$$A = \iint\limits_S F(z)ds, \tag{32}$$

$$B = \iint\limits_S \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} ds, \tag{33}$$

where $B_1 = \iint\limits_S [x, F']ds$, $B_2 = \iint\limits_S [x, F'']ds$. If $F = 0$, then $A = 0$, $B = 0$, $(Tu)^+ = 0, u' = a + [b, r]$, $u'' = c + [d, r]$. If the principal vector $A$ and the principal moment $B$ are equal to zero, we have $u' = 0$, $u'' = 0$, $h = 0$. Thus (28) is solvable for any right-hand side.

Consider the conjugate equation

$$g(z) + \frac{1}{2\pi} \iint\limits_S T(\partial y, n)\Gamma(y-z)^* g(y)ds - \frac{1}{2}\Gamma(z)\alpha - \frac{1}{4}\Gamma_0(z)\beta = f(z), \tag{34}$$

where

$$\alpha = \frac{1}{2\pi} \iint\limits_S [T(\partial y, n)\Gamma(y-z)]^* g(y)ds, \quad \beta = \frac{1}{2\pi} \iint\limits_S [T(\partial y, n)\Gamma_0(y)]^* g(y)ds.$$

The equation (34) is always solvable if $f \in C^{1,\alpha}(S)$, $S \in L_2(\alpha)$, $0 < \beta < \alpha \leq 1$. If the solution of the BVP $(1)^-$ is sought in the form

$$u(x) = \frac{1}{2\pi} \iint\limits_S [T(\partial y, n)\Gamma(y-z)]^* g(y) ds - \frac{1}{2}\Gamma(x)\alpha - \frac{1}{4}\Gamma_0(x)\beta, \qquad (35)$$

then to define the unknown vector $g$, we obtain the integral equation (34).

Thus we formulate the final result.

**Theorem 6.** *The problem* $(1)^-$ *is solvable for an arbitrary vector* $f(y) \in C^{1,\alpha}(S)$*, for* $S \in L_2(\alpha)$*, and its solution is representable by formula* (35)*.*

**Theorem 7.** *The problem* $(2)^+$ *is solvable for the vector* $F \in C^{0,\alpha}(S)$ *if and only if the principal vector and the principal moment of external stresses are equal to zero,* $A = 0$ *and* $B = 0$*. A solution is represented by formula* (29)*.*

For the proof of Theorems 1,2,3,4,5,6 and 7 we used the method given in [4] which is applied to the proof of analogous theorems for isotropic media.

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