Proceedings of A. Razmadze
Mathematical Institute
Vol. **145** (2007), 35–41

ON SOME ESTIMATES OF TOPOLOGICAL WEIGHTS OF LEFT-INVARIANT MEASURES ON UNCOUNTABLE SOLVABLE GROUPS

A. KIRTADZE

Abstract. A lower estimate of the topological weight of a nonseparable left-invariant measure given on an uncountable solvable group is obtained in terms of cardinalities of the factors of the composition series of this group.

რეზიუმე. არათვლად ამოხსნად ჯგუფებზე განსაზღვრული არასეპარაბელური მარცხენა-ინვარიანტული ზომებისათვის ტოპოლოგიური წონების ქვემოდან შეფასება მიღებულია კარდინალების ტერმინებში ამ ჯგუფების კომპოზიციური მწკრივების ფაქტორების გათვალისწინებით.

One of important topics in modern measure theory is the problem of the existence of a nontrivial measure on a sufficiently large class of subsets of an initial base set which is usually assumed to be uncountable. In general, it is impossible to define a nonzero σ-finite continuous (i.e. vanishing at singletons) measure on the family of all subsets of an uncountable base set (see, e.g., [1], [9]). So, it is reasonable to consider those properties of a given measure, which show the richness (in an appropriate sense) of its domain of definition. One of such properties can be expressed in terms of the topological weight of the metric space canonically associated with a given measure. Namely, let E be an uncountable set, $\mathcal{S}$ be a σ-algebra of subsets of E and let μ be a σ-finite measure on E with $\mathrm{dom}(\mu) = \mathcal{S}$. For any two sets $X \in \mathcal{S}$ and $Y \in \mathcal{S}$ satisfying the relations $\mu(X) < +\infty$ and $\mu(Y) < +\infty$, we can put $d(X,Y) = \mu(X \triangle Y)$. The function d is a quasi-metric and, after the corresponding factorization, yields a metrical space canonically associated with μ. The topological weight of this metric space is called the weight of μ. A measure μ is called nonseparable if the above-mentioned metric space is nonseparable (i.e. the weight of μ is strictly greater than the first infinite cardinal ω).

2000 *Mathematics Subject Classification.* 28A05, 28D05.
Key words and phrases. Quasi-metric, nonseparable left-invariant measure, independent family of sets, solvable group, composition series.

Example 1. Nonseparable probability measures naturally appear when one deals with uncountable products of probability measure spaces. For instance, if $\mathbf{T}$ denotes the one-dimensional unit torus endowed with the standard Lebesgue probability measure, then the product of uncountably many copies of $\mathbf{T}$ carries the product probability measure which is nonseparable and translation-invariant (actually, this product measure coincides with the Haar measure on the topological product group). Note that uncountable direct sums of probability measure spaces also yield examples of nonseparable measures. However, in the latter case the obtained measures fail to be σ-finite, and this is a weak side of the operation of taking direct sums of probability measure spaces.

Example 2. Many years ago, a problem was formulated about the existence of nonseparable translation-invariant extensions of the standard Lebesgue measure λ on the real line $\mathbf{R}$ or, respectively, about the existence of nonseparable translation-invariant extensions of the Lebesgue probability measure λ' on the torus $\mathbf{T}$. This problem was investigated by several authors and was solved positively in the classical works [3] and [8] (see also [4] and [5] where some related topics are discussed). Moreover, certain generalized versions of the above-mentioned problem were considered for the Haar measure on an infinite metrizable compact topological group (see, e.g., [2]).

In this short paper we obtain some estimates of the topological weight of the metric space canonically associated with a nonseparable left (right) invariant measure given on an uncountable solvable group. Note that the existence of a non-atomic nonseparable left (right) invariant measure on an uncountable solvable group was shown in [6] under the assumption of the validity of the Continuum Hypothesis.

In this article we will consider those σ-algebras of groups G on which nonzero σ-finite (left) invariant measures exist with appropriate topological weight (character).

For any cardinal number β, we will denote by β^+ the least cardinal among all those ones which are strictly greater than β.

Lemma 1. *Let $(G, +)$ be an arbitrary uncountable commutative group and let* $\mathrm{card}(G) = \alpha$. *For any infinite cardinal number β such that $\beta \leq \alpha$, there exists a commutative group $(H, +)$ such that the following conditions hold:*

(1) $\mathrm{card}(H) = \beta$;

(2) *H is a homomorphic image of G.*

The proof of this purely algebraic lemma is similar to the argument presented in [6].

Lemma 2. *Suppose that $\omega \leq \beta \leq \alpha$ and that every commutative group of cardinality β admits a non-atomic σ-finite invariant measure whose weight* *is γ. Then every commutative group of cardinality α admits a non-atomic σ-finite invariant measure whose weight is the same γ.*

Proof. Let $(G, +)$ be an arbitrary uncountable commutative group and let $\mathrm{card}(G) = \alpha$. In view of Lemma 1, there exists a surjective homomorphism

$$\phi : G \to H,$$

where H is some commutative group of cardinality β. According to our assumption, H admits a nonseparable nonatomic σ-finite invariant measure μ, whose weight is γ. Consider the family of sets

$$\mathcal{S} = \{\phi^{-1}(Y) : Y \in \mathrm{dom}(\mu)\}$$

and define a functional ν on this family by putting

$$\nu(\phi^{-1}(Y)) = \mu(Y) \qquad (Y \in \mathrm{dom}(\mu)).$$

It can easily be verified that the definition of ν is correct (since ϕ is a surjection) and ν is a G-invariant nonatomic σ-finite measure on S.

Let us check that ν is also a G-invariant measure, whose weight is the same γ.

Indeed, there exists an uncountable family $\{Y_i : i \in I\}$ of μ-measurable subsets of H such that $\mathrm{card}(I) = \gamma$ and

$$d(Y_i, Y_j) > \varepsilon \quad (i \in I,\ j \in I,\ i \neq j)$$

for some strictly positive real ε. Then we may write

$$\nu(\phi^{-1}(Y_i) \triangle \phi^{-1}(Y_j)) > \varepsilon \quad (i \in I,\ j \in I,\ i \neq j),$$

which shows that the topological weight of ν is equal to γ.

Take now an arbitrary set $X \in \mathcal{S}$ and an arbitrary element $g \in G$. Then $X = \phi^{-1}(Y)$ for some set $Y \in \mathrm{dom}(\mu)$. Denoting $h = \phi(g)$, we easily come to the equality

$$g + X = \phi^{-1}(h + Y),$$

which yields

$$\nu(g + X) = \mu(h + Y) = \mu(Y) = \nu(\phi^{-1}(Y)) = \nu(X),$$

i.e. ν turns out to be a G-invariant measure.

This ends the proof of Lemma 2. □

In fact, the proof of Lemma 2 is based on the method of homomorphisms (cf. [6]). In this context, note that the method of direct products which is often applied in the theory of invariant or quasi-invariant measures (see, e.g., [4], [5] and [7]) is a particular case of the method of homomorphisms.

Recall that a family $\{E_i : i \in I\}$ of subsets of an infinite set E is stochastically independent (in the generalized sense) if the relation

$$\bigcap\{\overline{E_i} : i \in J\} = \mathrm{card}(E)$$

holds, whenever J is a countable subset of I and

$$(\forall i)(i \in J \Rightarrow (\overline{E_i} = E_i) \vee (\overline{E_i} = E \setminus E_i)).$$

For constructing nonseparable measures on an uncountable set E, it is useful to apply Tarski's auxiliary proposition which states (under some assumptions on E) the existence of a stochastically independent family of subsets of E, having the cardinality greater than or equal to $\mathrm{card}(E)^+$ (see, for instance, [2], [3], [4], and [10]). For our purpose, it suffices to apply a certain variant of this proposition, which plays the key role in the sequel.

Lemma 3. *Let $(G,+)$ be an uncountable commutative group with* $\mathrm{card}(G) = \alpha$ *and let β be an infinite cardinal such that $\beta^\omega \leq \alpha$. Then there exists a stochastically independent family of subsets of G having cardinality $(\beta^\omega)^+$.*

Example 3. Assume the Generalized Continuum Hypothesis and define the cardinal α by the equality:

$$\alpha = \omega_\omega = \omega_1 + \omega_2 + \cdots + \omega_n + \cdots .$$

If $\beta < \alpha$, then there exists a natural number $n \in \omega$ such that $\beta = \omega_n$. Taking into account our assumption, we easily get

$$\beta^\omega = \omega_n^\omega = \omega_n < \alpha.$$

Therefore, by Lemma 3, we can assert that if $\mathrm{card}(G) = \alpha$, then for every infinite cardinal $\beta < \alpha$, there exists a stochastically independent family of subsets of G having cardinality $\beta^+ = 2^\beta$.

Theorem 1. *Let $(G,+)$ be an arbitrary uncountable commutative group with* $\mathrm{Card}(G) = \alpha$ *and let β be an infinite cardinal such that $\beta^\omega \leq \alpha$. Then there exists an invariant probability measure on G, whose weight is $(\beta^\omega)^+$.*

Proof. Let μ_0 denote the probability invariant measure on G which is defined at all countable and co-countable subsets of G and vanishes on all singletons in G. Let $S_0 = \mathrm{dom}(\mu_0)$ and let S be the algebra on the group G, generated by the union

$$S_0 \cup \Big\{ \bigcup_{i \in I} E_i \Big\},$$

where $(E_i : i \in I)$ is a stochastically independent family of subsets of G indicated in Lemma 3. All elements of the algebra S are representable in the form

$$\bigcup_{\psi \in \{0;1\}^p} ((E_{i_1}^{(1)} \cap \ldots E_{i_p}^{\psi(p)}) \cap Z_\psi),$$

where p is an arbitrary natural number, $\{0;1\}^p$ is the set of all functions from $\{1,2,\ldots,p\}$ into $\{0;1\}$, $(E_{i_r} : 1 \leq r \leq p)$ is a sequence of pairwise distinct elements from the family $(E_i : i \in I)$ and, finally, for arbitrary

$\psi \in \{0;1\}^p, r \in 1,2,\dots,p$, we have $Z_\psi \in S_0, E_{i_r}^{\psi(r)} = E_{i_r}$ if $\psi(r) = 0$ and $E_{i_r}^{\psi(r)} = G \backslash E_{i_r}$ if $\psi(r) = 1$.

This implies that we can define the functional μ by the formula

$$\mu\Big(\bigcup_{\psi \in \{0;1\}^p} (E_{i_1}^{\psi(1)} \cap \dots \cap E_{i_p}^{\psi(p)} \cap Z_\psi)\Big) = \frac{1}{2^p} \sum_{\psi \in \{0;1\}^p} \mu_0(Z_\psi).$$

It is easy to verify that the functional μ is a finitely-additive extension of the measure μ_0. Note that μ is also a countably-additive function and therefore is a measure defined on the algebra S. According to the Caratheodory theorem, this measure can be extended (in a unique way) on the σ-algebra generated by

$$S_0 \cup \Big(\bigcup_{i \in I} \{E_i\}\Big).$$

Denote by $\overline{\mu}$ the extended measure. The measure $\overline{\mu}$ is a nonatomic invariant measure on G such that

$$d(E_i, E_j) = \overline{\mu}(E_i \triangle E_j) = 1/2 \quad (i \in I,\ j \in I,\ i \neq j).$$

Finally, it is easy to verify that the topological weight of the metric space canonically associated with measure $\overline{\mu}$ is equal to $(\beta^\omega)^+$.

Theorem 1 has thus been proved (cf. [2] and [4]). □

As said above, the question of the existence of a nonseparable non-atomic σ-finite left-invariant measure on an uncountable solvable group was investigated in [6] and it was shown therein that such measures always exist if the Continuum Hypothesis holds. In this connection, it is interesting to obtain some lower estimates for the weight of a nonseparable invariant measure given on an uncountable solvable group.

If $(G, \cdot)$ is an uncountable solvable group, then there exists a finite sequence

$$\{e\} = G_0 \subset G_1 \subset G_2 \subset \dots \subset G_n = G$$

of subgroups of G, where e denotes the neutral element of G and, for each natural number $i \in [0, n-1]$, the group G_i is normal in G_{i+1} and the factor-group G_{i+1}/G_i is commutative. Since $G = G_n$ is uncountable, some of the factor-groups G_{i+1}/G_i are uncountable, too. We denote:

$$\theta = \min\{\mathrm{card}(G_{i+1}/G_i) : G_{i+1}/G_i \textit{ is uncountable}\}.$$

Consider two possible cases.

1. $\omega < \mathrm{card}(G_n/G_{n-1})$.

In this case, $\mathrm{card}(G_n/G_{n-1}) \geq \theta$. Using Lemma 1, there exists a commutative group H with $\mathrm{card}(H) = \theta$, which is a homomorphic image of G_n/G_{n-1}. According to Lemma 2 and Theorem 1, the group H (hence the group G_n/G_{n-1}) admits a non-atomic σ-finite invariant measure whose weight is $(\beta^\omega)^+$, where $\beta^\omega \leq \theta$. Let

$$\psi : G_n \to G_n/G_{n-1}$$

denote the canonical epimorphism. Then, applying Lemma 2 and Theorem 1 once more, we obtain a non-atomic σ-finite left-invariant measure on $G_n = G$, whose weight is $(\beta^\omega)^+$.

2. $\mathrm{card}(G_n/G_{n-1}) \leq \omega$.

In this case, $\mathrm{card}(G_n) = \mathrm{card}(G_{n-1})$, the group G_{n-1} is uncountable and the length of its composition series is strictly less than the length of the composition series of G_{n-1}. By inductive assumption, we may suppose that there exists a non-atomic σ-finite left-invariant measure on G_{n-1} whose weight is $(\beta^\omega)^+$, where $\beta^\omega \leq \theta$. Therefore, an analogous measure does exist on G_n (cf. [6], the proof of Lemma 3).

The above argument leads to the following statement.

Theorem 2. *let $(G, \cdot)$ be an uncountable solvable group and let θ be defined as above. Then there exists a non-atomic σ-finite left-invariant measure on G, whose weight is $(\beta^\omega)^+$, where β is an infinite cardinal such that $\beta^\omega \leq \theta$.*

References

1. V. V. Buldygin and A. B. Kharazishvili, Geometric aspects of probability theory and mathematical statistics. *Mathematics and its Applications*, 514. *Kluwer Academic Publishers, Dordrecht*, 2000.
2. E. Hewitt and K. Ross, Abstract harmonic analysis. Vol. 1, *Springer-Verlag, Berlin*, 1963.
3. S. Kakutani and J. Oxtoby, Construction of a non-separable invariant extension of the Lebesgue measure space. *Ann. of Math. (2)* **52**(1950), 580–590.
4. A. B. Kharazishvili,Invariant extensions of the Lebesgue measure. (Russian) *Tbilis. Gos. Univ., Tbilisi*, 1983.
5. A. B. Kharazishvili, Transformation groups and invariant measures. *Set-theoretical aspects. World Scientific Publishing Co., Inc., River Edge, NJ*, 1998.
6. A. Kharazishvili and A. Kirtadze, Nonseparable left-invariant measures on uncountable solvable groups. *Proc. A. Razmadze Math. Inst.* **139**(2005), 45–52.
7. A. Kirtadze, On the method of direct products in the theory of quasi-invariant measures. *Georgian Math. J.* **12**(2005), No. 1, 115–120.
8. K. Kodaira and S. Kakutani, A non-separable translation invariant extension of the Lebesgue measure space. *Ann. of Math. (2)* **52**(1950), 574–579.
9. J. C. Oxtoby, Measure and category. A survey of the analogies between topological and measure spaces. *Graduate Texts in Mathematics*, Vol. 2. *Springer-Verlag, New York-Berlin*, 1971.

10. G. R. Pantsulaia, An application of independent families of sets to the measure extension problem. *Georgian Math. J.* **11**(2004), No. 2, 379–390.

(Received 07.11.2007)

Author's address:

Department of Mathematics
Georgian Technical University
77, M. Kostava St., Tbilisi 0175
Georgia
E-mail: kirtadze2@yahoo.com