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ON THE CLASS OF FRECHET b-SPACES

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Abstract. We introduce the class of Fréchet b-spaces and study some of their properties. As an application, we give the bornological Mittag-Leffler lemma in the category of quotient bornological spaces [4].

რეზიუმე. შემოღებულია ფრეშეს b-ტიპის სივრცეები და შესწავლილია მათი ზოგიერთი თვისება. მიღებული შედეგები გამოყენებულია მიტაგ-ლეფლერის ბორნოლოგიური ლემის მისაღებად ბორნოლოგიური ფაქტორ-სივრცეების კატეგორიებში [4].

1. Introduction and Notations

The Mittag-Leffler lemma applies to Fréchet spaces and the von Neumann boundedness of a Fréchet space has a useful property that the majority of boundedness do not have. In [1] (see also [2]) we introduced the class of Fréchet b-spaces and we used this class to prove a bornological version of Mittag-Leffler lemma in the category of b-spaces. We observe that a Fréchet b-space is equipped with a bounded structure which has the properties of the bounded structure of a Fréchet space.

In this paper, we will define and study this class of b-spaces. For example, we will show that if E is a completely metrizable vector space, then the b-space $E_{\mathrm{co}} = (E, \beta_{\mathrm{co}})$ is a Fréchet b-space, where $B \in \beta_{\mathrm{co}}$ if its convex hull is bounded in the von Neumann boundedness of E. The same result will be true for the b-space $E_f = (E, \beta_f)$, if E is a completely metrizable locally quasi-convex vector space and β_f is the fine boundedness of E. Finally, we will generalize the bornological analogue of the well-known Mittag-Leffler Lemma that we established in [2], (Theorem 3.3) by proving that for each $n \in \mathbb{N}$, if $E_n|F_n$ is a quotient bornological space and $u_{n+1} : E_{n+1} \to e_n$ is a bounded linear mapping such that its restriction $v_{n+1} = u_{n+1/F_{n+1}}$, then $F_{n+1} \to F_n$ is an approximatively surjective bounded linear mapping. If F_n is a Fréchet b-space, then $\varprojlim_n (E_n|F_n) \simeq (\varprojlim_n E_n)|(\varprojlim_n F_n)$.

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Let us give some notation and recall some definitions that will be used in this paper. Let E be a real or complex vector space and let B be an absolutely convex set of E. Let E_B be the vector space generated by B i.e. $E_B = \cup_{\lambda>0}\lambda B$. The Minkowski functional of B is a semi-norm on E_B. It is a norm if and only if B does not contain any nonzero subspace of E. The set B is completant if its Minkowski functional is a Banach norm.

A bounded structure β on a vector space E is defined by a set of "bounded" subsets of E with the following properties :

(1) Every finite subset of E is bounded;

(2) every union of two bounded subsets is bounded;

(3) every subset of a bounded subset is bounded;

(4) a set homothetic to a bounded subset is bounded;

(5) each bounded subset is contained in a completant bounded subset.

A b-space (E, β) is a vector space E with a boundedness β. A subspace F of a b-space E is bornologically closed if $F \cap E_B$ is closed in E_B for every completant bounded B of E. If B is a bounded subset of E, then the completant hull of B is the bounded set $\lambda(B) = \left\{ \sum_n \lambda_n b : \sum_n |\lambda_n| \leq 1 \right\}$. A bounded subset B of E is completant if and only if $B = \lambda(B)$.

Given two b-spaces (E, β_E) and (F, β_F), a linear mapping $u : E \to F$ is bounded, if it maps bounded subsets of E into bounded subsets of F. The mapping $u : E \to F$ is bornologically surjective if for every $B' \in \beta_F$, there exists $B \in \beta_E$ such that $u(B) = B0$. We denote by **b** the category of b-spaces and bounded linear mappings. For more information about b-spaces we refer the reader to [3].

Let (E, β_E) be a b-space. A b-subspace of E is a subspace F with a boundedness β_F such that (F, β_F) is a b-space and $\beta_F \subseteq \beta_E$. A quotient bornological space $E|F$ is a vector space E/F, where E is a b-space and F a b-subspace of E. Given two quotient bornological spaces $E|F$ and $E_1|F_1$, a strict morphism $u : E|F \to E_1|F_1$ is induced by a bounded linear mapping $u_1 : E \to E_1$ whose restriction to F is a bounded linear mapping $F \to F_1$. Two bounded linear mappings $u_1,\ v_1 : E \to E_1$, both inducing a strict morphism, induce the same strict morphism $E|F \to E_1|F_1$ if and only if $u_1 - v_1$ is a bounded linear mapping $E \to F_1$. A strict morphism u is a class of equivalence of bounded linear mappings for the equivalence just defined. The class of quotient bornological spaces and strict morphisms is a category that we call $\widetilde{\mathbf{q}}$. A pseudo-isomorphism $u : E|F \to E_1|F_1$ is a strict morphism induced by a bounded linear mapping $u_1 : E \to E_1$ which is bornologically surjective and such that $u_1^{-1}(F_1) = F$ as b-spaces i.e. $B \in \beta_F$ if $B \in \beta_F$ and $u_1(B) \in \beta_{F_1}$).

The category $\widetilde{\mathbf{q}}$ is not abelian; in fact, if E is a Banach space and F is a closed subspace of E, it would be reasonable if the quotient Banach space $E|F$ is isomorphic to the quotient $(E/F)|\{0\}$. This is not the case in $\widetilde{\mathbf{q}}$

unless F is complemented in E. L. Waelbroeck [4] introduced an abelian category $\mathbf{q}$ generated by $\widetilde{\mathbf{q}}$ and inverses of pseudo-isomorphims, i.e. $\mathbf{q}$ has the same objects as $\widetilde{\mathbf{q}}$ and every morphism u of $\mathbf{q}$ can be expressed as $u = v \circ s^{-1}$, where s is a pseudo-isomorphism and v is a strict morphism. For more information about quotient bornological spaces we refer the reader to [4].

2. Main Results

The Mittag-Leffler lemma [5] applies to Fréchet spaces. The boundedness of a Fréchet space has a property that a general bornology does not have. b-Spaces whose boundedness have it will be called Fréchet b-spaces.

Definition 2.1. A b-space E is a Fréchet b-space if for all sequences of bounded subsets $(B_n)_n$ of E. There exists a sequence of positive real numbers $(\lambda_n)_n$ such that the set $\cup_n \lambda_n B_n$ is bounded in E.

Proposition 2.2. Let E be a Fréchet b-space and (Bn)n be a sequence of bounded subsets of E. Then there exists a completant bounded subset B' of E which absorbs all the B_n.

Proof. Let $(B_n)_n$ be a sequence of bounded subsets of E, there exists a sequence $(\lambda_n)_n$ such that, for all $n \in \mathbb{N}$, $\lambda_n \in \mathbb{R}_*^+$, and the set $\cup_n \lambda_n B_n$ is bounded in E. The set $\sum_n 2^{-n-1} \lambda_n B_n$ is completant, contained in the completant hull of $\cup_n \lambda_n B_n$ and absorbs all the bounded subsets B_n. $\square$

Now, we give some examples of Fréchet b-spaces :

Let E be a topological vector space. A subset B is bounded in the von Neumann boundedness of E if it is absorbed by each neighbourhood of the origin. The space E with its von Neumann boundedness will be called E_b.

If E is a locally convex space in which all bounded closed absolutely convex subsets are completants, then the space E_b is a b-space.

Proposition 2.3. If E is a Fréchet space, then the b-space E_b is a Fréchet b-space.

The following result is stronger.

Proposition 2.4. Let E be a completely metrizable vector space, then the space $E_{\text{co}} = (E, \beta_{\text{co}})$ is a Fréchet b-space, where $B \in \beta_{\text{co}}$ if its convex hull is bounded in E_b.

Proof. A fundamental sequence $(V_n)_n$ of neighbourhoods of the origin exists in E such that for each n, we have $V_{n+1} + V_{n+1} \subset V_n$. If $(x_n)_n$ is a sequence of elements of E such that for all n, $x_n \in V_n$, then the series $\sum_n x_n$ converges in E, and its sum $\sum_{n=1}^{\infty} x_n$ belongs to the closure of V_{n+1}.

Let $(B_k)_k$ be a sequence of completant bounded subsets of E_{co}. For each k, one can find λ_k such that $\lambda_k B_k \subset V_k$. We must show that $\sum_k \lambda_k B_k$ is bounded in the b-space E_{co}. This subset is convex. It should be proved we must show that it is bounded in E_b. As $\sum_n \lambda_n B_n \subset \overline{V}_{N-1}$, the set $\sum_n \lambda_n B_n$ is bounded in E_{co}, and then there exists $\varepsilon > 0$, such that

$$\varepsilon\left(\sum_{n=0}^{N-1} \lambda_n B_n\right) \subset \overline{V}_{n+1}.$$

We assume also that $\varepsilon \leq 1$. Then

$$\varepsilon\left(\sum_{n=0}^{N} \lambda_n B_n + \sum_{n=N+1}^{\infty} \lambda_n B_n\right) \subset \overline{V}_{n+1} + \overline{V}_{n+1} \subset \overline{V}_n.$$

This shows that $\sum_n \lambda_n B_n$ is bounded in E, and therefore in E_{co}.

A mapping $v_p : E \to \mathbb{R}^+$ is a p-semi-norm if $v_p(x+y) \leq v_p(x) + v_p(y)$ and $v_p(tx) = |t|^p v_p(x)$ if t is a scalar. A locally quasi-convex topological vector space E is a topological vector space whose topology is defined by a family of p-semi-norms v_p.

A sequence $(x_n)_n$ of elements of a locally quasi-convex topological vector space E decreases rapidly if for all $k \in \mathbb{N}$, the set $\{(1+n)^k x_n : n \in \mathbb{N}\}$ is bounded in E (if and only if, for all continuous p-semi-norm v_p $(0 < p \leq 1)$, the sequence $(v_p(x_n))_n$ decreases rapidly in $\mathbb{R}^+$).

If E is a b-space, then a sequence $(x_n)_n$ of elements of E is rapidly decreasing if for all $k \in \mathbb{N}$, the set $\{n^k x_n : n \in \mathbb{N}\}$ is bounded in E. If E is a locally quasi-convex topological vector space, then a sequence $(x_n)_n$ of elements of E decreases rapidly if it is rapidly decreasing in E_b. □

Another example of Fréchet b-space is giving by the following: If E is a locally quasi-convex topological vector space, then a subset of E is bounded for the fine boundedness if it is included in the closed absolutely convex hull of a rapidly decreasing sequence (i.e. a subset B of E is bounded for the fine boundedness if there exists a rapidly decreasing sequence $(x_n)_n$ such that $B \subset \lambda((x_n)_n)$).

The fine boundedness of a locally quasi-convex topological space is a convex vector boundedness and is stronger than the von Neumann boundedness. The space E_f will be E with the fine boundedness. It is a b-space if every bounded closed absolutely convex subset in E is completant.

Proposition 2.5. Let E be a completely metrizable locally quasi-convex vector space. Then the b-space E_f is a Fréchet b-space.

Proof. Let $(B_n)_n$ be a sequence of bounded subsets of E_f. For each n, there exists a rapidly decreasing sequence $(x_{nk})_k$ of elements of E such that B_n is contained in $\lambda(\{x_{nk} : k \in \mathbb{N}\})$. The topology of E is defined by a sequence $(v_r)_r$ of p_r-semi-norms such that $(v_r)^{\frac{1}{p_r}} = (v_{r+1})^{\frac{1}{p_r+1}}$.

We shall use the following double sequence of real positive numbers :

$$y_{rk} = \max\Big\{v_r^{\frac{1}{p_r}}((x_{nk})) : 0 \leq n \leq r\Big\}.$$

For Cantor's diagonal, we obtain a rapidly decreasing sequence $(z_k)_k$ of real positive numbers such that, for every r, we have $v_{rk} \leq 2^{-r} z_k$ except for finite numbers of k.

If it is done, then we find a sequence of numbers strictly positive $(\lambda_r)_r$ such that for all r there exists k such that $\lambda_r y_{rk} \leq 2^{-r} z_k$.

Let $x'_{rk} = \lambda_r x_{rk}$. The bijection

$$\mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (r,k) \longmapsto (r+k)\Big(\frac{r+k+1}{2}\Big) + k$$

changes it to a rapidly decreasing sequence $(t_n)_n$ of E, such that the set $\lambda(\{t_n : n \in \mathbb{N}\})$ absorbs all the subsets B_n. $\square$

Remark. We must describe the application of the Cantor diagonal. Consider a double sequence $(y_{rk})_{rk}$ such that, for all k, the sequence $(y_{rk})_r$ is increasing and for all r, the sequence $(y_{rk})_k$ is rapidly decreasing. For each r, we can choose n_r in such a way that $y_{rk} \leq 2^{-r}(1 + k^{-r})$ if $k > n_r$. The sequence (n_r) increases and tends to infinity. Let $z_k = k^{-l}$ when $n_r \leq l < n_{r+1}$. It decreases rapidly, and $y_{rk} \leq 2^{-r} z_k$ when $k \geq n_r$.

If (E, β_E) is a b-space, then a subset C of E is compact if there exists a bounded completant subset B in E such that C is compact in the Banach space E_B. We denote by C_E the family of compact subsets of the b-space E and E_c the space E with the boundedness C_E. It is a b-space, whenever E is a b-space. It is clear that the family C_E is another boundedness on E which is finer than β_E.

If E is a Fréchet space, then the family C_E is the set of relatively compact subsets of E, and E_c is the space E with the boundedness C_E. It is also a b-space.

Recall that if E is a Fréchet space, then a compact subset K of E is contained in the closed absolutely convex hull of $(x_n)_n$, where $(x_n)_n$ is a sequence of E which converges to 0 for the topology of E.

Proposition 2.6. If (E, β_E) is a Fréchet b-space, then the b-space E_c is a Fréchet b-space.

Proof. Let $(C_n)_n$ be a sequence of compact convex subsets of E. Then for every n, there exists $B_n \in \beta_E$ such that C_n is compact in E_{B_n}. Since (E, β_E) is a Fréchet b space, there exists a sequence of strictly positive numbers $(\lambda_n)_n$ such that the set $\sum_n \lambda_n B_n$ is bounded. Let $(\lambda'_n)_n$ be a new sequence such that $\lambda'_n > 0$, $\lambda'_n < \lambda_n$ and $\frac{\lambda'_n}{\lambda_n} \to 0$. The set $\sum_n \lambda'_n C_n$ is clearly relatively compact in the Banach space absorbed by $\sum_n \lambda_n B_n$

(i.e. having $\sum_n \lambda_n B_n$ as an unit ball). This shows that E_c is a Fréchet b-space. □

As application, we give the bornological Mittag-Leffler Lemma in the category of b-spaces. But we need to recall from [2] the definition of approximatively surjective mappings.

Definition 2.7. Let E and F be b-spaces. A bounded linear mapping $u : E \to F$ is approximatively surjective if for every bounded subset B of F, there exists a bounded completant subset B_1 of F and there exists a bounded completant subset C of E such that $B \subset B_1$, u $u(C) \subset B_1$ and for every $\varepsilon > 0$ we have $B_1 \subset \varepsilon B_1 + \cup_{Mn\in\mathbb{R}} Mu(C)$.

In the Banach case, it is clear that a mapping is approximatively surjective if and only if it has a dense range.

The following result was established in [2] (Theorem 3.3).

Aplication 2.8. For each $n \in \mathbb{N}$, let E_n be a b-space and F_n be a bornologically closed subspace of E_n. For all $n \in \mathbb{N}$, let $u_{n+1} : E_{n+1} \to E_n$ be a bounded linear mapping such that $v_{n+1} = u_{n+1/Fn+1} : F_{n+1} \to F_n$ is an approximatively surjective bounded linear mapping. If F_n is a Fréchet b-space, then $\varprojlim_n (E_n|F_n) \simeq (\varprojlim_n E_n)|(\varprojlim_n F_n)$.

In the category of quotient bornological spaces, we obtain

Aplication 2.9. For each $n \in \mathbb{N}$, let $En|Fn$ be a quotient bornological space and let $u_{n+1} : E_{n+1} \to E_n$ be a bounded linear mapping such that its restriction $v_{n+1} = u_{n+1}/F_{n+1} : F_{n+1} \to F_n$ is an approximatively surjective bounded linear mapping. If F_n is a Fréchet b-space, then $\varprojlim_n (E_n|F_n) \simeq (\varprojlim_n E_n)|(\varprojlim_n F_n)$.

Proof. In fact, it follows from Application 2.8 that the right exact functor $\varprojlim_n(\cdot) : \mathbf{b} \to \mathbf{b}$ is exact, and hence, Theorem 4.1 of [4] implies that we can extend it to an exact functor $\varprojlim_n(\cdot) : \mathbf{q} \to \mathbf{q}$. Since each quotient bornological space $E_n|F_n$ defines an exact sequence

$$0 \to F_n \to E_n \to E_n|F_n \to 0$$

its image by the exact functor $\varprojlim_n(\cdot) : \mathbf{q} \to \mathbf{q}$ is the following exact sequence:

$$0 \to \varprojlim_n F_n \to \varprojlim_n E_n \to \varprojlim_n (E_n \to F_n) \to 0.$$

As consequence, we deduce that

$$\varprojlim_n (E_n|F_n) \simeq (\varprojlim_n E_n) \,|\, (\varprojlim_n F_n). \quad \square$$

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