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SOME APPLICATIONS OF SURJECTIVE HOMOMORPHISMS IN THE THEORY OF INVARIANT AND QUASI-INVARIANT MEASURES

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Abstract. Some applications of the method of surjective homomorphisms in the theory of invariant and quasi-invariant measures are given.In this context non-separable invariant measures and absolutely negligible sets are considered.

რეზიუმე. მოცემულია სურექციული ჰომომორფიზმების მეთოდის ზოგიერთი გამოყენება ინვარიანტულ და კვაზიინვარიანტულ ზომათა თეორიაში. ამ კონტექსტში განხილულია არასეპარაბელური ინვარიანტული ზომები და აბსოლუტურად უგულებელყოფადი სიმრავლეები.

In the present paper an approach to some questions in the theory of invariant (quasi-invariant) measures is discussed. It is useful in certain situations, where given topological groups or topological vector spaces are equipped with various nonzero σ-finite left invariant (left quasi-invariant) measures. In this direction, there is a deep methodology which enables to investigate some important properties of such measures (see, e.g., [1], [2]). We would like to consider a general method which is oriented to the study of invariant (quasi-invariant) measures; is purely algebraic and turns out to be helpful in various questions of measure theory. We call it the method of surjective homomorphisms. Note that the method of direct products (see [2], [3], [4], [8]) is a particular case of the method of surjective homomorphisms.

Throughout this article it is convenient to use the following notation:

$\mathrm{dom}(\mu)$ = the domain of a measure μ;

$\mathbf{R}$ = the real line;

$\mathbf{N}$ = the set of all natural numbers ;

ω_1 = the first uncountable cardinal.

Let (E_1, G_1, μ_1) and (E_2, G_2, μ_2) be two spaces equipped with their transformation groups and with invariant (quasi-invariant) measures and let $X \subset E_1$. If X has some "nice" (from the measure-theoretical viewpoint)

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property P, then in certain situations the set $X \times E_2$ has the same property P in the product space $(E_1 \times E_2, G_1 \times G_2, \mu_1 \times \mu_2)$.

The method of surjective homomorphisms can be described in a similar way. Namely, let (G_1, μ_1) and (G_2, μ_2) be two groups endowed with invariant (quasi-invariant) measures and let $\phi : G_1 \to G_2$ be a surjective homomorphism. In many cases it turns out that if a set $X \subset G_2$ has a "nice" measure-theoretical property P, then the pre-image $\phi^{-1}(X)$ has the same property P in the group G_1. This is the main feature of the method of surjective homomorphisms. In particular, if the canonical epimorphism

$$pr_1 : G_1 \times G_2 \to G_1$$

is taken as ϕ, then the method of surjective homomorphisms is reduced to the method of direct products (see above).

To illustrate this methodology, let us present several facts and statements.

Example 1. The method of direct products was extensively developed and exploited in [2]. In particular, in the same work [2] Sierpiński's problem (concerning proper extensions of invariant and quasi-invariant measures) was solved with the aid of this method and without the help of additional set-theoretical assumptions.

Example 2. The method of direct products is essential for studying the property of metrical transitivity of given measures. In particular, if an invariant (quasi-invariant) measure μ_1 is metrically transitive with respect to a countable transformation group G_1 and an invariant (quasi-invariant) measure μ_2 is metrically transitive with respect to a transformation group G_2, then the product measure $\mu_1 \times \mu_2$ is metrically transitive with respect to the product group $G_1 \times G_2$ (see, e.g., [11]). Since the metrical transitivity of a measure is closely connected with the uniqueness property, one can conclude that the method of direct products turns out to be helpful for establishing the uniqueness property of a given measure.

Example 3. The method of direct products is useful for obtaining some generalizations of a well-known W. Sierpinski's result for an uncountable group $(G, \cdot)$, whose $card(G) = \alpha$ is a regular cardinal(see,e.g.,[6],[8]). In particular, let $(G, \cdot)$ is an arbitrary group such that $G = G' \times G''$ and $G' \bigcap G'' = \{e\}$ where G' and G'' are subgroups of G with $card(G') = \omega_1$ and e denotes the neutral element of G. If μ is a nonzero σ-finite G-quasi-invariant measure on G, then for each uncountable set $X \subset G'$ there exist a G-quasi-invariant measure μ' on G extending μ and a set $Y \in I(\mu')$ for which we have $X \cdot Y = G \notin I(\mu')$, where $I(\mu')$ is the σ-ideal generated by all μ'-measure zero sets in G. In particular, if $X \in I(\mu')$, then G is representable in the form of an algebraic product of two μ'-measure zero sets.

Example 4. The method of direct products is also useful for constructing non-separable extensions of invariant (quasi-invariant) measures given on infinite-dimensional topological vector spaces. In particular, in [5] in the infinite-dimensional vector space $\mathbf{R}^{\mathbf{N}}$ a σ-finite Borel measure χ invariant with respect to a dense vector subspace R^N was constructed. On the other hand, in the n-dimensional Euclidean space $\mathbf{R}^n$, $n \in \mathbf{N}$, there exists a non-separable invariant measure μ. By applying the method of direct products it can be shown that the product measure $\chi \times \mu$ is a non-separable measure on the space $\mathbf{R}^{\mathbf{N}}$ invariant under a dense vector subspace (see, e. g.,[7]).

Example 5. Let $(G, \cdot)$ be an arbitrary group and let $Y \subset G$. We say that Y is G-absolutely negligible in G if for any σ-finite G-invariant (G-quasi-invariant) measure μ on G there exists a G-invariant (G-quasi-invariant) measure μ' on G extending μ such that $\mu'(Y) = 0$. If $\phi : G_1 \to G_2$ is a surjective homomorphism of the group G_1 into the group G_2 and Y is a G_2-absolutely negligible subset of G_2, then $X = \phi^{-1}(Y)$ is G_1-absolutely negligible in G_2(see, e.g,[10]).

Example 6. By using the method of surjective homomorphisms it was shown that for any uncountable commutative group $(G, +)$ there exist two G-absolutely negligible subsets A and B of G such that their algebraic sum $A + B$ coincides with the whole group G (see [10]).

Example 7. The method of surjective homomorphisms is crucial for establishing the existence of a non-atomic non-separable σ-finite left invariant measure on an arbitrary uncountable solvable group (see [9]).

Proposition. *Let $(G_1, \cdot)$ and $(G_2, \cdot)$ be arbitrary uncountable groups; let the group G_2 be equipped with a σ-finite left G_2-invariant (left G_2-quasi-invariant) measure μ and let*

$$\phi : G_1 \to G_2$$

be a surjective homomorphisms of the group G_1 onto the group G_2. Consider the family of sets

$$S = \{\phi^{-1}(Y) : Y \in \mathrm{dom}(\mu)\},$$

and define a functional ν on this family by putting

$$\nu(\phi^{-1}(Y)) = \mu(Y),$$

where $Y \in \mathrm{dom}(\mu)$. Then this functional is a measure satisfying the following relations:

a) *S is a G_1-invariant σ-algebra of subsets of G_1;*

b) *ν is a nonatomic σ-finite G_1-invariant measure on S.*

Proof. Suppose that for sets $Y_1 \in \mathrm{dom}(\mu)$ and $Y_2 \in \mathrm{dom}(\mu)$ the following assertion is true:

$$\phi^{-1}(Y_1) = \phi^{-1}(Y_2).$$

Note that the mapping ϕ is surjective; consequently, $Y_1 \triangle Y_2 = \emptyset$. Therefore,

$$\nu(Y_1 \triangle Y_2) = 0.$$

Hence

$$\nu(Y_1) = \nu(Y_2).$$

This implies that the definition of ν is correct.

Take now an arbitrary set $X \in S$ and an arbitrary element $g \in G_1$. Then $X = \phi^{-1}(Y)$ for some set $Y \in \mathrm{dom}(\mu)$. If $h = \phi(g)$, then we easily come to the equality

$$g \cdot X = \phi^{-1}(h \cdot Y),$$

which yields

$$\nu(g \cdot X) = \mu(h \cdot Y) = \mu(Y) = \nu(\phi^{-1}(Y)) = \nu(X),$$

i. e., ν turns out to be a G_1-invariant measure. □

Now let $\{Y_i : i \in I\}$ is an uncountable family of μ-measurable subsets of G_1. Applying Proposition, we may write

$$\nu(\phi^{-1}(Y_j) \odot \phi^{-1}(Y_k)) = \nu(\phi^{-1}(Y_j \odot Y_k)) = \mu(Y_j \odot Y_k),$$

where $j \in I$, $k \in I$ and the symbol " $\odot$ " denotes the basic operation of set theory (union, intersection, difference, symmetrical difference, etc.).

In many cases we have the following statement.

Let $(G_1, \cdot)$ and $(G_2, \cdot)$ be arbitrary uncountable groups; let the group G_2 be equipped with a σ-finite left G_2-invariant (left G_2-quasi-invariant) measure μ and let

$$\phi : G_1 \to G_2$$

be a surjective homomorphisms of the group G_1 onto the group G_2. Let measure ν be defined as above. If the measure μ has some ,,nice" (from the measure-theoretical viewpoint) property, then the measure ν has the same property.

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