Proceedings of A. Razmadze
Mathematical Institute
Vol. **144** (2007), 1–4

CANCELLATION OF DIRECT SUMS OF COUNTABLE ABELIAN GROUPS WITH FINITE TORSION-FREE RANK

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Abstract. We prove a cancellation theorem for direct sums of mixed countable abelian groups of finite torsion-free rank under certain restrictions on the complementary group. This improves a classical result due to Warfield (1976) obtained in the countable case as well as it generalizes in some aspect a recent result due to Goebel-May (2003).

რეზიუმე. დამტკიცებულია შეკვეცის თეორემა სასრული გრეხვისგან თავისუფალი რანგის შერეული თვლადი აბელის ჯგუფებისთვის დამატებით ჯგუფზე გარკვეული შეზღუდვების პირობებში. ეს აუმჯობესებს უორფილდის (1976) მიერ მიღებულ კლასიკურ შედეგს თვლად შემთხვევაში და აგრეთვე ზოგიერთ ასპექტში ანზოგადებს გოებელ-მეის (2003) შედეგს.

1. Introduction

The cancellation property or the so-called Kaplansky Test Problem for abelian groups says that (e.g. [2] or [5]) if A is an abelian group so that $A \oplus G = A \oplus H$ for any other abelian groups G and H, then $G \cong H$. This property was studied by many authors who proved significant results in this subject. A brief history of the most important of them is the following: Walker and Cohn were the first who independently established in [6] and [1], respectively, a version of the aforementioned problem for finitely generated groups (in particular, for finite groups). Their result was further extended by Crawley [2] to countable torsion abelian groups with finite Ulm-Kaplansky invariants and to torsion-complete abelian p-groups of finite Ulm-Kaplansky invariants. Later on, Goebel jointly with May generalized in [4] the first Crawley's achievement to direct sums of countable abelian p-groups having special Ulm-Kaplansky invariants. On the other hand, concerning the cancellation theorems for mixed abelian groups, precisely thirty years ago, Warfield stated in [7] the following remarkable claim.

2000 *Mathematics Subject Classification.* Primary 20K10, 20K15, 20K21.
Key words and phrases. Countable groups, direct sums, finite torsion-free rank, cancellation.

Theorem (Warfield, 1976). Let A, G and H be countable abelian groups of finite torsion-free rank such that the Ulm-Kaplansky invariants of A are finite and $A = pA$ (i.e. A is p-divisible) for almost all prime numbers p. Then

$$A \oplus G \cong A \oplus H \Leftrightarrow G \cong H.$$

The purpose of the present paper is to strengthen under the same circumstances on A the cited Warfield's result to direct sums of mixed countable abelian groups of finite torsion-free rank by the usage of a global variant of our recent isomorphism theorem, given in details in [3].

We note that Warfield pointed out in [7] (p. 16) that Crawley showed in [2] that a p-group with finite Ulm-Kaplansky invariants has the cancellation property general. Nevertheless, we specify that this is so only when A is countable or, equivalently (see, for example, [4]), simply presented; the simply presented groups with finite Ulm-Kaplansky invariants are of necessity countable.

2. Main Result

In this section, we use the idea from [7] plus the technique developed in [3], addressing the question when two direct sums of mixed countable abelian groups of finite torsion-free rank are isomorphic to prove a cancellation theorem for them.

Before doing that, we recall the major instrument. Specifically, we state the following (all unexplained notions and notation are in agreement with [5]).

Criterion (Isomorphism). Suppose that G and H are reduced direct sums of countable abelian groups with the same Ulm-Kaplansky invariants and suppose that there are torsion groups T and S with $G \oplus T \cong H \oplus S$.

If G is of finite torsion-free rank, then $G \cong H$.

Proof. Clearly, by [5] (Vol I, p. 105, Ex. 4), we calculate that $r_0(G) = r_0(G \oplus T) = r_0(G) + r_0(T)$ and $r_0(H) = r_0(H \oplus S) = r_0(H) + r_0(S)$ since $r_0(T) = r_0(S) = 0$. Thus $r_0(H) = r_0(G) < \chi_0$. Next, we write $G = \oplus_{i \in I} G_i$, where, for each index $i \in I$, the direct summands G_i are countable and $|I| > \chi_0$; otherwise G has to be countable and everything is known by virtue of [7] (Th. 2). Since $r_0(G) = \sum_{i \in I} r_0(G_i) < \chi_0$, it easily follows that $r_0(G_i) = 0$ for all but a finite number of indices $i \in I$. Therefore, all G_i are torsion groups for these indices $i \in I$. Consequently, $G = \oplus_{j \in J} G_j \oplus U$, where $J \subseteq I$ and, for each index $j \in J$, every G_j is torsion, whereas U is countable mixed of finite torsion-free rank. Similarly, one can decompose $H = \oplus_{m \in M} H_m \oplus V$, where $M \subseteq I$ and, for each index $m \in M$, every H_m is torsion while V is countable mixed of finite torsion-free rank. Further, the proof goes on as in [3]. □

Remark. A modification in the p-mixed case of the previous isomorphism assertion was exploited in [3] for studying group algebras of direct sums of countable mixed abelian groups.

And so, we have at our disposal all the information needed to proceed by proving the following central statement.

Theorem (Cancellation). *Let A be a countable abelian group of finite torsion-free rank and with finite Ulm-Kaplansky invariants such that it is p-divisible for almost all primes p. If G is of finite torsion-free rank a direct sum of countable abelian groups, then $A \oplus G \cong A \oplus H$ for any group H if and only if $G \cong H$.*

Proof. It is obvious that H has to be abelian. Besides, since $A \oplus G$ is a direct sum of countable groups, it follows by result of Kaplansky-C. Walker (e.g. [5], Vol. I, p. 63, Proposition 9.10) that so is H as being an isomorphic copy of a direct summand of a direct sum of countable groups. Thus H is a direct sum of countable abelian groups as is G.

Furthermore, according to [5] (Vol. I, p. 105, Ex. 4), we obtain that $r_0(A) + r_0(G) = r_0(A \oplus G) = r_0(A \oplus H) = r_0(A) + r_0(H)$. Because both $r_0(A) < \chi_0$ and $r_0(G) < \chi_0$, we derive that $r_0(H) = r_0(G)$.

Moreover, owing to [5], (Vol. I, p. 185, Exercise 8), we compute for the Ulm-Kaplansky invariants that $f(A) + f(G) = f(A) + f(H)$. Since $f(A) < \chi_0$, we have that $f(G) = f(H)$.

Now, in order to illustrate that G and H are isomorphic, we must show that all the conditions from our basic Isomorphism Criterion, alluded to above, are satisfied. Foremost, we may without loss of generality assume that G and H are both reduced (see, e.g., [7, Theorem 3] and [4]). So, what remains to demonstrate is that there exist torsion groups T and S, respectively, such that $T \oplus G \cong S \oplus H$. In fact, this follows in the same may as [7], pp. 15–16, §2. $\square$

Remark. Notice that the previous Theorem refines [7] (Th. 3) as well as, in some way, the main result from [4].

As an immediate consequence, we yield the following.

Corollary (Power). *Suppose A is a countable abelian group of finite torsion-free rank such that its Ulm-Kaplansky invariants are finite. If G is of finite torsion-free rank a direct sum of countable abelian groups, then $A \oplus G \cong A \oplus H$ for another group H implies that $\oplus_n G \cong \oplus_n H$ for some $n \in \mathbb{N}$.*

Proof. Follows directly by the proof of the preceding main theorem and the method described in [7] (Th. 4). $\square$

As a final discussion, we recollect once again an important problem going from Warfield (see, for instance, [7], Problem 6).

Problem (Warfield, 1976). Show that if A is a countable group of finite torsion-free rank such that the Ulm-Kaplansky invariants of A are finite and such that A is p-divisible for all but a finite number of primes p, and if G and H are arbitrary abelian groups with $A \oplus G \cong A \oplus H$, then $G \cong H$.

It is worthwhile noticing that in the torsion case this was proved by Crawley in [1]. Moreover, it is worth finding whether or not the foregoing problem may be enlarged to direct sums of countable groups.

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(Received 10.01.2007; revised 11.06.07)

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