# ON ONE PROBLEM OF THE PLANE THEORY OF ELASTICITY WITH A PARTIALLY UNKNOWN BOUNDARY 

G. KAPANADZE


#### Abstract

The problem of finding an equi-strong contour in a rectangular plate to each side of which are attached absolutely smooth rigid punches with rectilinear bases, is considered. An unknown part of the boundary is free from stresses. The condition for the contour to be equi-strong is that the tangential normal stress are stable on it.

Using the methods of the theory of analytic functions, an elastic equilibrium of the plate and analytic form of the equi-strong contour are defined under different physical and geometrical conditions.








 Ћлддм



In the present work we consider the problem of finding an equi-strong contour in a rectangular plate to each side of which are attached absolutely smooth rigid punches with rectilinear bases. It is assumed that normal compressive forces with principal vectors $p$ and $q$ are applied to the punches. Our purpose is to find elastic equilibrium of the plate and analytic form of an unknown contour under the condition that tangential normal stress takes on that contour constant value.

Using the methods of complex analysis, the problem is reduced to the mixed problem of the theory of analytic functions. The solution of the

[^0]latter allows one to construct both the equation of the unknown contour and Kolosov-Muskhelishvili's complex potentials effectively (analytically).

Similar problems of the plane theory of elasticity for an infinite plane weakened by equi-strong holes have been investigated in [1]-[4], and for a finite doubly-connected region in [5]-[8].

Statement of the Problem. Let a homogeneous elastic plate on a plane $z=x+i y$ of a complex variable occupy a doubly-connected region whose external boundary is a rectangle with the sides $2 a$ and $2 b$, and internal one is a closed smooth contour (an unknown part of the boundary). Suppose that absolutely smooth rigid punches with rectilinear bases are applied to the opposite sides of the rectangle; the punches are under the action of compressive forces with principal vectors $p$ and $q$, and the interior boundary is free from stresses.

Consider the problem: Find an elastic equilibrium of the plate and analytic form of an unknown contour under the condition that the tangential normal stress takes on that contour constant value $\sigma_{s}=k=$ const.

Solution of the Problem. Due to the axial symmetry, we restrict ourselves to the consideration of elastic equilibrium on a quarter of the plate only and denote it by $S$ (curvilinear pentagon). By $L$ we denote its boundary consisting of rectilinear segments $L_{1}=\bigcup_{1}^{4} L_{j}^{(1)}=\bigcup_{1}^{4} A_{j} A_{j+1}$ and arc $L_{2}=$ $A_{5} A_{1}$ (the unknown part of the boundary). Assume that $A_{1} A_{2} \| A_{3} A_{4}$ and $A_{2} A_{3} \| A_{4} A_{5}$. Suppose also that normal compressive forces with the principal vectors $A_{2} A_{3}=b$ and $A_{3} A_{4}=a$ act on the sides $q / 2$ and $p / 2$, while forces, opposite to the above, i.e., $-q / 2$ and $-p / 2$, act on the sides $A_{4} A_{5}$ and $A_{1} A_{2}$. It is not difficult to see that under these conditions the normal displacements $v_{n}(t)$ on every segment $L_{j}^{(1)}(j=1, \ldots, 4)$ are constant, while tangential stresses $\tau_{n s}$ on the whole boundary $L$ and normal stress $\sigma_{n}$ on $A_{5} A_{1}$ are equal to zero.

On the basis of the well-known Kolosov-Muskhelishvili's formulas ([9]) the problem under consideration is reduced to finding two holomorphic in $S$ functions $\varphi(z)$ and $\psi(z)$ with the following boundary conditions on $L$ :

$$
\begin{align*}
& \operatorname{Re}\left[e^{-i \alpha(t)}\left(\varkappa \varphi(t)-t \overline{\varphi^{\prime}(t)}-\overline{\psi(t)}\right)\right]=2 \mu v_{n}(t), \quad t \in L_{1}  \tag{1}\\
& \operatorname{Re}\left[e^{-i \alpha(t)}\left(\varphi(t)+t \overline{\varphi^{\prime}(t)}+\overline{\psi(t)}\right)\right]=c(t), \quad t \in L_{1}  \tag{2}\\
& \quad \varphi(t)+t \overline{\varphi^{\prime}(t)}+\overline{\psi(t)}=0, \quad t \in L_{2}  \tag{3}\\
& \operatorname{Re}\left(\varphi^{\prime}(t)\right)=\frac{k}{4}, \quad t \in L_{2} \tag{4}
\end{align*}
$$

where $\alpha(t)$ is the angle lying between the outer normal to the contour $\Gamma$ and the $o x$-axis, i.e., $\alpha(t)=\alpha_{k}=-\frac{\pi}{2}+\frac{\pi(k-1)}{2}, k \in L_{k}^{(1)}(k=1, \ldots, 4)$.
$c(t)=\operatorname{Re} \int_{A_{1}}^{t} i \sigma_{n}\left(s_{0}\right) \exp i\left[\alpha\left(t_{0}\right)-\alpha(t)\right] d s_{0}$. It is easy to show that $c(t)$ is a piecewise constant function

$$
c(t)= \begin{cases}0, & t \in L_{1}^{(1)} \cup L_{4}^{(1)}, \\ -p / 2, & t \in L_{2}^{(1)}, \\ -q / 2, & t \in L_{3}^{(1)} .\end{cases}
$$

We require for the function $\varphi(z)$ to be continuous in a closed region $S+L$, and for the functions $\varphi^{\prime}(z)$ and $\psi(z)$ to be everywhere continuously extendable on the boundary $L$, except possibly for the points $A_{k}$ in the vicinity of which they satisfy the condition

$$
\begin{equation*}
\left|\varphi^{\prime}(z)\right|,|\psi(z)|<M\left|z-A_{k}\right|^{-\delta_{k}} \tag{5}
\end{equation*}
$$

where $M=$ const, $0 \leq \delta_{k} \leq 1, k=2,3,4,0 \leq \delta<\frac{1}{2}, k=1,5$.
Summing equalities (1) and (2), then differentiating with respect to the arc abscissa $s$ and taking into account that the function $v_{n}(t)$ is piecewise continuous, we obtain

$$
\begin{equation*}
\operatorname{Im} \varphi^{\prime}(t)=0, \quad t \in L_{1} \tag{6}
\end{equation*}
$$

It is proved (see [7]) that the problem (4), (6) under the condition (5) has a unique solution $\varphi^{\prime}(z)=\frac{k}{4}$, and hence

$$
\begin{equation*}
\varphi(t)=\frac{k}{4} z \tag{7}
\end{equation*}
$$

(an arbitrary constant of integration is assumed to be equal to zero).
By virtue of the relations (2), (4) and (7), with respect to the potential $\psi(t)$ we obtain the boundary conditions

$$
\begin{gather*}
\operatorname{Re}\left[e^{-i \alpha(t)}\left(\frac{k}{2} t+\overline{\psi(t)}\right)\right]=c(t), \quad t \in L_{1},  \tag{8}\\
\frac{k}{2} t+\overline{\psi(t)}=0, \quad t \in L_{2} .
\end{gather*}
$$

Let the function $z=\omega(\varsigma)$ map conformally the unit semi-circle $D_{0}=$ $\{|\varsigma|<1, \operatorname{Im} \varsigma>0\}$ onto the region $S$. By $a_{k},(k=1, \ldots, 5)$ we denote preimages of the points $A_{k}$ and assume that $a_{1}=1, a_{3}=i, a_{5}=-1$ (i.e., the contour $L_{2}$ transforms into the segment $[-1,1]$ ).

Taking into account that the equality $\operatorname{Re}\left[e^{-i \alpha(t)} t\right]=\operatorname{Re}\left[e^{-i \alpha(t)} A(t)\right]$ holds on $L_{1}\left(A(t)=A_{k}\right.$ for $\left.t \in L_{k}^{(1)}(k=1, \ldots, 4)\right)$, from the conditions (8) with respect to the functions $\omega(\varsigma)$ and $\psi_{0}(\varsigma)=\psi[\omega(\varsigma)]$ we obtain the following boundary value problem:

$$
\begin{align*}
& \operatorname{Re}\left[e^{-i \alpha(t)} \omega(t)\right]=\operatorname{Re}\left[e^{-i \alpha(t)} A(t)\right], \quad t \in l_{1}, \\
& \operatorname{Re}\left[e^{-i \alpha(t)} \overline{\psi(t)}\right]=c(t)-\frac{k}{2} \operatorname{Re}\left[e^{-i \alpha(t)} A(t)\right], \quad t \in l_{1}, \tag{9}
\end{align*}
$$

$$
\frac{k}{2} \omega(t)=-\overline{\psi(t)}, \quad t \in l_{2}
$$

where $l_{1}=\bigcup_{1}^{4} l_{1}^{(k)}\left(l_{1}^{(k)}(k=1, \ldots, 4)\right.$ are the parts of the semicircle $l_{0}=$ $\{|t|=1, \operatorname{Im} t>0\}$ corresponding to the segments $L_{k}^{(1)}$ under the mapping $z=\omega(\varsigma)) ; l_{2}=[-1,1]$.

Consider the function

$$
W_{0}(\varsigma)= \begin{cases}\frac{k}{2} \omega(\varsigma), & |\varsigma|<1,  \tag{10}\\ -\operatorname{Im} \varsigma>0 \\ -\psi_{0 *}(\varsigma), & |\varsigma|<1, \\ \operatorname{Im} \varsigma<0\end{cases}
$$

where $\psi_{0 *}(\varsigma)=\overline{\psi_{0}(\zeta)}$.
On the basis of (9) we conclude that the function $W_{0}(\varsigma)$ is holomorphic in the circle $D=\{|\varsigma|<1\}$, continuously extendable up to the boundary $l=\{|t|=1\}$ and satisfies the boundary conditions

$$
\begin{align*}
& \operatorname{Re}\left[i W_{0}(\sigma)\right]=0, \quad \sigma \in l_{1}^{(1)} ; \quad \operatorname{Re}\left[i W_{0}(\sigma)\right]=0, \quad \sigma \in l_{1 *}^{(1)} ; \\
& \operatorname{Re}\left[W_{0}(\sigma)\right]=\frac{k a}{2}, \quad \sigma \in l_{1}^{(2)} ; \quad \operatorname{Re}\left[W_{0}(\sigma)\right]=\frac{k a+p}{2}, \quad \sigma \in l_{1 *}^{(2)} ; \\
& \operatorname{Re}\left[i W_{0}(\sigma)\right]=-\frac{k b}{2}, \quad \sigma \in l_{1}^{(3)} ; \quad \operatorname{Re}\left[i W_{0}(\sigma)\right]=-\frac{k b+q}{2}, \quad \sigma \in l_{1 *}^{(3)} ;  \tag{11}\\
& \operatorname{Re}\left[W_{0}(\sigma)\right]=0, \quad \sigma \in l_{1}^{(4)} ; \quad \operatorname{Re}\left[W_{0}(\sigma)\right]=0, \quad \sigma \in l_{1 *}^{(4)},
\end{align*}
$$

where $l_{1 *}^{(k)}(k=1, \ldots, 4)$ is the image of the arc $l_{1}^{(k)}$ under the mapping $\varsigma_{1}=\bar{\zeta}$.

We will now seek for a solution of the problem (11) of the class $h\left(a_{1}, \ldots, a_{5}\right)$ (for details, see [10]).

We introduce into consideration holomorphic in the circle $D$ functions $\Psi_{1}(\varsigma)$ and $\Psi_{2}(\varsigma)$ defined by the formulas

$$
\begin{equation*}
\Psi_{1}(\varsigma)=\left[W_{0}(\varsigma)+W_{0 *}(\varsigma)\right] / 2 ; \quad \Psi_{2}(\varsigma)=-i\left[W_{0}(\varsigma)-W_{0 *}(\varsigma)\right] / 2 \tag{12}
\end{equation*}
$$

where $W_{0 *}(\varsigma)=\overline{W_{0}(\bar{\varsigma})}$.
It is easily seen that the functions $\Psi_{j}(\varsigma)(j=1,2)$ satisfy the condition

$$
\begin{equation*}
\Psi_{j}(\varsigma)=\Psi_{j *}(\varsigma) \quad(j=1,2) \tag{13}
\end{equation*}
$$

and the function $W_{0}(\varsigma)$ is defined through the above functions by the formula

$$
\begin{equation*}
W_{0}(\varsigma)=\Psi_{1}(\varsigma)+i \Psi_{2}(\varsigma) . \tag{14}
\end{equation*}
$$

Substituting (12) into (11), for the functions $\Psi_{1}(\varsigma)$ and $\Psi_{2}(\varsigma)$ we obtain the boundary value problems

$$
\begin{align*}
& \Psi_{1}(\sigma)-\Psi_{1}\left(\frac{1}{\sigma}\right)=0, \quad \sigma \in l_{1}^{(1)} ; \\
& \Psi_{1}(\sigma)+\Psi_{1}\left(\frac{1}{\sigma}\right)=\frac{2 k a+p}{2}, \quad \sigma \in l_{1 *}^{(2)} ;  \tag{15}\\
& \Psi_{1}(\sigma)-\Psi_{1}\left(\frac{1}{\sigma}\right)=-\frac{i q}{2}, \quad \sigma \in l_{1}^{(3)} ; \\
& \Psi_{1}(\sigma)+\Psi_{1}\left(\frac{1}{\sigma}\right)=0, \quad \sigma \in l_{1}^{(4)} ; \\
& \Psi_{2}(\sigma)+\Psi_{2}\left(\frac{1}{\sigma}\right)=0, \quad \sigma \in l_{1}^{(1)} ; \\
& \Psi_{2}(\sigma)-\Psi_{2}\left(\frac{1}{\sigma}\right)=\frac{i p}{2}, \quad \sigma \in l_{1}^{(2)} ;  \tag{16}\\
& \Psi_{2}(\sigma)+\Psi_{1}\left(\frac{1}{\sigma}\right)=\frac{2 k b+q}{2}, \quad \sigma \in l_{1}^{(3)} \\
& \Psi_{2}(\sigma)-\Psi_{2}\left(\frac{1}{\sigma}\right)=0, \quad \sigma \in l_{1}^{(4)} .
\end{align*}
$$

The problems (15) and (16) are of the same type. For the solution of these problems we use the method of conformal sewing (see [11]). Under the sewing function we mean Zhukovski's function $\xi=\varsigma+\frac{1}{\varsigma}$ which maps the circle $D$ onto the plane with a cut along the segment $I=[-2 ; 2]$ of the real axis in such a way that the upper semicircle $l_{1}$ is mapped onto the upper contour and the lower semicircle $l_{1 *}$ onto the lower contour of the segment $I$. The positive direction on $I$ is assumed to coincide with that of the real axis. We introduce the functions

$$
\Phi_{j}(\xi)=\Psi_{j}[\varsigma(\xi)]=\Psi_{j}\left[\left(\xi-\sqrt{\xi^{2}-4}\right) / 2\right] \quad(j=1,2),
$$

where under the square root is understood that branch which is positive on the real axis outside the segment $I$. Then for $\sigma \in l_{1}$ we have

$$
\begin{aligned}
& \Psi_{j}(\sigma)=\Psi_{j}\left[\left(\tau-\sqrt{\tau^{2}-4}\right) / 2\right]=\Phi_{j}^{+}(\tau), \quad \tau \in I \\
& \Psi_{j}\left(\frac{1}{\sigma}\right)=\Psi_{j}\left[\left(\tau+\sqrt{\tau^{2}-4}\right) / 2\right]=\Phi_{j}^{-}(\tau), \quad \tau \in I, \quad(j=1,2)
\end{aligned}
$$

and the boundary conditions (15) and (16) take the form

$$
\begin{align*}
& \Phi_{1}^{+}(\tau)-\Phi_{1}^{-}(\tau)=0, \quad \tau \in\left[\delta_{1} ; 2\right] ; \\
& \Phi_{1}^{+}(\tau)+\Phi_{1}^{-}(\tau)=\frac{2 k a+p}{2}, \quad \tau \in\left[0 ; \delta_{1}\right]  \tag{17}\\
& \Phi_{1}^{+}(\tau)-\Phi_{1}^{-}(\tau)=-\frac{i q}{2}, \quad \tau \in\left[-\delta_{2} ; 0\right] ; \\
& \Phi_{1}^{+}(\tau)+\Phi_{1}^{-}(\tau)=0, \quad \tau \in\left[-2 ;-\delta_{2}\right]
\end{align*}
$$

$$
\begin{align*}
& \Phi_{2}^{+}(\tau)+\Phi_{2}^{-}(\tau)=0, \quad \tau \in\left[\delta_{1} ; 2\right] \\
& \Phi_{2}^{+}(\tau)-\Phi_{2}^{-}(\tau)=\frac{i p}{2}, \quad \tau \in\left[0 ; \delta_{1}\right]  \tag{18}\\
& \Phi_{2}^{+}(\tau)+\Phi_{2}^{-}(\tau)=\frac{2 k b+q}{2}, \quad \tau \in\left[-\delta_{2} ; 0\right] \\
& \Phi_{2}^{+}(\tau)-\Phi_{2}^{-}(\tau)=0, \quad \tau \in\left[-2 ;-\delta_{2}\right]
\end{align*}
$$

where $2 ; \delta_{1} ; 0 ;-\delta_{2} ;-2$ are the images of the points $a_{k}(k=1, \ldots, 5)$, respectively, under the mapping $\xi=\varsigma+\frac{1}{\varsigma}$.

We will seek for bounded at infinity solutions of the problems (17) and (18) of the class $h\left(-2 ;-\delta_{2} ; 0 ; \delta_{1} ; 2\right)$, satisfying the condition

$$
\begin{equation*}
\Phi_{j}(\xi)=\overline{\Phi_{j}(\bar{\xi})} \quad(j=1,2) \tag{19}
\end{equation*}
$$

The indices for the given class of problems are equal to -2 .
Consider the problem (17). The necessary and sufficient condition for the solvability of the problem has the form

$$
\begin{equation*}
(2 k a+p) \int_{0}^{\delta_{1}} \frac{d \tau}{\chi_{1}(\tau)}-i q \int_{-\delta_{2}}^{0} \frac{d \tau}{\chi_{1}(\tau)}=0 \tag{20}
\end{equation*}
$$

and the solution itself is given by the formula

$$
\begin{equation*}
\Phi_{1}(\xi)=\frac{\chi_{1}(\xi)}{2 \pi i}\left[\frac{2 k a+p}{2} \int_{0}^{\delta_{1}} \frac{d \tau}{\chi_{1}(\tau)(\tau-\xi)}-\frac{i q}{2} \int_{-\delta_{2}}^{0} \frac{d \tau}{\chi_{1}(\tau)(\tau-\xi)}\right] \tag{21}
\end{equation*}
$$

where $\chi_{1}(\xi)=\sqrt{(\xi+2)\left(\xi+\delta_{2}\right) \xi\left(\xi-\delta_{1}\right)}$.
Introduce the functions $\zeta_{2}(\xi)=\Phi_{2}(-\xi)$; the boundary conditions (18) yield

$$
\begin{aligned}
& \zeta_{2}^{+}(\tau)+\zeta_{2}^{-}(\tau)=0, \quad \tau \in\left[-2 ;-\delta_{1}\right] ; \quad \zeta_{2}^{+}(\tau)-\zeta_{2}^{-}(\tau)=-\frac{i p}{2}, \quad \tau \in\left[-\delta_{1} ; 0\right] \\
& \zeta_{2}^{+}(\tau)+\zeta_{2}^{-}(\tau)=\frac{2 k b+q}{2}, \quad \tau \in\left[0 ; \delta_{2}\right] ; \quad \zeta_{2}^{+}(\tau)-\zeta_{2}^{-}(\tau)=0, \quad \tau \in\left[\delta_{2} ; 2\right]
\end{aligned}
$$

The necessary and sufficient condition for the existence of a bounded at infinity solution of the class $h\left(-2 ;-\delta_{2} ; 0 ; \delta_{1} ; 2\right)$ has the form

$$
\begin{equation*}
(2 k b+q) \int_{0}^{\delta_{2}} \frac{d \tau}{\chi_{2}(\tau)}-i p \int_{-\delta_{1}}^{0} \frac{d \tau}{\chi_{2}(\tau)}=0 \tag{22}
\end{equation*}
$$

and the solution itself is given by the formula

$$
\begin{equation*}
\zeta_{2}(\xi)=\frac{\chi_{2}(\xi)}{2 \pi i}\left[\frac{2 k b+q}{2} \int_{0}^{\delta_{2}} \frac{d \tau}{\chi_{2}(\tau)(\tau-\xi)}-\frac{i p}{2} \int_{-\delta_{1}}^{0} \frac{d \tau}{\chi_{2}(\tau)(\tau-\xi)}\right] \tag{23}
\end{equation*}
$$

where $\chi_{2}(\xi)=\sqrt{(\xi+2)\left(\xi+\delta_{1}\right) \xi\left(\xi-\delta_{2}\right)}$.
It is easy be verify that the functions $\Phi_{1}(\xi)$ and $\Phi_{2}(\xi)=\zeta_{2}(-\xi)$ satisfy the condition (19).

The integrals appearing in formulas (20)-(23) are the first and third kind elliptic integrals (see [12]).

If the approximations

$$
\begin{aligned}
F(\varphi, k) & =\int_{0}^{\varphi} \frac{d \varphi}{\sqrt{1-k^{2} \sin ^{2} \varphi}} \approx \int_{0}^{\varphi}\left(1+\frac{1}{2} k^{2} \sin ^{2} \varphi\right) d \varphi ; \\
\Pi(\varphi ; n ; k) & =\int_{0}^{\varphi} \frac{d \varphi}{\left(1-n \sin ^{2} \varphi\right) \sqrt{1-k^{2} \sin ^{2} \varphi}} \approx \int_{0}^{\varphi}\left[1+\left(\frac{k^{2}}{2}+n\right) \sin ^{2} \varphi\right] d \varphi
\end{aligned}
$$

are satisfied, then the conditions (20) and (22) take the form

$$
\begin{align*}
(2 k a+p)\left[10 \delta_{1}+8 \delta_{2}-\delta_{1} \delta_{2}\right] & =-q\left[8 \delta_{1}+10 \delta_{2}+\delta_{1} \delta_{2}\right] \\
(2 k b+q)\left[8 \delta_{1}+10 \delta_{2}-\delta_{1} \delta_{2}\right] & =-p\left[10 \delta_{1}+8 \delta_{2}+\delta_{1} \delta_{2}\right] . \tag{24}
\end{align*}
$$

and the solution of the problem (11) by virtue of (14) takes the form

$$
\begin{align*}
W_{0}[\varsigma(\xi)] & =\Phi_{1}(\xi)+i \Phi_{2}(\xi)=\frac{\delta_{1} \delta_{2}}{8\left(\delta_{1}+\delta_{2}\right) \sqrt{2\left(\delta_{1}+\delta_{2}\right)}} \times \\
& \times\left[\sqrt{\left(1+\frac{2}{\xi}\right)\left(1+\frac{\delta_{2}}{\xi}\right)\left(1-\frac{\delta_{1}}{\xi}\right)}(2 k a+p-q)+\right. \\
& \left.+i \sqrt{\left(1-\frac{2}{\xi}\right)\left(1-\frac{\delta_{1}}{\xi}\right)\left(1+\frac{\delta_{2}}{\xi}\right)}(2 k b+q-p)\right] . \tag{25}
\end{align*}
$$

Thus owing to formula (10), the equation of an unknown contour has the form (assuming $\xi=\sigma \in(-\infty ;-2] \cup[2 ; \infty)$ )

$$
\begin{align*}
& w_{0}(\sigma)= \omega[\varsigma(\sigma)]=\frac{2}{k} W_{0}[\varsigma(\sigma)]=\frac{\delta_{1} \delta_{2}}{4 k\left(\delta_{1}+\delta_{2}\right) \sqrt{2\left(\delta_{1}+\delta_{2}\right)}} \times \\
& \times\left[(2 k a+p-q) \sqrt{\left(1+\frac{2}{\sigma}\right)\left(1+\frac{\delta_{2}}{\sigma}\right)\left(1-\frac{\delta_{1}}{\sigma}\right)}+\right. \\
&+\left.i(2 k b+q-p) \sqrt{\left(1-\frac{2}{\sigma}\right)\left(1-\frac{\delta_{1}}{\sigma}\right)\left(1+\frac{\delta_{2}}{\sigma}\right)}\right]  \tag{26}\\
& \quad \sigma \in(-\infty ;-2] \cup[2 ; \infty)
\end{align*}
$$

and the function $\psi_{0}[\varsigma(\xi)]$ is given by the formula

$$
\psi_{0}[\varsigma(\xi)]=\overline{W_{0}[\overline{\varsigma(\xi)}]}=-\frac{k}{2} \overline{\omega_{0}(\xi)}=-\frac{\delta_{1} \delta_{2}}{8\left(\delta_{1}+\delta_{2}\right) \sqrt{2\left(\delta_{1}+\delta_{2}\right)}} \times
$$

$$
\begin{align*}
& \times\left[(2 k a+p-q) \sqrt{\left(1+\frac{2}{\xi}\right)\left(1+\frac{\delta_{2}}{\xi}\right)\left(1-\frac{\delta_{1}}{\xi}\right)}-\right. \\
& \left.-i(2 k b+q-p) \sqrt{\left(1-\frac{2}{\xi}\right)\left(1-\frac{\delta_{1}}{\xi}\right)\left(1+\frac{\delta_{2}}{\xi}\right)}\right] \tag{27}
\end{align*}
$$

Having found a part of the unknown equi-strong contour and bearing the axial symmetry in mind, we will be able to find the remaining parts of that contour by means of axisymmetric mapping.

Revert now to the conditions (24). These condition provide us with $\delta_{2}=\alpha \delta_{1}$, where

$$
\begin{align*}
\alpha & =\frac{8 k^{2} a b+4 k[(a-b)(p-q)+9(-a p+b q)]-4(p-q)(5 p+4 q)}{8 k^{2} a b+4 k[(a-b)(-p+q)+9(a p-b q)]+4(p-q)(5 q+4 p)}  \tag{28}\\
k & =-\frac{p(10+8 \alpha)+q(8+10 \alpha)}{a(10+8 \alpha)+b(8+10 \alpha)-(a+b) \delta_{2}} . \tag{29}
\end{align*}
$$

Since $0<\delta_{2}<2$, from (29) we obtain the estimate

$$
\begin{equation*}
\frac{10(p+\alpha q)+8(q+\alpha p)}{8(b+\alpha a)+10(a+\alpha b)}<-k<\frac{10(p+\alpha q)+8(q+\alpha p)}{8(b+\alpha a)+10(a+\alpha b)-2(a+b)} \tag{30}
\end{equation*}
$$

Thus for determination of the parameters $k, \delta_{1}$ and $\delta_{2}$ we have obtained the conditions (28)-(30). However, it is practically impossible to determine the above parameters in a general case, and thus we are obliged to restrict ourselves to the consideration of particular cases of practical value for which the problem can be solved once and for all.

1. Let us consider some of the cases, for example, the case in which all sides of the square, weakened by an equi-strong hole, are compressed, i.e., we put $p=q$ and $a=b$.

Assuming $\delta_{1}=\delta_{2}=\delta$, two conditions (24) are reduced to one

$$
(2 k a+p)(18-\delta)=-p(18+\delta)
$$

which in turn yields

$$
\begin{aligned}
& k=\frac{-36 p}{\lambda(18-\delta)} \quad(\lambda=2 a-\text { is the side length of the square }), \\
& \delta=\frac{18(k \lambda-2 p)}{k \lambda} .
\end{aligned}
$$

Since $0<\delta<2$, for $k$ we obtain the estimate $-k \in\left(\frac{2 p}{\lambda} ; \frac{9 p}{4 \lambda}\right)$.
On the basis of formula (26), the equation for the quarter of the unknown contour takes the form

$$
\begin{gather*}
w_{0}(\sigma)=\frac{\sqrt{\delta} \lambda}{16} \sqrt{1-\frac{\delta^{2}}{\sigma^{2}}}\left(\sqrt{1+\frac{2}{\sigma}}+i \sqrt{1-\frac{2}{\sigma}}\right),  \tag{31}\\
\sigma \in(-\infty ;-2] \cup[2 ; \infty)
\end{gather*}
$$

If by $x_{1}$ and $x^{*}$ we denote respectively abscissas of the points $A_{1}$ and $A$ ( $A$ is the midpoint of the contour $A_{1} A_{5}$ ), then we will get

$$
x_{1}=\frac{\sqrt{2}}{32} \lambda \sqrt{\delta} \sqrt{4-\delta^{2}}, \quad x^{*}=\frac{\sqrt{\delta}}{16} \lambda
$$

This allows us to conclude that for $-k \in\left(\frac{2 p}{\lambda} ; \frac{36 p}{(18-\sqrt{2}) \lambda}\right)$ we have $0<\delta<$ $\sqrt{2} ; x_{1}>x^{*}$; and for $-k=\frac{36 p}{(18-\sqrt{2}) \lambda}, \delta=\sqrt{2}$ we have $x_{1}=x^{*}=\frac{\sqrt[4]{2}}{16} \lambda$. For $\left.-k=\frac{54 p}{(27-\sqrt{3}) \lambda}\right)$, the coordinate $x_{1}$ reaches its maximum $x_{1 \max }=\frac{\lambda}{2^{5 / 2} 3^{3 / 4}}$, $x^{*}=\frac{\sqrt{3}}{2} x_{1 \text { max }}, \delta=\frac{2 \sqrt{3}}{3}$. For $-k \rightarrow \frac{2 p}{\lambda}, x_{1}, x^{*} \rightarrow 0$ (the hole transforms into a point), and for $-k \rightarrow \frac{9 p}{4 \lambda}, x_{1} \rightarrow 0, x^{*} \rightarrow \frac{\sqrt{2}}{16} \lambda$.
2. Consider the case $p=q, a>b$. Then from (28) we obtain

$$
\alpha=\frac{2 k a b-9 p(a-b)}{2 k a b+9 p(a-b)},
$$

(which in its turn allows one to conclude that $\alpha>1$ ) and

$$
\begin{equation*}
-k=\frac{9(\alpha+1)(a-b) p}{2(\alpha-1) a b} . \tag{32}
\end{equation*}
$$

In this case, the condition (30) takes the form

$$
\frac{9(\alpha+1) p}{a(5+4 \alpha)+b(4+5 \alpha)}<\frac{9(\alpha+1)(a-b) p}{2(\alpha-1) a b}<\frac{9(\alpha+1) p}{4 a(\alpha+1)+b(3+5 \alpha)}
$$

or

$$
\frac{2}{a(5+4 \alpha)+b(4+5 \alpha)}<\frac{a-b}{(\alpha-1) a b}<\frac{2}{4 a(\alpha+1)+b(3+5 \alpha)}
$$

The above condition results in the system

$$
\left\{\begin{array}{l}
4 \alpha\left(a-\frac{5}{4} b\right)+5\left(a-\frac{4}{5} b\right)>0 \\
\alpha\left(a-\frac{5}{4} b\right)+\left(a-\frac{3}{4} b\right)<0
\end{array}\right.
$$

for the existence of which it is necessary and sufficient that the inequality $a<\frac{5}{4} b$ to take place, and hence $b<a<\frac{5}{4} b$; for $\alpha$ we obtain the estimate

$$
\begin{equation*}
1<\alpha<\frac{5 a-4 b}{5 b-4 a} \tag{33}
\end{equation*}
$$

Choosing $\alpha$ by virtue of the condition (33), we define $k$ by formula (32), and for $\delta_{2}$ and $\delta_{1}$ we obtain

$$
\delta_{2}=\frac{k[a(10+8 \alpha)+b(8+10 \alpha)]+18 p(\alpha+1)}{k(a+b)}, \quad \delta_{1}=\frac{\delta_{2}}{\alpha}
$$

Then the functions $\omega_{0}(\sigma)$ and $\psi_{0}[\varsigma(\xi)]$ are defined by formulas (26) and (27) in which we put $p=q$.
3. Let us now find out what condition should satisfy the values $p, q, a$ and $b$ in order to assume $\delta_{1}=\delta_{2}=\delta$ a priori. Obviously, in this case $\alpha=1$, and from (28) we get

$$
\begin{equation*}
-k=\frac{p^{2}-q^{2}}{2(a p-b q)} . \tag{34}
\end{equation*}
$$

Since $-k>0$, the above formula allows us to conclude that we have to require either $\left\{\begin{array}{l}p>q \\ a p>b q\end{array}\right.$ or $\left\{\begin{array}{l}p<q \\ a p<b q\end{array}\right.$.

The relation (30) takes in the case under consideration the form

$$
\begin{equation*}
\frac{1}{a+b}<\frac{p-q}{2(a p-b q)}<\frac{9}{8(a+b)} \tag{35}
\end{equation*}
$$

Assuming $\left\{\begin{array}{l}p>q \\ a p>b q\end{array}\right.$, from (35) we obtain the system

$$
\left\{\begin{array}{l}
(p+q)(a-b)<0, \\
a p-b q>\frac{4}{9}(b+a)(p-q) .
\end{array}\right.
$$

Similarly, under the assumption $\left\{\begin{array}{l}p<q \\ a p<b q\end{array}\right.$ we have

$$
\left\{\begin{array}{l}
(p+q)(a-b)>0 \\
a p-b q<\frac{4}{9}(b+a)(p-q)
\end{array}\right.
$$

Thus we have obtained the following result: if

$$
\left\{\begin{array} { l } 
{ p > q , } \\
{ b > a , } \\
{ a p - b q > 0 , } \\
{ 4 ( b p - a q ) < 5 ( a p - b q ) , }
\end{array} \quad \text { or } \quad \left\{\begin{array}{l}
p<q, \\
b<a, \\
b q-a p>0, \\
4(b p-a q)>5(a p-b q),
\end{array}\right.\right.
$$

then for the condition $\delta_{1}=\delta_{2}=\delta$ the above formulated problem has a solution, the values $k$ and $\delta$ are defined by the formulas

$$
\begin{aligned}
-k & =\frac{p^{2}-q^{2}}{2(a p-b q)}, \\
\delta & =\frac{18 k(a+b)+18(p+q)}{k(a+b)}
\end{aligned}
$$

and the functions $\omega_{0}(\sigma)$ and $\psi_{0}[\varsigma(\xi)]$ are defined by formulas (26) and (27) in which we have to write $\delta_{1}=\delta_{2}=\delta$.

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Author's address:
I. Javakhishvili Tbilisi Stete University

University St. 2, Tbilisi 0143
Georgia


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