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ON BASES OF MULTIDIMENSIONAL HAAR TYPE WAVELET SYSTEMS IN THE SPACES $L^p_Q(d\mu), 1 \le p < \infty$

Z. MELIKIDZE

Abstract. In the paper necessary and sufficient conditions on a Borel measure μ for which the Haar type wavelet system of functions $\{\chi_n(x)\}_{n=1}^{\infty}$ is a basis in $L^p_Q(d\mu), 1 \le p < \infty$ are established.

რეზიუმე. ნაშრომში ნაპოვნია აუცილებელი და საკმარისი პირობები ბორელის μ ზომაზე რომლისთვისაც ჰაარის ტიპის ვეივლეტთა $\{\chi_n(x)\}_{n=1}^{\infty}$ სისტემა ბაზისია $L^p_Q(d\mu)$, $1 \le p < \infty$, სივრცეში.

1. Introduction

Let E be a Borel set in R^n. Suppose that μ is a finite positive Borel measure on E. By the symbol $L^p_E(d\mu), 1 \le p \le \infty$, we denote the Banach space of all functions f such that

$$\|f\|_{L^p_E(d\mu)} = \left(\int_E |f(x)|^p d\mu \right)^{1/p} < \infty \tag{1}$$

for $1 \le p < \infty$ and

$$\|f\|_{L^\infty_E(d\mu)} = \operatorname*{ess\,sup}_{x \in E} |f(x)| \qquad \text{(relative to } \mu) \tag{2}$$

for $p = \infty$. If μ is the Lebesgue measure, then we write L^p_E instead of $L^p_E(d\mu)$.

In the sequel by the symbol Z^n will be denoted the space of all integer vectors γ whose dimension equals n.

Let $A : R^n \to R^n$ be a fixed linear map such that $A(Z^n) \subset Z^n$ and that all (complex) eigenvalues of A have absolute values greater than 1. Further, assume that $|\det A| = \delta$. From the properties of A mentioned above it follows that $\delta \ge 2$ is a natural number.

We will treat with Z^n as an additive group. Then $A(Z^n)$ is a normal subgroup, so we can form the cosets of $A(Z^n)$ in Z^n. They naturally form

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a group. Let us take only one element from each coset of $A(Z^n)$. A subset of Z^n defined by this way will be called a set of digits. It is well known (see [10], Proposition 5.5) that the number of different cosets of $A(Z^n)$ in Z^n equals $|\det A| = \delta$.

Let us fix a set of digits $S = \{k_1, \dots, k_\delta\}$ and define the set

$$\overline{Q} = \left\{ x \in R^n : x = \sum_{j=1}^{\infty} A^{-j} s_j, \text{ where } s_j \in S \right\}. \tag{3}$$

It is easy to check that the series in (3) is absolutely convergent.

It can be happened, however, that different sets of digits give sets $\overline{Q}$ which are different. For some sets of digits the sets $\overline{Q}$ can be "very" irregular. Such sets are called *fractals*.

Now we define the vectors

$$k_0^{(1)} = \Theta \text{ (where } \Theta \text{ is the zero vector)},$$

$$k_{n+1}^{(\delta(m-1)+i)} = k_n^{(m)} + A^{-(n+1)}(k_i),\ n = 0, 1, \dots,\ m = 1, \dots, \delta^n,\ i = 1, \dots, \delta$$

and the sets

$$\overline{Q}_n^{(m)} = A^{-n}(\overline{Q}) + k_n^{(m)}, n = 0, 1, \dots, \quad m = 1, \dots, \delta^n$$

for the sets of digits $S = \{k_1, \dots, k_\delta\}$.

Let $\overline{Q}$ be the set introduced by (3). Then (see [10], Proposition 5.19) the following properties of $\overline{Q}$ hold:

(i) $\overline{Q}$ is a compact subset of R^n;

(ii) $\overline{Q} = \bigcup\limits_{m=1}^{\delta^n} \overline{Q}_n^{(m)}, n = 1, 2, \dots;$

(iii) $\bigcup\limits_{\gamma \in Z^n} (\overline{Q} + \gamma) \equiv R^n;$

(iv) $\overline{Q}$ contains an open set.

In addition, let us suppose that the set of digits $S = \{k_1, \dots, k_\delta\}$ are chosen so that $\overline{Q}$ satisfies the condition

(v) $|\overline{Q}| = 1$, where $|\overline{Q}|$ denotes the Lebesgue measure of $\overline{Q}$.

The interior part of $\overline{Q}$ (measure of its boundary equals zero) will be denoted by Q. By the same manner, we assign to the each set $\overline{Q}_n^{(m)}$, $\quad n = 0, 1, \dots, \quad m = 1, \dots, \delta^n$, its interior part $Q_n^{(m)}$. The sets $Q_n^{(m)}$, $\quad n = 0, 1, \dots, \quad m = 1, \dots, \delta^n$ are called *Haar sets*.

Let $A_\delta = (\alpha_{i,j}), i = 0, \dots, \delta - 1, \quad j = 1, \dots, \delta$ be the matrix, such that $\alpha_{0,j} = 1, \quad j = 1, \dots, \delta$ and $\sum\limits_{j=1}^{\delta} \alpha_{n,j}\alpha_{m,j} = \delta\delta_{n,m}$, where $\delta_{n,m}$ is the Kronecker delta.

Let us define the following function system

$$\chi_0^{(0)}(x) = 1, \quad x \in \overline{Q},$$

$$\chi_n^{(m)}(x) = \begin{cases} \delta^{n/2}\alpha_{i,j}, & x \in Q_{n+1}^{(l\delta+j)} \\ 0, & x \in \overline{Q}\backslash\overline{Q}_n^{(l+1)}, \end{cases} \tag{4}$$
$$n = 0, 1, \dots, \quad m = 1, \dots, \delta^n(\delta - 1), \quad j = 1, \dots, \delta,$$

where $1 \le i \le \delta - 1$ and l are chosen so that $i - 1 \equiv (m-1)(\mathrm{mod}(\delta - 1))$ and $l = \left[\frac{m-1}{\delta-1}\right]$ ($[a]$ is an entire part of the real number a).

In this paper we deal with general Borel measures. Hence, it should be taken into account values of functions (4) at the points of discontinuity. Thus, we define values of functions (4) so that a closed system in $C_{\overline{Q}}$ can be obtained. Namely, the values of the functions $\chi_n^{(m)}(x)$, $n = 0, 1, \dots,$ $m = 1, \dots, \delta^n(\delta - 1)$, at the point of discontinuity x_0 is equal to the arithmetic mean of the numbers $\delta^{n/2}\alpha_{i,j}$, for which $x_0 \in Q_{n+1}^{(l\delta+j)}$, $j = 1, \dots, \delta$, i and l have the same meaning as in (4).

Let us write

$$\chi_1(x) = \chi_0^{(0)}(x) \text{ and for } n = \delta^k + j \quad \chi_n(x) = \chi_k^{(j)}(x). \tag{5}$$

System (5) is called *Haar type wavelet system* and the latter is an orthonormal basis in L_Q^p, $1 \le p < \infty$.

A system of functions $\{f_n(x)\}$ is said to be *closed* in $L^p(d\mu)$, $1 \le p < \infty$, if every function from $L^p(d\mu)$ can be approximated in the norm by finite linear combinations of f_n. A system $\{f_n(x)\}$ in $L^q(d\mu)$ (q denotes the conjugate exponent of p, $1 \le p < \infty$) is called *total* with respect to $L^p(d\mu)$ if only the zero function in $L^p(d\mu)$ is orthogonal to f_n for any n. By A. I. Markushevitch [9], we call the system of functions $\{f_n(x)\}$ a *basis in a wide sense* for $L^p(d\mu)$, $1 \le p < \infty$, if that system is minimal closed in $L^p(d\mu)$ and the system conjugate to $\{f_n(x)\}$ is total with respect to $L^p(d\mu)$.

The problem of the existence of a conditional basis in a Hilbert space was open for a long time. The latter problem has been solved by K.I. Babenko [1] who showed that the system obtained by product of the trigonometric system $\{e^{inx}\}_{n=-\infty}^{\infty}$ and the function $M_\alpha(x) = |x|^\alpha$, $0 < \alpha < 1/2$, forms a conditional basis in $L_{[-\pi,\pi]}^p$.

Observe that if the system $\{f_n(x)M(x)\}$, where M is some function, forms a basis in L^p for some $1 \le p < \infty$, then the system $\{f_n\}$ is a basis itself in $L^p(\psi(x)dx)$ and vice versa, where $\psi(x) = |M(x)|^p$.

In 1972 R. Hunt, B. Muckenhoupt and R. Wheeden [3] derived a characterization of the class of all weight functions W for which $W(x)e^{inx}$ is a basis in $L_{[0,2\pi]}^p$. Namely, the next statement holds.

Theorem (Hunt, Muckenhoupt, Wheeden). *Let $W(x)$ be a non-negative 2π-periodic function. The trigonometric system $\{e^{inx}\}_{n=-\infty}^{\infty}$ is a basis in $L_{[0,2\pi]}^p(W(x)dx)$, $1 \le p < \infty$, if and only if there exists an absolute* *constant K_p such that for any interval I*

$$\left(\frac{1}{|I|}\int\limits_I W(x)dx\right)\left(\frac{1}{|I|}\int\limits_I (W(x))^{-1/(p-1)}dx\right)^{p-1} \le K_p,$$

holds, where $|I|$ denotes a length of I.

In 1971 A. S. Krantzberg [8] described a class of all positive Borel measures μ for which the Haar system is a basis in $L^p(d\mu)$, $1 \le p < \infty$. In particular the following statement holds.

Theorem (Krantzberg). *Let μ be a positive Borel measure on $[0,1]$. The Haar system $\{\chi_n(x)\}_{n=1}^{\infty}$ is a basis in $L^p(d\mu)$, $1 \le p < \infty$, if and only if μ has the form*

$$d\mu(x) = \psi(x)dx, \tag{6}$$

where $\psi(x)$ is a non-negative Lebesgue integrable function satisfying the condition

$$\left(\frac{1}{|\Delta|}\int\limits_\Delta \psi(x)dx\right)\left(\frac{1}{|\Delta|}\int\limits_\Delta (\psi(x))^{-1/(p-1)}dx\right)^{p-1} \le K_p$$

for any dyadic interval $\Delta = ((m-1)/2^n, m/2^n)$ $(n=1,2,\dots;m=1,\dots,2^n)$, where K_p depends only on p.

In the sequel the symbol $L^q(d\mu)$ will denote the dual space of $L^p(d\mu)$, $1 \le p < \infty$: $1/p + 1/q = 1$ (as usual, we assume $1/\infty = 0$ and $1/0 = \infty$. So that $p = 1$ implies $q = \infty$).

2. The system $\{\varphi_n(x)\}_{n=1}^{\infty}$ as a basis in a wide sense

Theorem 1. *Suppose that $\{\varphi_n(x)\}_{n=1}^{\infty} \in L_Q^{\infty}$ is minimal in L_Q^{∞}. Assume that $\{\psi_n(x)\}_{n=1}^{\infty}$ is biorthonormal to $\{\varphi_n(x)\}$ and, besides, $\{\varphi_n(x)\}_{n=1}^{\infty}$ is total with respect to L_Q^1. Then the system $\{\varphi_n(x)\}_{n=1}^{\infty}$ is a basis in a wide sense in $L_Q^p(\psi(x)dx)$, $1 \le p < \infty$ ($\psi(x) \in L_Q^1$, $\psi(x) > 0$ a.e. on Q) if and only if for any natural number n*

$$|\psi_n(x)|^p[\psi(x)]^{-1} \in L_Q^{1/(p-1)}.$$

Proof. Necessity. Since the system $\{\varphi_n(x)\}_{n=1}^{\infty}$ is a basis in a wide sense in $L_Q^p(\psi(x)dx)$, there exists a system $\{f_n(x)\}_{n=1}^{\infty}$ in $L_Q^q(\psi(x)dx)$ biorthonormal to $\{\varphi_n(x)\}_{n=1}^{\infty}$ $(1/p + 1/q = 1)$ such that

$$\int\limits_Q \varphi_n(x)f_k(x)\psi(x)dx = \delta_{n,k}, \quad n = 1,2,\dots, \quad k = 1,2,\dots.$$

It is clear that

$$\int\limits_Q \left[f_k(x) - \psi_k(x)[\psi(x)]^{-1}\right] \varphi_n(x)\psi(x)dx = 0$$

for any natural number n. Hence

$$\int\limits_Q \left[f_k(x)\psi(x) - \psi_k(x)\right] \varphi_n(x)dx = 0.$$

Taking into account that the system $\{\varphi_n(x)\}_{n=1}^{\infty}$ is total with respect to L_Q^1, we have

$$f_k(x) = \psi_k(x)[\psi(x)]^{-1}$$

Consequently

$$\psi_n(x)[\psi(x)]^{-1} \in L_Q^q(\psi(x)dx).$$

Hence,

$$|\psi_n(x)|^p[\psi(x)]^{-1} \in L_Q^{1/(p-1)}$$

for any natural number n. Necessity has been proved.

Sufficiency. Suppose that

$$|\psi_n(x)|^p[\psi(x)]^{-1} \in L_Q^{1/(p-1)}.$$

Hence,

$$f_n(x) = \psi_n(x)[\psi(x)]^{-1} \in L_Q^q(\psi(x)dx).$$

Further, note that $\{f_n(x)\}_{n=1}^{\infty}$ is a system biorthonormal to $\{\varphi_n(x)\}_{n=1}^{\infty}$. Consequently the system $\{\varphi_n(x)\}_{n=1}^{\infty}$ is minimal in $L_Q^p(\psi(x)dx)$.

Now assume that $\{\varphi_n(x)\}_{n=1}^{\infty}$ is not closed in $L_Q^p(\psi(x)dx)$. Then there exists a function $f \in L_Q^q(\psi(x)dx)$, $f(x) \not\equiv 0$, such that

$$\int\limits_Q f(x)\varphi_n(x)\psi(x)dx = 0$$

for any natural number n. Since the system $\{\varphi_n(x)\}_{n=1}^{\infty}$ is total with respect to L_Q^1, it follows that

$$f(x)\psi(x) \equiv 0$$

Which contradics the assumption $f(x)\psi(x) \not\equiv 0$. Finally we conclude that $\{\varphi_n(x)\}_{n=1}^{\infty}$ is closed in $L_Q^p(\psi(x)dx)$.

It remains to prove that the system $\{f_n(x)\}_{n=1}^{\infty}$ conjugate to $\{\varphi_n(x)\}_{n=1}^{\infty}$ is total with respect to $L_Q^p(\psi(x)dx)$. Let $f(x) \in L_Q^p(\psi(x)dx)$ and

$$\int\limits_Q f(x)f_n(x)\psi(x)dx = 0$$

for any natural n. Hence

$$\int\limits_Q f(x)\varphi_n(x)dx = 0, \quad n = 1, 2, \dots .$$

Taking into account that the system $\{\varphi_n(x)\}_{n=1}^{\infty}$ is total with respect to L_Q^1, we obtain

$$f(x) \equiv 0,$$

which clearly means that the system $\{f_n(x)\}_{n=1}^{\infty}$ is total with respect to $L_Q^p(\psi(x)dx)$.

The theorem has been proved. □

3. The system $\{\chi_n(x)\}_{n=1}^{\infty}$ as a basis

Theorem 2. *The system $\{\chi_n(x)\}_{n=1}^{\infty}$ is a basis in the space $L_Q^p(d\mu)$, $1 \le p < \infty$, if and only if*

(a) *there exists a Lebesgue integrable function $\psi(x)$ such, that $d\mu(x) = \psi(x)dx$;*

(b) $\psi(x) > 0$ *a.e. on Q;*

(c) $[\psi(x)]^{-1} \in L_Q^{1/(p-1)}$;

(d) *there exists a number $M_p > 0$ such, that for every Haar set $Q_n^{(m)}$, $n = 0, 1, \dots$, $m = 1, \dots, \delta^n$, we have*

$$\Bigg(\frac{1}{|Q_n^{(m)}|}\int\limits_{Q_n^{(m)}} \psi(x)dx\Bigg)\Bigg(\frac{1}{|Q_n^{(m)}|}\int\limits_{Q_n^{(m)}} [\psi(x)]^{-1/(p-1)}dx\Bigg)^{p-1} \le M_p.$$

To prove Theorem 2 we use the following fact, which is a direct consequence of Theorem 1 and of Lemmas 1 and 2 from [7] (we only notice that these Lemmas in multi- dimensional case can be derived by the same manner as in the case of one-dimensional case).

Theorem 3. *The system $\{\chi_n(x)\}_{n=1}^{\infty}$ forms a basis in a wide sense in $L^p(d\mu)$, $1 \le p < \infty$, if and only if conditions (a), (b) and (c) of Theorem 2 hold.*

Assume that the measure μ satisfies conditions (a), (b), (c) of Theorem 2 which means that the system $\{\chi_n(x)\}_{n=1}^{\infty}$ is a basis in a wide sense in $L^p(d\mu)$, $1 \le p < \infty$. Denote by $S_n(f, x)$ the partial sums for $f \in L^p(d\mu)$ in terms of that system. Consider the Haar sets of maximal measure on which all the functions $\chi_n(x)$, $n = 1, \dots, N$ ($N \equiv 1(\mathrm{mod}(\delta - 1))$), are constant. It is obvious that for a given integer $N \ge 1$ there are N number of such sets . Let us denote the latter sets by $\Gamma_1, \dots, \Gamma_N$. Using this notation we can formulate the next lemma which will be useful to prove Theorem 2.

Lemma 1. *Let μ be a measure satisfying conditions (a), (b), (c) of Theorem 2. Then for every $N \geq 1$ $(N \equiv 1(\bmod\ (\delta-1)))$ we have*

$$S_N(f,x) = \frac{1}{|\Gamma_n|} \int\limits_{\Gamma_n} f(t)dt \quad \text{if } x \in \Gamma_n, \quad 1 \leq n \leq N. \tag{7}$$

Proof. By the assumption we have that $\{\chi_i(x)\}_{i=1}^{\infty}$ is a basis in a wide sense in $L^p(d\mu)$. We will construct the conjugate system $\{\psi_i(x)\}_{i=1}^{\infty}$. Obviously the latter can be written as follows

$$\psi_i(x) = \chi_i(x)[\psi(x)]^{-1}. \tag{8}$$

For any $N \geq 1$ $(N \equiv 1(\mathrm{mod}(\delta-1)))$ and $f \in L^p(d\mu)$ we have

$$S_N(f(x)) = \sum_{i=1}^{N} c_i \chi_i(x),$$

where

$$c_i = \int\limits_Q f(x)\psi_i(x)d\mu. \tag{9}$$

Now we use an induction. For $N=1$ we have

$$c_1 = \int\limits_Q f(x)\chi_1(x)[\psi(x)]^{-1}\psi(x)dx = \int\limits_Q f(x)dx.$$

Hence

$$S_1(f,x) = \int\limits_Q f(x)dx.$$

Let us now assume that (7) holds for some $N \equiv 1(\mathrm{mod}(\delta-1))$ and calculate $S_{N+\delta-1}(f,x)$. Indeed, it is easy to see that

$$S_{N+\delta-1}(f,x) = S_N(f,x) + \sum_{i=1}^{\delta-1} c_{N+i}\chi_{N+i}(x).$$

Denote by $P_1, \dots, P_\delta$ the Haar sets on which $\chi_{N+i}(x)$, $1 \leq i \leq \delta-1$, takes the values $\alpha_{i,1}|\Gamma_k|^{-1/2}, \dots, \alpha_{i,\delta}|\Gamma_k|^{-1/2}$ respectively, where $\left|\bigcup\limits_{j=1}^{\delta} P_j \setminus \Gamma_k\right| = 0$. Hence, due to (8) and (9) we have

$$c_{N+i} = \int\limits_Q f(x)\varphi_{N+i}(x)[\psi(x)]^{-1}\psi(x)dx = \int\limits_Q f(x)\varphi_{N+i}(x)dx =$$

$$= \sum_{j=1}^{\delta} \alpha_{i,j}|\Gamma_k|^{-1/2} \int\limits_{P_j} f(x)dx.$$

Further, if $x \in P_n$, $1 \leq n \leq \delta$, then

$$\sum_{i=1}^{\delta-1} c_{N+i}\varphi_{N+i}(x) = \sum_{i=1}^{\delta-1}\sum_{j=1}^{\delta} \alpha_{i,j}|\Gamma_k|^{-1/2}\int\limits_{P_j} f(x)dx\alpha_{i,n}|\Gamma_k|^{-1/2} =$$

$$= |\Gamma_k|^{-1}\sum_{j=1}^{\delta}\sum_{i=1}^{\delta-1}\alpha_{i,j}\alpha_{i,n}\int\limits_{P_j} f(x)dx = |\Gamma_k|^{-1}\sum_{j=1}^{\delta}\int\limits_{P_j} f(x)dx\sum_{i=1}^{\delta-1}\alpha_{i,j}\alpha_{i,n}.$$

Since $\sum\limits_{i=1}^{\delta-1}\alpha_{i,j}\alpha_{i,n} = \delta\delta_{j,n} - 1$, we have

$$\sum_{i=1}^{\delta-1} c_{N+i}\varphi_{N+i}(x) = |\Gamma_k|^{-1}\sum_{j=1}^{\delta}(\delta\delta_{j,n}-1)\int\limits_{P_j} f(x)dx =$$

$$= |\Gamma_k|^{-1}\sum_{j=1}^{\delta}\delta\delta_{j,n}\int\limits_{P_j} f(x)dx - |\Gamma_k|^{-1}\sum_{j=1}^{\delta}\int\limits_{P_j} f(x)dx =$$

$$= |\Gamma_k|^{-1}\delta\int\limits_{P_n} f(x)dx - |\Gamma_k|^{-1}\int\limits_{\Gamma_k} f(x)dx =$$

$$= |P_n|^{-1}\int\limits_{P_n} f(x)dx - |\Gamma_k|^{-1}\int\limits_{\Gamma_k} f(x)dx.$$

Consequently

$$S_{N+\delta-1}(f,x) = \frac{1}{|P_n|}\int\limits_{P_n} f(x)dx \quad \text{for } x \in P_n,\ 1 \leq n \leq \delta. \tag{10}$$

Taking into account that χ_{N+i} $(1 \leq i \leq \delta-1)$ equals zero outside of $\overline{\Gamma_k}$ and (7), (10), finally we have the desired result. Lemma 1 has been proved. $\square$

Proof of Theorem 2. It is well known that a basis in a wide sense in $L^p(d\mu)$ is a basis if and only if norms of partial sums of functions from this basis are uniformly bounded. Thus, in order to obtain sufficiency of (a)-(d), by Theorem 3 it suffices to estimate norms of $S_N(f,x)$. Let $N \equiv 1(\mathrm{mod}(\delta-1))$ and $0 \leq k < \delta-1$. Without loss of generality we assume that Γ_n (for the Haar sets $\Gamma_1, \cdots, \Gamma_N$ considered above) denotes that Haar set for which the functions $\chi_{N+k}(x)$ $(0 \leq k < \delta-1)$ equal zero outside of $\overline{\Gamma_N}$. According to Lemma 1 we have

$$\int\limits_Q |S_{N+k}(f,x)|^p\psi(x)dx = \sum_{i=1}^{N}\int\limits_{\Gamma_i} |S_{N+k}(f,x)|^p\psi(x)dx =$$

$$= \sum_{i=1}^{N-1} \left| \frac{1}{|\Gamma_i|} \int_{\Gamma_i} f(x)dx \right|^p \int_{\Gamma_i} \psi(x)dx +$$

$$+ \int_{\Gamma_N} \left| \frac{1}{|\Gamma_N|} \int_{\Gamma_N} f(x)dx + \sum_{j=1}^{k} \int_{\Gamma_N} f(x)\chi_{N+j}(x)dx\chi_{N+j}(x) \right|^p \psi(x)dx \le$$

$$\le \sum_{i=1}^{N-1} \frac{1}{|\Gamma_i|^p} \int_{\Gamma_i} |f(x)|^p \psi(x)dx \left(\int_{\Gamma_i} [\psi(x)]^{-q/p} dx \right)^{p/q} \int_{\Gamma_i} \psi(x)dx +$$

$$+ \frac{(1+B^2k)^p}{|\Gamma_N|^p} \int_{\Gamma_N} |f(x)|^p \psi(x)dx \left(\int_{\Gamma_N} [\psi(x)]^{-q/p} dx \right)^{p/q} \int_{\Gamma_N} \psi(x)dx \le$$

$$\le \left(1 + (1 + B^2(\delta-2))^p \right) M_p \int_Q |f(x)|^p \psi(x)dx,$$

where $B = \max\{|\alpha_{i,j}| : 0 \le i \le \delta - 1,\ 1 \le j \le \delta\}$. Sufficiency of Theorem 2 has been proved.

To show necessity, we use the fact that if the system $\{\chi_i(x)\}_{i=1}^{\infty}$ is a basis in $L^p(d\mu)$, then

$$||S_N|| \le M < +\infty. \tag{11}$$

On the other hand, by virtue of Theorem 3, Lemma 1 and the boudedness of the operators S_N $(N \equiv 1(\mathrm{mod}(\delta-1)))$ we have

$$||S_N|| = \sup_{||f||_{L^p_Q(d\mu)} \le 1} ||S_N(f,x)||_{L^p_Q(d\mu)} \ge \max_{1 \le i \le N} \sup ||S_N(f,x)||_{L^p_Q(d\mu)}, \tag{12}$$

where the latter supremum is taken over all f provided

$$||f||_{L^p(d\mu)} \le 1 \quad \text{and} \quad f(x) = 0 \quad \text{for} \quad x \in Q \backslash \overline{\Gamma_i}.$$

Equality (8) shows that the supremum is equal to

$$\sup_{||f||_{L^p_Q(d\mu)} \le 1} ||S_N(f,x)||_{L^p_{\Gamma_i}(d\mu)} =$$

$$= \left(\int_{\Gamma_i} \psi(x)dx \right)^{1/p} \sup_{||f||_{L^p_Q(d\mu)} \le 1} \left| \frac{1}{\Gamma_i} \int_{\Gamma_i} f(x)dx \right| \quad \text{for} \quad i = 1, \dots, N \tag{13}$$

Using the equality

$$f(x) = f(x)[\psi(x)]^{-1}\psi(x)$$

we obtain

$$\sup_{||f||_{L^p_Q(d\mu)} \le 1} \left| \int_{\Gamma_i} f(x)dx \right| = ||[\psi(x)]^{-1}||_{L^p_{\Gamma_i}(d\mu)} = \left(\int_{\Gamma_i} [\psi(x)]^{-1/(p-1)} dx \right)^{(p-1)/p}.$$

Hence, (11)-(13) lead to necessity of (d). $\square$

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Author's address

Department of Mechanics and Mathematics
I. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0143
Georgia