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NONSEPARABLE LEFT-INVARIANT MEASURES ON UNCOUNTABLE SOLVABLE GROUPS

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Abstract. It is proved, under the Continuum Hypothesis, that every uncountable solvable group admits a nonseparable nonatomic σ-finite left-invariant measure. Some related questions concerning properties of such measures are also discussed.

რეზიუმე. დამტკიცებულია, რომ კონტინუუმ–ჰიპოთეზის მიღების შემთხვევაში ყოველ არათვლად ამოხსნად ჯგუფზე არსებობს არასეპარაბელური არაატომური σ-სასრული მარცხნიდან ინვარიანტული ზომა. განხილულია ასეთი ზომების თვისებებთან დაკავშირებული ზოგიერთი საკითხი.

Let E be a nonempty set, $\mathcal{S}$ be a σ-algebra of subsets of E and let μ be a measure on E with $\mathrm{dom}(\mu) = \mathcal{S}$. As usual, the triple $(E, \mathcal{S}, \mu)$ is called a measure space.

For any two sets $X \in \mathcal{S}$ and $Y \in \mathcal{S}$ having finite μ-measure, one can define $d(X, Y) = \mu(X \triangle Y)$. The function d is a pseudometric and, after the corresponding factorization, yields a metrical space canonically associated with μ.

A measure μ is called nonseparable if the above-mentioned metrical space is nonseparable.

It is well known that the following two assertions are equivalent:

(1) μ is nonseparable;

(2) there exist a real $\varepsilon > 0$ and an uncountable family $\{Y_i : i \in I\}$ of μ-measurable subsets of E such that

$$d(Y_i, Y_j) > \varepsilon \quad (i \in I,\ j \in I,\ i \neq j).$$

Nonseparable probability measures naturally appear when one deals with uncountable products of probability measure spaces. For example, if $\mathbb{T}$ denotes the one-dimensional torus endowed with the standard Lebesgue probability measure, then the product of uncountably many copies of $\mathbb{T}$ carries the invariant product measure which is nonseparable.

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Uncountable direct sums of probability measure spaces also yield examples of nonseparable measures. However, in the latter case the obtained measures fail to be σ-finite, and this is a weak side of the operation of taking direct sums of probability measure spaces.

Many years ago a problem was formulated about the existence of nonseparable invariant extensions of the standard Lebesgue measure λ on the real line $\mathbb{R}$ or about the existence of nonseparable invariant extensions of the Lebesgue probability measure λ' on the torus $\mathbb{T}$ (see [1]). This problem was intensively investigated by several authors and was solved positively in the classical works [2] and [3]. Moreover, some generalized versions of the above-mentioned problem were considered for a Haar measure on a σ-compact locally compact topological group (see, e.g., [4]).

For further information on various methods of extending invariant (and, more generally, quasiinvariant) measures which are given on abstract spaces equipped with their transformation groups, see e.g. [5]-[9].

Let $(G, \cdot)$ be an arbitrary uncountable group. In our consideration below we do not assume that G is equipped with a topology compatible with its algebraic structure.

There are trivial examples of probability left-invariant measures defined on some σ-algebras of subsets of G. For instance, we can introduce the probability measure μ_0 defined on the family of all countable and co-countable subsets of G and vanishing on all singletons in G. Obviously, the measure μ_0 is invariant under the group of all left (right) translations of G. In this context, the following natural question arises: does there exist a nonseparable nonatomic σ-finite left-invariant measure on G?

Our goal is to show (under the Continuum Hypothesis) that if G is solvable, then the answer to this question is positive. Moreover, it turns out that if G is uncountable and commutative, then there always exists a nonseparable nonatomic invariant probability measure on G.

We need several auxiliary propositions.

The first lemma is purely algebraic in its nature and, possibly, is well-known. For the sake of completeness, we give its proof here.

Lemma 1. *Let $(G, +)$ be an uncountable commutative group. There exists a commutative group $(H, +)$ such that:*

1) $\mathrm{card}(H) = \omega_1$;
2) *H is a homomorphic image of G.*

Proof. Since the original group G is uncountable, it contains a subgroup G_1 with $\mathrm{card}(G_1) = \omega_1$. This G_1 can be isomorphically embedded in some divisible commutative group G_2 (see, e.g., [10]). It is not difficult to choose such a group G_2 of the same cardinality ω_1. Consider now the canonical embedding

$$Id_{(G_1,G_2)} : G_1 \to G_2$$

and identify G_1 with its image under $Id_{(G_1,G_2)}$. Since G_2 is divisible, we can extend $Id_{(G_1,G_2)}$ to a group homomorphism

$$\phi : G \to G_2.$$

Now, let us put $H = \phi(G)$. Clearly, H is a homomorphic image of G. By virtue of the relations

$$G_1 \subset \phi(G) \subset G_2, \quad \mathrm{card}(G_1) = \mathrm{card}(G_2) = \omega_1,$$

we also have $\mathrm{card}(H) = \omega_1$, which completes the proof of Lemma 1. □

Remark 1. One application of the above lemma can be found in paper [11] where algebraic sums of so-called absolutely negligible sets in uncountable commutative groups are discussed. These sets play an essential role in some questions concerning extensions of nonzero σ-finite left-invariant measures on groups.

If a commutative group $(G, +)$ is given and ν is a left-invariant measure on G, then ν is also a right-invariant measure on G. In this case, we simply say that ν is an invariant measure on G.

Lemma 2. *Suppose that every commutative group of cardinality ω_1 admits a nonseparable nonatomic σ-finite invariant measure. Then every uncountable commutative group admits a nonseparable nonatomic σ-finite invariant measure.*

Proof. Let $(G, +)$ be an arbitrary uncountable commutative group. In view of Lemma 1, there exists a surjective homomorphism

$$\phi : G \to H,$$

where H is some commutative group of cardinality ω_1. According to our assumption, H admits a nonseparable nonatomic σ-finite invariant measure μ. Consider the family of sets

$$\mathcal{S} = \{\phi^{-1}(Y) : Y \in \mathrm{dom}(\mu)\}$$

and define a functional ν on this family by putting

$$\nu(\phi^{-1}(Y)) = \mu(Y) \quad (Y \in \mathrm{dom}(\mu)).$$

It can easily be verified that:

a) the definition of ν is correct (since ϕ is a surjection);

b) $\mathcal{S}$ is a G-invariant σ-algebra of subsets of G;

c) ν is a nonatomic σ-finite measure on $\mathcal{S}$.

Let us check that ν is also nonseparable and G-invariant.

Indeed, since the measure μ is nonseparable, there exists an uncountable family $\{Y_i : i \in I\}$ of μ-measurable subsets of H such that

$$d(Y_i, Y_j) > \varepsilon \quad (i \in I,\ j \in I,\ i \neq j)$$

for some strictly positive real ε. Then we may write

$$\nu\big(\phi^{-1}(Y_i) \triangle \phi^{-1}(Y_j)\big) > \varepsilon \quad (i \in I,\ j \in I,\ i \neq j),$$

which shows the nonseparability of ν.

Take now an arbitrary set $X \in \mathcal{S}$ and an arbitrary element $g \in G$. Then $X = \phi^{-1}(Y)$ for some set $Y \in \mathrm{dom}(\mu)$. Denoting $h = \phi(g)$, we easily come to the equality

$$g + X = \phi^{-1}(h + Y),$$

which yields

$$\nu(g + X) = \mu(h + Y) = \mu(Y) = \nu\big(\phi^{-1}(Y)\big) = \nu(X),$$

i.e., ν turns out to be a G-invariant measure. □

This ends the proof of Lemma 2.

Lemma 3. *Suppose again that every commutative group of cardinality ω_1 admits a nonseparable nonatomic σ-finite invariant measure. Then every uncountable solvable group admits a nonseparable nonatomic σ-finite left-invariant measure.*

Proof. Let $(G, \cdot)$ be an arbitrary uncountable solvable group. There exists a composition series

$$\{e\} = G_0 \subset G_1 \subset \cdots \subset G_{n-1} \subset G_n = G$$

of subgroups of G, where e denotes the neutral element of G and, for each natural number $i \in [0, n-1]$, the group G_i is normal in G_{i+1} and the factor-group G_{i+1}/G_i is commutative.

We must show that the group $G = G_n$ admits a nonseparable nonatomic σ-finite left-invariant measure on some σ-algebra of its subsets. For this purpose, we use the induction method on n.

If $n = 1$, then $G = G_1/G_0$ is commutative and we may apply Lemma 2.

Suppose now that our assertion has already been proved for uncountable solvable groups whose composition series contains at most n subgroups, and let us establish the validity of our assertion for those uncountable solvable groups G whose composition series consists of $n+1$ subgroups, i.e., $G = G_n$.

Let $\psi : G_n \to G_n/G_{n-1}$ denote the canonical epimorphism. Consider two possible cases.

1. The commutative group G_n/G_{n-1} is uncountable.

In this case we apply again Lemma 2 and equip the group G_n/G_{n-1} with a nonseparable nonatomic σ-finite invariant measure μ. Further, as in the proof of Lemma 2, we define

$$\mathcal{S} = \{\psi^{-1}(Y) : Y \in \mathrm{dom}(\mu)\},$$
$$\nu(\psi^{-1}(Y)) = \mu(Y) \quad (Y \in \mathrm{dom}(\mu)).$$

In this manner we obtain a σ-algebra $\mathcal{S}$ of subsets of $G_n = G$ and a nonseparable nonatomic σ-finite left-invariant measure ν on $\mathcal{S}$ (cf. the proof of Lemma 2). Therefore, we claim that our group G admits a nonseparable nonatomic σ-finite left-invariant measure.

2. The commutative group G_n/G_{n-1} is at most countable.

In this case, taking into account the uncountability of our group G_n, we claim that the subgroup G_{n-1} is uncountable, too. According to the inductive assumption, G_{n-1} admits a nonseparable nonatomic σ-finite left-invariant measure μ. Denote by $\{g_k : k \in K\}$ a selector of the countable family of sets G_n/G_{n-1}. For each index $k \in K$, we have a canonical bijection between the sets G_{n-1} and $g_k \cdot G_{n-1}$. By using this bijection, we transfer the measure μ to the set $g_k \cdot G_{n-1}$ and denote the obtained measure by μ_k. Let us stress that one and only one of the measures μ_k $(k \in K)$ coincides with the original measure μ.

Now, let us put:

$\mathcal{S}'$ = the family of all subsets X of G_n such that $X \cap g_k \cdot G_{n-1} \in \mathrm{dom}(\mu_k)$ for each $k \in K$.

It can easily be seen that $\mathcal{S}'$ is a σ-algebra of sets in G_n. Further, for any set $X \in \mathcal{S}'$, we define

$$\nu(X) = \sum_{k \in K} \mu_k(X \cap g_k \cdot G_{n-1}).$$

A direct verification shows that ν is a nonatomic σ-finite left-invariant measure on the group $G_n = G$. Finally, ν is nonseparable since it contains (as a part) the nonseparable measure μ. We thus conclude that our G admits a nonseparable nonatomic σ-finite left-invariant measure. □

This completes the proof of Lemma 3.

Lemma 4. *Assume the Continuum Hypothesis. Then every commutative group* $(H, +)$ *with* $\mathrm{card}(H) = \omega_1$ *admits a nonseparable nonatomic invariant probability measure.*

Proof. It is well known (see, e.g., [7]) that there exists a partition $\{H_\xi : \xi < \omega_1\}$ of H satisfying the following conditions:

(a) $\mathrm{card}(H_\xi) \le \omega$ for any ordinal $\xi < \omega_1$;

(b) for each subset Ξ of ω_1, the set $Z(\Xi) = \cup\{H_\xi : \xi \in \Xi\}$ is almost invariant in H, i.e., we have

$$(\forall h \in H)\big(\mathrm{card}((h + Z(\Xi)) \triangle Z(\Xi)) \le \omega\big).$$

Also, according to an old result on independent families of sets (due to Tarski), if E is a set of cardinality continuum, then there exists an uncountable family $\{E_i : i \in I\}$ of subsets of E such that:

(c) for any injective sequence $\{i_n : n < \omega\} \subset I$, we have

$$\mathrm{card}\left(\cap\{E'_{i_n} : n < \omega\}\right) = \mathbf{c},$$

where $\mathbf{c}$ denotes the cardinality continuum and

$$E'_{i_n} = E_{i_n} \quad \vee \quad E'_{i_n} = E \setminus E_{i_n} \quad (n < \omega).$$

Note that the existence of $\{E_i : i \in I\}$ can be established directly (by using the method of transfinite induction and without appealing to the above-mentioned result of Tarski).

In view of our assumption $\mathbf{c} = \omega_1$, we immediately obtain an analogous uncountable family $\{\Xi_i : i \in I\}$ of subsets of ω_1, i.e., the following relation holds:

(d) for any injective sequence $\{i_n : n < \omega\} \subset I$, we have

$$\mathrm{card}\left(\cap\{\Xi'_{i_n} : n < \omega\}\right) = \mathbf{c},$$

where

$$\Xi'_{i_n} = \Xi_{i_n} \quad \vee \quad \Xi'_{i_n} = \omega_1 \setminus \Xi_{i_n} \quad (n < \omega).$$

For every index $i \in I$, let us put

$$A_i = \cup\{H_\xi : \xi \in \Xi_i\}$$

and consider the family $\{A_i : i \in I\}$ of subsets of H. This family is independent, too, and each set A_i $(i \in I)$ is almost invariant in H (in view of (b)).

Let μ_0 denote the probability invariant measure on H which is defined on all countable and co-countable subsets of H and vanishes on all singletons in H. Applying the standard method of extending invariant measures (cf. [2], [4], [7]), we can extend μ_0 to a nonatomic invariant measure μ on H such that

$$\{A_i : i \in I\} \subset \mathrm{dom}(\mu),$$
$$d(A_i, A_j) = 1/2 \quad (i \in I,\ j \in I,\ i \neq j).$$

In particular, the last relation shows that μ is nonseparable. □

Lemma 4 has thus been proved.

Remark 2. Unfortunately, the role of the Continuum Hypothesis in Lemma 4 is unclear. In other words, we do not know whether it is possible to prove this lemma within the theory **ZFC**.

Remark 3. Lemma 4 can be easily generalized to those infinite groups $(H, \cdot)$ (not necessarily commutative) which satisfy the equality $\mathrm{card}(H)^\omega = \mathrm{card}(H)$. The proof can be carried out by the same method of independent families of sets. However, this method does not work for those uncountable groups $(H, \cdot)$ whose cardinality is cofinal with ω. Also, it is reasonable to point out in this place that, under the Generalized Continuum Hypothesis, the following two assertions are equivalent for an infinite group H:

(1) $\mathrm{card}(H)^{\omega} = \mathrm{card}(H)$;

(2) $\mathrm{card}(H)$ is not cofinal with ω.

Finally, let us recall that if H is an uncountable σ-compact locally compact topological group, then the equality $\mathrm{card}(H)^{\omega} = \mathrm{card}(H)$ is valid (see, for instance, [12]).

Taking into account Lemmas 1 - 4, we come to the following statement.

Theorem 1. *If the Continuum Hypothesis holds, then every uncountable solvable group admits a nonseparable nonatomic σ-finite left-invariant measure.*

We also have a certain analog of this theorem for commutative groups (within the theory **ZFC**).

Theorem 2. *Any commutative group $(G, +)$ with $\mathrm{card}(G) \geq \mathbf{c}$ admits a nonseparable nonatomic invariant probability measure.*

The proof of Theorem 2 follows from the corresponding analogs of Lemmas 1, 2 and 4.

Remark 4. In connection with the results presented in this paper, the following two open questions seem to be of interest.

I. Is it true that every uncountable group admits a nonseparable nonatomic σ-finite left-invariant measure? More generally, is it true that any nonzero σ-finite left-invariant measure given on an uncountable group admits a nonseparable left-invariant extension?

II. Is it true that every uncountable solvable group admits a nonseparable nonatomic σ-finite left-invariant measure having the uniqueness property? More generally, is it true that any nonzero σ-finite left-invariant measure with the uniqueness property on an uncountable solvable group admits a nonseparable left-invariant extension with the same property?

A detailed information about the uniqueness property for invariant measures can be found in [4], [7], [8].

Note that a nonseparable invariant extension of the Lebesgue measure, constructed by Kakutani and Oxtoby [2], does not possess this property. On the other hand, a nonseparable invariant extension of the Lebesgue measure, constructed by Kodaira and Kakutani [3], has the uniqueness property (for more details, see [8]).

References

1. E. Szpilrajn, On problems of the theory of measure. (Russian) *Uspekhi Mat. Nauk* (*N.S.*) **1**(1946), No. 2(12), 179–188.
2. K. Kakutani and J. Oxtoby, Construction of a nonseparable invariant extension of the Lebesgue measure space. *Ann. Math.* **52**(1950), 580–590.
3. K. Kodaira and S. Kakutani, A nonseparable translation-invariant extension of the Lebesgue measure space. *Ann. Math.* **52**(1950), 574–579.
4. E. Hewitt and K. A. Ross, Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory, group representations. *Die Grundlehren der mathematischen Wissenschaften, Bd.* 115 *Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg*, 1963.
5. S. S. Pkhakadze, The theory of Lebesgue measure. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze* **25**(1958), 3–271.
6. A. Hulanicki, Invariant extensions of the Lebesgue measure. *Fund. Math.* **51**(1962/1963), 111–115.
7. A. B. Kharazishvili, Invariant extensions of the Lebesgue measure. (Russian) *Tbilis. Gos. Univ., Tbilisi*, 1983.
8. A. B. Kharazishvili, Transformation groups and invariant measures. Set-theoretical aspects. *World Scientific Publishing Co., Inc., River Edge, NJ,* 1998.
9. G. Pantsulaia, An application of independent families of sets to the measure extension problem. *Georgian Math. J.* **11**(2004), No. 2.
10. A. G. Kurosh, The theory of groups. (Russian) *Izd. Nauka, Moscow*, 1967.
11. A. Kharazishvili and A. Kirtadze, On algebraic sums of absolutely negligible sets. *Proc. A. Razmadze Math. Institute*, **136**(2004), 55–61.
12. Handbook of set-theoretic topology, edited by K. Kunen and J.E. Vaughan, *North-Holland Publishing Co., Amsterdam*, 1984.

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