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ON A NUMERICAL SOLUTION OF THE TWO-DIMENSIONAL FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND

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Abstract. For a two-dimensional Fredholm integral equation of the second kind the problems of belonging of the exact solution to the Sobolev-Slobodetskii spaces are considered. The corresponding mesh scheme is constructed and the rate of convergence of an approximate solution to an averaged exact solution is estimated.

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Introduction

A great many problems of mathematical physics and engineering are reduced to the solution of linear and nonlinear integral equations. One can manage to find an exact solution of an integral equation analytically only in some individual cases, therefore the methods of approximate solution of integral equations do not give rise to doubts. Mesh schemes of numerical solution of integral and integro-differential equations with non-smooth input data in Sobolev spaces have been studied in [1-7].

In the present work we consider the two-dimensional Fredholm integral equation of the second kind. The sufficient conditions are established under which the exact solution $u(x)$ belongs to the Sobolev-Slobodetskii spaces. An a priori estimate of the solution is obtained.

The mesh schemes are constructed by means of special averaging the kernel and the right-hand side. It is proved that the rate of convergence of an approximate solution to an averaged exact solution is equal to h^m for $u \in W_2^m(\Omega)$, $0 \le m \le 2$.

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1. Notation and Auxiliary Results

In this section we introduce the notation and present the results which will be necessary for our further exposition [10-11].

Let R^2 be a two-dimensional Euclidean space of points $x = (x_1, x_2)$, $x' = (x_1', x_2'), \dots$ with the distance

$$|x - x'| = ((x_1 - x_1')^2 + (x_2 - x_2')^2)^{1/2}.$$

By Ω we denote the unit square in R^2:

$$\Omega = \{x = (x_1, x_2) : 0 < x_\alpha < 1, \quad \alpha = 1, 2\}.$$

The use will be made of the notation

$$D^\gamma \equiv \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \partial x_2^{\gamma_2}},$$

where $\gamma = (\gamma_1, \gamma_2)$ is a multi-index, γ_1, $\gamma_2 \geq 0$ are integers, $|\gamma| = \gamma_1 + \gamma_2$.

$W_2^m(\Omega)$ denotes the Sobolev-Slobodetskii space. For integers $m \geq 0$, the norm in $W_2^m(\Omega)$ is given by the formula

$$\|u\|_{W_2^m(\Omega)} = \left(\sum_{s=0}^{m} |u|^2_{W_2^s(\Omega)} \right)^{1/2},$$

where

$$|u|_{W_2^s(\Omega)} = \left(\sum_{|\gamma|=s} \|D^\gamma u\|^2_{L_2(\Omega)} \right)^{1/2}.$$

For $m = \overline{m} + \alpha$, where $\overline{m}$ is an integer and $0 < \alpha < 1$,

$$\|u\|_{W_2^m(\Omega)} = \left(\|u\|^2_{W_2^{\overline{m}}(\Omega)} + |u|^2_{W_2^m(\Omega)} \right)^{1/2},$$

where

$$\|u\|_{W_2^m(\Omega)} = \left(\sum_{|\gamma|=m} \int_\Omega \int_\Omega \frac{|D^\gamma u(x) - D^\gamma u(y)|^2}{|x-y|^{2+2\alpha}} dx dy \right)^{1/2}.$$

We will also consider an anisotropic Sobolev-Slobodetskii space $W_2^{m,0}(\Omega \times \Omega)$. For integers m, the norm in $W_2^{m,0}(\Omega \times \Omega)$ is defined by the equality

$$\|K\|_{W_2^{m,0}(\Omega\times\Omega)} = \left(\sum_{s=0}^{m} |K|_{W_2^{s,0}(\Omega\times\Omega)} \right)^{1/2},$$

where

$$|K|_{W_2^{s,0}(\Omega\times\Omega)} = \left(\sum_{|\gamma|=s} \int_\Omega \int_\Omega |D_x^\gamma K(x,t)|^2 dx dt \right)^{1/2},$$

and for $m = \overline{m} + \alpha$, $0 < \alpha < 1$, $\overline{m}$ is an integer,

$$\|K\|_{W_2^{m,0}(\Omega\times\Omega)} = \left(\|K\|_{W_2^{\overline{m},0}}(\Omega \times \Omega) + |K|_{W_2^{m,0}(\Omega\times\Omega)} \right)^{1/2},$$

where

$$|K|_{W_2^{m,0}(\Omega\times\Omega)} = \left(\sum_{|\gamma|=\overline{m}} \int_\Omega\int_\Omega\int_\Omega \frac{|D_x^\gamma K(x,\xi) - D_t^\gamma K(t,\xi)|^2}{|x-t|^{2+2\alpha}} dxdtd\xi\right)^{1/2}.$$

Introduce the notation $\omega_h = \{x_{ij} = (ih, jh),\ i,j = 1,2,\dots N-1,\ h = 1/N\}$ is the uniform mesh on Ω, where h is the step of the mesh ω_h, and x_{ij} is the node of the mesh ω_h.

For the mesh functions we use the notation

$$v_{ij} = v(ih, jh).$$

We introduce the mesh norm in $L_2(\Omega)$ on a two-dimensional mesh

$$\|v\|_{0,2,\omega_h} = \left\{h^2 \sum_{i,j=1}^{N} (v_{i,j})^2\right\}^{1/2}.$$

By $\|M\|$ is denoted the Euclidean norm of the matrix $M = (m_{ij})_{i,j=1}^n$,

$$\|M\| = \left(\sum_{i,j=1}^{n} m_{ij}^2\right)^{1/2}.$$

2. Statement of the Problem and Its Investigation

Let us consider the two-dimensional Fredholm integral equation of the second kind

$$u(x) = \lambda \int_\Omega K(x,\xi)u(\xi)d\xi + f(x), \quad x \in \Omega, \tag{1}$$

where λ is the real parameter which is not a characteristic number.

It is known [8] that for $f \in L_2(\Omega)$ and $K \in L_2(\Omega\times\Omega)$ there exists the unique solution $u \in L_2(\Omega)$ of equation (1) for which the estimate

$$\|u\|_{L_2(\Omega)} \le C\|f\|_{L_2(\Omega)} \tag{2}$$

is valid.

We prove the theorem on the smoothness of the solution of equation (1).

Theorem 1. *Let $f(x) \in W_2^m(\Omega)$, and the kernel satisfy the condition*

$$K \in W_2^{m,0}(\Omega\times\Omega).$$

Then the solution of equation (1) *belongs to the class $W_2^m(\Omega)$ for which the estimate*

$$\|u\|_{W_2^m(\Omega)} \le C\|f\|_{W_2^m(\Omega)}, \quad 0 \le m \le 2. \tag{3}$$

is valid.

Proof. Consider the integral operator

$$Tu(x)=\int\limits_{\Omega}K(x,\xi)u(\xi)d\xi.$$

and show that

$$\|Tu\|_{W_2^m(\Omega)}\leq\|K\|_{W_2^{m,0}(\Omega\times\Omega)}\|u\|_{L_2(\Omega)},\quad 0\leq m\leq 2. \tag{4}$$

Indeed, for $m=a$, $\alpha\in(0,1)$, we have

$$\|Tu\|_{W_2^\alpha(\Omega)}=\Bigg\{\int\limits_{\Omega}\bigg(\int\limits_{\Omega}K(x,\xi)u(\xi)d\xi\bigg)^2dx+$$
$$+\int\limits_{\Omega}\int\limits_{\Omega}\frac{\Big(\int\limits_{\Omega}K(x,\xi)u(\xi)d\xi-\int\limits_{\Omega}K(t,\xi)u(\xi)d\xi\Big)^2}{|x-t|^{2+2\alpha}}dxdt\Bigg\}^{1/2}\leq$$
$$\leq\|K\|_{W_2^{\alpha,0}(\Omega\times\Omega)}\|u\|_{L_2(\Omega)},$$

but if $m=1+\alpha$, $\alpha\in(0,1)$, then

$$\|Tu\|_{W_2^{1+\alpha}(\Omega)}=\Bigg\{\int\limits_{\Omega}\bigg(\int\limits_{\Omega}K(x,\xi)u(\xi)d\xi\bigg)^2dx+$$
$$+\int\limits_{\Omega}\bigg(\int\limits_{\Omega}\frac{\partial K(x,\xi)}{\partial x_1}u(\xi)d\xi\bigg)^2dx+\int\limits_{\Omega}\bigg(\int\limits_{\Omega}\frac{\partial K(x,\xi)}{\partial x_2}u(\xi)d\xi\bigg)^2dx+$$
$$+\int\limits_{\Omega}\int\limits_{\Omega}\frac{\Big(\int\limits_{\Omega}K'_{x_1}(x,\xi)u(\xi)d\xi-\int\limits_{\Omega}K'_{t_1}(t,\xi)u(\xi)d\xi\Big)^2}{|x-t|^{2+2\alpha}}dxdt+$$
$$+\int\limits_{\Omega}\int\limits_{\Omega}\frac{\Big(\int\limits_{\Omega}K'_{x_2}(x,\xi)u(\xi)d\xi-\int\limits_{\Omega}K'_{t_2}(t,\xi)u(\xi)d\xi\Big)^2}{|x-t|^{2+2\alpha}}dxdt\Bigg\}^{1/2},$$

i.e.,

$$\|Tu\|_{w_2^{1+\alpha}(\Omega)}\leq\|K\|_{W_2^{1+\alpha,0}(\Omega\times\Omega)}\|u\|_{L_2(\Omega)}=\|K\|_{W_2^{m,0}(\Omega\times\Omega)}\|u\|_{L_2(\Omega)}.$$

However, if $m=0,1,2$, estimate (4) becomes evident.

Next, from equation (1) it follows that

$$\|u\|_{W_2^m(\Omega)}\leq\|Tu\|_{W_2^m(\Omega)}+\|f\|_{W_2^m(\Omega)}. \tag{5}$$

Therefore, taking into account inequality (4), from (5) we obtain

$$\|u\|_{W_2^m(\Omega)}\leq\|K\|_{W_w^{m,0}(\Omega\times\Omega)}\|u\|_{L_2(\Omega)}+\|f\|_{W_2^m(\Omega)}. \tag{6}$$

Substituting into (6) the obtained from (2) estimate

$$\|u\|_{L_2(\Omega)} \le C\|f\|_{W_2^m(\Omega)}, \tag{7}$$

we get inequality (3).

Thus the theorem is complete. □

3. Construction of the Mesh Scheme

We proceed to constructing the mesh scheme. Towards this end, we transform equation (1),

$$\frac{1}{h^2}\int\limits_{\Omega_{il}} u(x)dx = \frac{\lambda}{h^2}\sum_{j,k=1}^{N}\int\limits_{\Omega_{il}}\int\limits_{\Omega_{jk}}(K(x,\xi)+\frac{1}{h^2}\int\limits_{\Omega_{jk}}K(x,s)ds-$$
$$-\frac{1}{h^2}\int\limits_{\Omega_{jk}}K(x,s)ds)u(\xi)dxd\xi+\frac{1}{h^2}\int\limits_{\Omega_{il}}f(x)dx, \quad i,l=1,2,\dots,N,$$

and hence

$$\frac{1}{h^2}\int\limits_{\Omega_{il}} u(x)dx = \lambda\sum_{j,k=1}^{N} h^2 A_{il}^{jk}\Bigg(\frac{1}{h^2}\int\limits_{\Omega_{jk}}u(\xi)d\xi\Bigg)+$$
$$+\frac{1}{h^2}\int\limits_{\Omega_{il}}f(x)dx+\lambda R_{il}, \quad i,l=1,2,\dots,N. \tag{8}$$

where

$$R_{il} = \frac{1}{h^2}\sum_{j,k=1}^{N}\int\limits_{\Omega_{il}}\int\limits_{\Omega_{jk}}(K(x,\xi)-\frac{1}{h^2}\int\limits_{\Omega_{jk}}K(x,s)ds)u(\xi)dxd\xi,$$
$$A_{il}^{jk} = \frac{1}{h^4}\int\limits_{\Omega_{il}}\int\limits_{\Omega_{jk}}K(k,s)dxds, \quad i,j,l,k=1,2,\dots,N.$$

Rejecting the residual, we reduce expression (8) to the linear algebraic system

$$v_{il} = \lambda\sum_{j,k=1}^{N} h^2 A_{il}^{jk}v_{jk}+\frac{1}{h^2}\int\limits_{\Omega_{il}}f(x)dx, \quad i,l=1,2,\dots,N. \tag{9}$$

Subtracting now (9) from (8), for the error

$$z_{il} = \frac{1}{h^2}\int\limits_{\Omega_{il}}u(x)dx - v_{il}, \quad i,l=1,2,\dots,N \tag{10}$$

we have

$$(I-A)Z = \lambda R, \tag{11}$$

where

$$Z=(z_1,\dots,z_N),\quad R=(R_1,\dots,R_N);\quad A=(\lambda h^2 B_{ij})_{i,j=1}^N,$$

but

$$z_i=(z_{i1},\dots,z_{1N});\quad R_i=(R_{i1},\dots,R_{iN});\quad B_{ij}=(A_{il}^j k)_{l,k=1}^N,$$

and I is the unit matrix.

The lemma below proves that the matrix $I-A$ is nondegenerated.

Lemma 1. *If the kernel $K(x,\xi)$ is symmetric and satisfies the condition*

$$|\lambda|\,\|K\|_{L_2(\Omega\times\Omega)}<1, \tag{12}$$

then the matrix $I-A$ is nondegenerated, invertible, and the estimate

$$\|(I-A)^{-1}\|\le\frac{1}{1-q},$$

where $q=\|A\|<1$, is valid.

Proof. The equality

$$\|A\|^2=\sum_{i,l=1}^N\sum_{j,k=1}^N\lambda^2h^4(A_{il}^{jk})^2=\sum_{i,l=1}^N\sum_{j,k=1}^N\frac{\lambda^2}{h^4}\Big(\int\limits_{\Omega_{il}}\int\limits_{\Omega_{jk}}K(x,s)dxds\Big)^2$$

is true.

Using the Cauchy-Bunjakovskii's inequality, we have

$$\|A\|^2\le\sum_{i,l=1}^N\sum_{j,k=1}^N\lambda^2\int\limits_{\Omega_{il}}\int\limits_{\Omega_{jk}}K^2(x,s)dxds=\lambda^2\int\limits_{\Omega}\int\limits_{\Omega}K^2(x,s)dxds,$$

or, by means of the condition (12) we obtain

$$\|A\|<1.$$

Thus by Theorem 7.1.1 (see [9], p. 195) we can prove the validity of Lemma 1. □

As the corollary of the above lemma it follows that the problem (11) is uniquely solvable, and for its solution

$$\|Z\|_{0,2,\omega_h}\le|\lambda|(1-q)^{-1}\|R\|_{0,2,\omega_h}. \tag{13}$$

4. Error Estimation

In this section we will estimate a discrete norm of the error (10) which is, in its turn, a solution of the system (11). The norm of the error $\|Z\|_{0,2,\omega_h}$ is estimated in the following two theorems.

Theorem 2. *Let the kernel $K(x,t)$ be symmetric and satisfy the condition* (12)*. Moreover, let*

$$f \in W_2^{\alpha}(\Omega), \quad K \in W_2^{\alpha,0}(\Omega \times \Omega), \quad 0 < \alpha \leq 1.$$

Then the convergence of the mesh scheme (9) *is characterized by the estimate*

$$\|Z\|_{0,2,\omega_h} \leq Ch^{\alpha}|K|_{W_2^{(\alpha,0)}(\Omega\times\Omega)}\|f\|_{L_2(\Omega)}. \tag{14}$$

Proof. Let us estimate the norm $\|R\|_{0,2,\omega_h}$. To this end, we use the Cauchy-Bunjakovskii's inequality and obtain

$$\|R\|^2_{0,2,\omega_h} = h^2 \sum_{i,l=1}^{N} (R_{il})^2 = h^2 \sum_{i,l=1}^{N} \Big\{ \frac{1}{h^2} \sum_{j,k=1}^{N} \int\limits_{\Omega_{il}} \int\limits_{\Omega_{jk}} \int\limits_{\Omega_{jk}} (K(x,\xi)-$$

$$-K(x,s))u(\xi)dxd\xi ds \Big\}^2 \leq \frac{1}{h^6} \sum_{i,l=1}^{N} \Big\{ \sum_{j,k=1}^{N} \Big(\int\limits_{\Omega_{il}} \int\limits_{\Omega_{jk}} \int\limits_{\Omega_{jk}} (K(x,\xi)-$$

$$-K(x,s))^2 dxd\xi ds \Big)^{1/2} \Big(\int\limits_{\Omega_{il}} \int\limits_{\Omega_{jk}} \int\limits_{\Omega_{jk}} u^2(\xi)dxd\xi ds \Big)^{1/2} \Big\}^2.$$

Applying now the Cauchy-Bunjakovskii's inequality for the sum, we find that

$$\|R\|^2_{0,2,\omega_h} \leq \frac{1}{h^6} \sum_{i,l=1}^{N} \sum_{j,k=1}^{N} \int\limits_{\Omega_{il}} \int\limits_{\Omega_{jk}} \int\limits_{\Omega_{jk}} \frac{(K(x,\xi)-K(x,s))^2}{|\xi - s|^{2+2\alpha}} |\xi - s|^{2+2\alpha} dxd\xi ds \times$$

$$\times \sum_{j,k=1}^{N} \int\limits_{\Omega_{il}} \int\limits_{\Omega_{jk}} \int\limits_{\Omega_{jk}} u^2(\xi)d\xi \leq h^{2\alpha} \int\limits_{\Omega} \int\limits_{\Omega} \int\limits_{\Omega} \frac{|K(x,\xi)-K(x,s)|^2}{|\xi - s|^{2+2\alpha}} dxd\xi ds \|u\|^2_{L_2(\Omega)},$$

or, taking into account the fact that the kernel $K(x,y)$ is symmetric,

$$\|R\|_{0,2,\omega_h} \leq h^{\alpha}|K|_{W_2^{\alpha,0}(\Omega\times\Omega)}\|u\|_{L_2(\Omega)}. \tag{15}$$

Substituting (15) into (13) and taking into account estimate (2), we obtain (14).

Thus the theorem is complete. □

Theorem 3. *Let the kernel $K(x,t)$ be symmetric and satisfy the condition* (12)*. Moreover, let*

$$f \in W_2^{1+\alpha(\Omega)}, \quad K \in W_2^{1+\alpha,0}(\Omega \times \Omega).$$

Then the convergence of the mesh scheme is characterized by the estimate

$$\|Z\|_{0,2,\omega_h} \leq C_1 h^{1+\alpha}\|K\|_{W_2^{1+\alpha,0}(\Omega\times\Omega)}\|f\|_{W_2^{1+\alpha}(\Omega)}, \tag{16}$$

where $C_1 = |\lambda|(1-q)^{-1}2^{2+\alpha/2}$, $0 < \alpha \leq 1$.

Proof. We transform the residual of the system (8) and obtain

$$R_{il}=R_{il}^{(1)}+R_{il}^{(2)},$$

where

$$R_{il}^{(1)}=\frac{1}{h^6}\sum_{j,k=1}^{N}\int\limits_{\Omega_{jk}}u(t)dt\int\limits_{\Omega_{il}}\int\limits_{\Omega_{jk}}\int\limits_{\Omega_{jk}}(K(x,\xi)-K(x,s))dxd\xi ds,$$

$$R_{il}^{(2)}=\frac{1}{h^6}\sum_{j,k=1}^{N}\int\limits_{\Omega_{il}}\int\limits_{\Omega_{jk}}\int\limits_{\Omega_{jk}}\int\limits_{\Omega_{jk}}(K(x,\xi)-K(x,s))(u(\xi)-u(t))dxd\xi dsdt.$$

Estimating the norm $\|R\|_{0,2,\omega_h}$, we find that

$$\|R\|_{0,2,\omega_h}^2\le 2h^2\sum_{i,j=1}^{N}(R_{il}^{(1)})^2+2h^2\sum_{i,l=1}^{N}(R_{il}^{(2)})^2. \tag{17}$$

In (17) we estimate $h^2\sum\limits_{i,l=1}^{N}(R_{il}^{(1)})^2$. Since the expression

$$K(x;\xi_1,\xi_2)-K(x;s_1,s_2)=K(x;\xi_1,\xi_2)-K(x;s_1,\xi_2)+K(x;x_1,\xi_2)-$$
$$-K(x;s_1,s_2)=\int\limits_{s_1}^{\xi_1}\frac{\partial K(x;\rho,\xi_2)}{\partial\rho}+\int\limits_{s_2}^{\xi_2}\frac{\partial K(x;s_1,\eta)}{\partial\eta}d\eta$$

is valid, we have

$$\int\limits_{\Omega_{il}}\int\limits_{\Omega_{jk}}(K(x;\xi)-K(x;s))d\xi ds=$$
$$=\int\limits_{\Omega_{jk}}\int\limits_{\Omega_{jk}}\Bigg\{\int\limits_{s_1}^{\xi_1}\Bigg(\frac{\partial K(x;\rho_1,\xi_2)}{\partial\rho_1}-\frac{1}{h}\int\limits_{(j-1)h}^{jh}\frac{\partial K(x;\eta_1,s_2)}{\partial\eta_1}d\eta_1\Bigg)d\rho_1+$$
$$+\int\limits_{s_2}^{\xi_2}\Bigg(\frac{\partial K(x;s_1,\eta_2)}{\partial\eta_2}-\frac{1}{h}\int\limits_{(k-1)h}^{kh}\frac{\partial K(x,\xi_1,\rho_2)}{\partial\rho_2}d\rho_2\Bigg)d\eta_2\Bigg\}d\xi ds.$$

Here we use the obvious identity

$$\int\limits_{\Omega_{jk}}\int\limits_{\Omega_{jk}}\Bigg(\int\limits_{s_1}^{\xi_1}d\rho_1\Bigg)d\xi ds=\int\limits_{\Omega_{jk}}\int\limits_{jk}\Bigg(\int\limits_{s_2}^{\xi_2}d\eta_2\Bigg)d\eta ds=0.$$

Consequently, we have

$$h^2\sum_{i,l=1}^{N}(R_{il}^{(1)})^2=$$

$$=h^2\sum_{i,l=1}^{N}\Big(\frac{1}{h^6}\sum_{j,k=1}^{N}\int\limits_{\Omega_{jk}}u(t)dt\int\limits_{\Omega_{il}}\int\limits_{\Omega_{jk}}\int\limits_{\Omega_{jk}}\Big[\frac{1}{h}\int\limits_{s_1}^{\xi_1}\int\limits_{(j-1)h}^{jh}\Big(\frac{\partial K(x;\rho_1,\xi_2)}{\partial\rho_1}-$$

$$-\frac{\partial K(x;\eta_1,s_2)}{\partial\eta_1}\Big)d\eta_1d\rho_1+$$

$$+\frac{1}{h}\int\limits_{s_2}^{\xi_2}\int\limits_{(k-1)h}^{kh}\Big(\frac{\partial K(x;s_1,\eta_2)}{\partial\eta_2}-\frac{\partial K(x;\xi_1,\rho_2)}{\partial\rho_2}\Big)d\xi_2d\eta_2\Big]dxd\xi ds\Big)^2\le$$

$$\le\frac{1}{h^4}\sum_{i,l=1}^{N}\Big(\sum_{j,k=1}^{N}\int\limits_{\Omega_{jk}}u^2(t)dt\sum_{j,k=1}^{N}\int\limits_{\Omega_{il}}\int\limits_{\Omega_{jk}}\int\limits_{\Omega_{jk}}\Big(\int\limits_{s_1}^{\xi}\int\limits_{(j-1)h}^{jh}\Big(\frac{\partial K(x;\rho_1,\xi_2)}{\partial\rho_1}-$$

$$-\frac{\partial K(x;\eta_1,s_2)}{\partial\eta_1}\Big)d\eta_1d\rho_1+$$

$$+\int\limits_{s_2}^{\xi_2}\int\limits_{(k-1)h}^{kh}\Big(\frac{\partial K(x;s_1,\eta_2)}{\partial\eta_2}-\frac{\partial K(x;\xi_1,\rho_2)}{\partial\rho_2}\Big)d\rho_2d\eta_2\Big)^2dxd\xi ds\le\frac{2}{h^2}\int\limits_{\Omega}u^2(t)dt\times$$

$$\times\sum_{i,l=1}^{N}\sum_{j,k=1}^{N}\int\limits_{\Omega_{il}}\int\limits_{\Omega_{jk}}\int\limits_{\Omega_{jk}}\Big[\int\limits_{s_1}^{\xi_1}\int\limits_{(j-1)h}^{jh}\frac{(\frac{\partial K(x;\rho_1,\xi_2)}{\partial\rho_1}-\frac{\partial K(x;\eta_1,s_2)}{\partial\eta_1})^2}{|(\rho_1-\eta_1)^2++(\xi_2-s_2)^2|^{1+\alpha}}|(\rho_1-\eta_1)^2+$$

$$+(\xi_2-s_2)^2|^{1+\alpha}\Big]d\eta_1d\rho_1+$$

$$+\int\limits_{s_2}^{\xi_2}\int\limits_{(k-1)h}^{kh}\Big(\frac{(\frac{\partial K(x;s_1,\eta_2)}{\partial\eta_2}-\frac{\partial K(x;\xi_1,\rho_2)}{\partial\rho_2})^2}{|(s_1-\xi_1)^2+(\eta_2-\rho_2)^2|^{1+\alpha}}|(s_1-\xi_1)^2+$$

$$+(\eta_2-\rho_2)^2|^{1+\alpha}d\eta_2d\rho_2\Big)dxd\xi ds,$$

i.e.,

$$h^2\sum_{i,l=1}^{N}(R_{il}^{(1)})^2\le 2^{2+\alpha}h^{2+2\alpha}\int\limits_{\Omega}u^2(t)dt\int\limits_{\Omega}\int\limits_{\Omega}\int\limits_{\Omega}\Big\{\Big(\frac{\partial K(x;\xi)}{\partial\xi_1}-\frac{\partial K(x;s)}{\partial s_1}\Big)^2+$$

$$+\Big(\frac{\partial K(x;\xi)}{\partial\xi_2}-\frac{\partial K(x,s)}{\partial s_2}\Big)^2\Big\}|\xi-s|^{-2-2\alpha}dxd\xi ds.$$

Taking into account the fact that the kernel is symmetric, we finally obtain

$$h^2\sum_{i,j=1}^{N}(R_{il}^{(1)})^2\le 2^{\alpha+2}h^{2+2\alpha}\|K\|^2_{W_2^{1+\alpha,0}(\Omega\times\Omega)}\|u\|^2_{L_2(\Omega)}. \tag{18}$$

Estimate now $h^2\sum_{i,l=1}^{N}(R_{il}^{(2)})^2$. Applying here the Cauchy-Bunjakovskii's inequality, we arrive at

$$h^2\sum_{i,l=1}^{N}(R_{il}^{(2)})^2\le\frac{1}{h^{10}}\sum_{i,l=1}^{N}\sum_{j,k=1}^{N}\int\limits_{\Omega_{il}}\int\limits_{\Omega_{jk}}\int\limits_{\Omega_{jk}}\int\limits_{\Omega_{jk}}(K(x,\xi)-K(x;s))^2dxd\xi dsdt\times$$
$$\times\sum_{j,k=1}^{N}\int\limits_{\Omega_{il}}\int\limits_{\Omega_{jk}}\int\limits_{\Omega_{jk}}\int\limits_{\Omega_{jk}}(u(\xi)-u(t))^2dxd\xi dsdt\le$$
$$\le\frac{1}{h^4}\sum_{i,l=1}^{N}\sum_{j,k=1}^{N}\int\limits_{\Omega_{il}}\int\limits_{\Omega_{jk}}\int\limits_{\Omega_{jk}}\frac{(K(x;\xi)-K(x;t))^2}{|\xi-t|^{2+2\alpha}}|\xi-t|^{2+2\alpha}dxd\xi dt\times$$
$$\times\sum_{j,k=1}^{N}\int\limits_{\Omega_{jk}}\int\limits_{\Omega_{jk}}(u(\xi)-u(t))^2d\xi dt\le$$
$$\le\frac{2^{1+\alpha}}{h^4}h^{2+2\alpha}\int\limits_{\Omega}\int\limits_{\Omega}\int\limits_{\Omega}\frac{(K(x,\xi)-K(x;t))^2}{|\xi-t|^{2+2\alpha}}dxd\xi dt\times$$
$$\times\sum_{j,k=1}^{N}\int\limits_{\Omega_{jk}}\int\limits_{\Omega_{jk}}(u(\xi)-u(t))^2d\xi dt. \tag{19}$$

In inequality (19) we transform the difference $u(\xi)-u(t)$ for which we have

$$u(\xi)-u(t)=\int\limits_{t_1}^{\xi_1}\frac{\partial u(\eta,\xi_2)}{\partial\eta}d\eta+\int\limits_{t_2}^{\xi_2}\frac{\partial u(t_1,\rho)}{\partial\rho}d\rho.$$

Hence (19) yields

$$h^2\sum_{i,l=1}^{N}(R_{il}^{(2)})^2\le\frac{2^{1+\alpha}}{h^4}h^{2+2\alpha}\int\limits_{\Omega}\int\limits_{\Omega}\int\limits_{\Omega}\frac{(K(x;\xi)-K(x;t))^2}{|\xi-t|^{2+2\alpha}}dxd\xi dt\times$$
$$\times\sum_{j,k=1}^{N}\int\limits_{\Omega_{jk}}\int\limits_{\Omega_{jk}}\Big(\int\limits_{t_1}^{\xi_1}\frac{\partial u(\rho_1,\xi_2)}{\partial\rho_1}d\rho_1+\int\limits_{t_2}^{\xi_2}\frac{\partial u(t_1,\eta_2)}{\partial\eta_2}d\eta_2\Big)^2d\xi dt\le$$

$$\leq 2^{2+\alpha}h^{2+2\alpha}\int\limits_{\Omega}\int\limits_{\Omega}\int\limits_{\Omega}\frac{(K(x,\xi)-K(x;s))^2}{|\xi-s|^{2+2\alpha}}dxd\xi ds\times$$

$$\times\sum_{j,k=1}^{N}\int\limits_{\Omega_{j,k}}\Big(\Big(\frac{\partial u(\xi)}{\partial\xi_1}\Big)^2+\Big(\frac{\partial u(\xi)}{\partial\xi_2}\Big)^2\Big)d\xi,$$

i.e.,

$$h^2\sum_{i,j=1}^{N}(R_{il}^{(2)})^2\leq 2^{2+\alpha}h^{2+2\alpha}\|K\|^2_{W_2^{\alpha,0}(\Omega\times\Omega)}|u|^2_{W_2^1(\Omega)}. \tag{20}$$

Using estimates (18) and (20), inequality (17) is transformed into

$$\|R\|_{0,2,\omega_h}\leq$$

$$\leq 2^{\frac{\alpha+3}{2}}h^{1+\alpha}\Big\{\|K\|^2_{W_2^{1+\alpha,0}(\Omega\times\Omega)}\|u\|^2_{L_2(\Omega)}+\|K\|^2_{W_2^{\alpha,0}(\Omega\times\Omega)}|u|^2_{W_2^1(\Omega)}\Big\}^{1/2}. \tag{21}$$

Therefore if we take into account estimates (3) and (21), then from (13) we obtain (16).

The theorem has been proved. □

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