Proceedings of A. Razmadze
Mathematical Institute
Vol. **139** (2005), 5–31

TWO-WEIGHTED INEQUALITY FOR SOME SUBLINEAR OPERATORS IN LEBESGUE SPACES, ASSOCIATED WITH THE LAPLACE-BESSEL DIFFERENTIAL OPERATORS

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Abstract. In this paper several general theorems for the boundedness of sublinear operators, associated with the Laplace–Bessel differential operator on a weighted Lebesgue space are established. Sufficient conditions on weighted functions ω and ω_1 are given so that certain sublinear operator is bounded from the weighted Lebesgue spaces $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$ into $L_{p,\omega_1,\gamma}(\mathbb{R}^n_+)$.

რეზიუმე. ნაშრომში დადგენილია საკმარისი პირობები წონათა წყვილზე, რომლებიც უზრუნველყოფს ლაპლას-ბესელის დიფერენციალურ ოპერატორთან დაკავშირებული ნახევრადწრფივი ოპერატორის შემოსაზღვრულობას წონიან ლებეგის სივრცეებში.

1. Introduction

The singular integral operators that have been considered by Mihlin [21] and Calderon and Zygmund [9] are playing an important role in the theory Harmonic Analysis and in particular, in the theory of partial differential equations. Klyuchantsev [19] and Kipriyanov and Klyuchantsev [20] have firstly introduced and investigated by the boundedness in L_p-spaces of multidimensional singular integrals, generated by the Laplace-Bessel differential operator $\Delta_{B_n} = \sum\limits_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + B_n$, $B_n = \frac{\partial^2}{\partial x_n^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}$, $\gamma > 0$ (B_n singular integrals). Aliev and Gadjiev [5] and Gadjiev and Guliyev [7] have studied the boundedness of B_n singular integrals in weighted L_p-spaces with radial and general weights consequently. The maximal functions, singular integrals, potentials and related topics associated with the Laplace-Bessel differential operator Δ_{B_n} which is known as an important differential operator in analysis and its applications, have been the research areas many mathematicans such as I. Kipriyanov and M. Klyuchantsev [19]–[20], L. Lyakhov [23]–[24],

2000 *Mathematics Subject Classification.* 42B25.
Key words and phrases. Weighted Lebesgue space, sublinear operator, singular integral, Laplace–Bessel differential operator, two-weighted inequality.

I. A. Gadjiev and I. A. Aliev [6]–[5], I. A. Aliev, S. Bayrakci [3, 4], V. S. Guliyev [12]–[14] and others.

In the paper we shall prove the boundedness of some sublinear operators, generated by the B_n Bessel differential operators on a weighted L_p spaces. Sufficient conditions on weighted functions ω and ω_1 are given so that certain sublinear operator is bounded from the weighted Lebesgue spaces $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$ into $L_{p,\omega_1,\gamma}(\mathbb{R}^n_+)$. The condition (2) (see below) is satisfied by many interesting operators in harmonic analysis, such as the B_n singular integrals (for example, see [19, 20]), B_n Hardy–Littlewood maximal operators (see also [12, 14]– [25]) and so on.

2. Notations and Background

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space, $x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$ are vectors in $\mathbb{R}^n$, $(x, \xi) = x_1\xi_1 + \ldots + x_n\xi_n$, $|x| = \sqrt{(x,x)}$. Let $\mathbb{R}^n_+ = \{x = (x_1, \dots x_n) : \ x_n > 0\}$, $\gamma > 0$. $E(x,r) = \{y \in R^n_+ : \ |x-y| < r\}$, $\Sigma_+ = \{x \in \mathbb{R}^n_+ : \ |x| = 1\}$.

For measurable set $E \subset \mathbb{R}^n_+$ let $|E|_\gamma = \int_E x_n^\gamma dx$. Then $|E(0,r)|_\gamma = \omega(n,\gamma) r^{n+\gamma}$, where $\omega(n,\gamma) = |E(0,1)|_\gamma$.

An almost everywhere positive and locally integrable function $\omega : \ \mathbb{R}^n_+ \to \mathbb{R}$ will be called a weight. We shall denote by $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$ the set of all measurable function f on $\mathbb{R}^n_+$ such that the norm

$$\|f\|_{L_{p,\omega,\gamma}(\mathbb{R}^n_+)} \equiv \|f\|_{p,\omega,\gamma;\mathbb{R}^n_+} = \left(\int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p \omega(x) x_n^\gamma \, dx \right)^{1/p}, \quad 1 \le p < \infty,$$

is finite. For $\omega = 1$ the space $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$ is denoted by $L_{p,\gamma}(\mathbb{R}^n_+)$, and the norm $\|f\|_{L_{p,\omega,\gamma}(\mathbb{R}^n_+)}$ by $\|f\|_{L_{p,\gamma}(\mathbb{R}^n_+)}$.

The operator of generalized shift (B_n shift operator) is defined by the following way (see [19], [22]):

$$T^y f(x) = C_\gamma \int\limits_0^\pi f\big(x' - y', \sqrt{x_n^2 - 2x_n y_n \cos\alpha + y_n^2}\big) \sin^{\gamma-1} \alpha d\alpha,$$

where $C_\gamma = \pi^{-\frac{1}{2}} \Gamma(\gamma + \frac{1}{2}) \Gamma^{-1}(\gamma)$.

Note that this shift operator is closely connected with B_n Bessel's singular differential operators (see [19], [22]).

The translation operator T^y generated the corresponding B_n-convolution

$$(f \otimes g)(x) = \int\limits_{\mathbb{R}^n_+} f(y) \big[T^y g(x)\big] y_n^\gamma \, dy,$$

for which the Young inequality

$$\left\|f \otimes g\right\|_{L_{r,\gamma}} \le \left\|f\right\|_{L_{p,\gamma}} \left\|g\right\|_{L_{q,\gamma}}, \quad 1 \le p, q, r \le \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

holds.

Lemma 1 ([5, 23]). *Let $1 \le p \le \infty$. Then for all $y \in \mathbb{R}^n_+$, $T^y f$ belongs $L_{p,\gamma}(\mathbb{R}^n_+)$ and*

$$\left\|T^y f(\cdot)\right\|_{L_{p,\gamma}} \le \left\|f\right\|_{L_{p,\gamma}}. \tag{1}$$

Definition 1. A function K defined on $\mathbb{R}^n_+$, is said to be B_n singular kernel in the space $\mathbb{R}^n_+$ if

i) $K \in C^\infty(\mathbb{R}^n_+)$;

ii) $K(rx) = r^{-n-\gamma} K(x)$ for each $r > 0$, $x \in \mathbb{R}^n_+$;

iii) $\int_\Sigma K(x) x_n^\gamma d\sigma(x) = 0$, where $d\sigma$ is the element of area of the Σ_+.

First, we establish the boundedness in weighted L_p spaces for a large class of sublinear operators, generated by the B_n Bessel differential operators.

Theorem 1. *Let $p \in (1, \infty)$ and let T be a sublinear bounded operator from $L_{p,\gamma}(\mathbb{R}^n_+)$ to $L_{p,\gamma}(\mathbb{R}^n_+)$ such that for any $f \in L_{1,\gamma}(\mathbb{R}^n_+)$ with compact support and $x \notin \operatorname{supp} f$*

$$\left|Tf(x)\right| \le c_0 \int_{\mathbb{R}^n_+} T^y |x|^{-n-\gamma} \left|f(y)\right| y_n^\gamma \, dy, \tag{2}$$

where c_0 is independent of f and x.

Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n_+$ and the following three conditions are satisfied:

(a) *there exist $b > 0$ such that*

$$\sup_{|x|/4 < |y| \le 4|x|} \omega_1(y) \le b\, \omega(x) \quad \text{for a.e. } x \in \mathbb{R}^n_+,$$

(b)

$$\mathcal{A} \equiv \sup_{r>0} \left(\int_{\mathbb{R}^n_+ \setminus E(0,2r)} \omega_1(x) |x|^{-(n+\gamma)p} x_n^\gamma dx \right) \left(\int_{E(0,r)} \omega^{1-p'}(x) x_n^\gamma \, dx \right)^{p-1} < \infty,$$

(c)

$$\mathcal{B} \equiv \sup_{r>0} \left(\int_{E(0,r)} \omega_1(x) x_n^\gamma dx \right) \left(\int_{\mathbb{R}^n_+ \setminus E(0,2r)} \omega^{1-p'}(x) |x|^{-(n+\gamma)p'} x_n^\gamma dx \right)^{p-1} < \infty.$$

Then there exists a constant c, independent of f, such that for all $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_+)$

$$\int_{\mathbb{R}^n_+} \left|Tf(x)\right|^p \omega_1(x) x_n^\gamma \, dx \le c \int_{\mathbb{R}^n_+} \left|f(x)\right|^p \omega(x) x_n^\gamma \, dx. \tag{3}$$

Moreover, condition (a) can be replaced by the condition

(a') *there exist $b > 0$ such that*

$$\omega_1(x)\Big(\sup_{|x|/4\le|y|\le4|x|}\frac{1}{\omega(y)}\Big)\le b \quad \text{for a.e. } x\in\mathbb{R}^n.$$

Proof. For $k\in Z$ we define $E_k=\{x\in\mathbb{R}^n_+:\ 2^k<|x|\le 2^{k+1}\}$, $E_{k,1}=\{x\in\mathbb{R}^n_+:\ |x|\le 2^{k-1}\}$, $E_{k,2}=\{x\in\mathbb{R}^n_+:\ 2^{k-1}<|x|\le 2^{k+2}\}$, $E_{k,3}=\{x\in\mathbb{R}^n_+:\ |x|>2^{k+2}\}$. Then $E_{k,2}=E_{k-1}\cup E_k\cup E_{k+1}$ and the multiplicity of the covering $\{E_{k,2}\}_{k\in Z}$ is equal to 3.

Given $f\in L_{p,\omega,\gamma}(\mathbb{R}^n_+)$, we write

$$\begin{aligned}\big|Tf(x)\big|=\sum_{k\in Z}\big|Tf(x)\big|\chi_{E_k}(x)\le\sum_{k\in Z}\big|Tf_{k,1}(x)\big|\chi_{E_k}(x)+\\+\sum_{k\in Z}\big|Tf_{k,2}(x)\big|\chi_{E_k}(x)+\sum_{k\in Z}\big|Tf_{k,3}(x)\big|\chi_{E_k}(x)\equiv\\ \equiv T_1f(x)+T_2f(x)+T_3f(x),\end{aligned}\tag{4}$$

where χ_{E_k} is the characteristic function of the set E_k, $f_{k,i}=f\chi_{E_{k,i}}$, $i=1,2,3$.

First we estimate $\|T_1f\|_{L_{p,\omega_1,\gamma}}$. Note that for $x\in E_k$, $y\in E_{k,1}$ we have $|y|\le 2^{k-1}\le|x|/2$. Moreover, $E_k\cap\operatorname{supp}f_{k,1}=\emptyset$ and $|x-y|\ge|x|/2$. Consequently by (2)

$$\begin{aligned}T_1f(x)&\le c_0\sum_{k\in Z}\Bigg(\int_{\mathbb{R}^n_+}T^y|x|^{-n-\gamma}\ |f_{k,1}(y)|y_n^\gamma\,dy\Bigg)\chi_{E_k}\le\\ &\le c_0\int_{\{y\in\mathbb{R}^n_+:\ |y|\le|x|/2\}}|x-y|^{-n-\gamma}\big|f(y)\big|\ y_n^\gamma\,dy\le\\ &\le 2^{n+\gamma}c_0|x|^{-n-\gamma}\int_{\{y\in\mathbb{R}^n_+:\ |y|\le|x|/2\}}\big|f(y)\big|\ y_n^\gamma\,dy\end{aligned}$$

for any $x\in E_k$. Hence, we have

$$\begin{aligned}&\int_{\mathbb{R}^n_+}\big|T_1f(x)\big|^p\omega_1(x)x_n^\gamma\,dx\le\\ &\le\big(2^{n+\gamma}c_0\big)^p\int_{\mathbb{R}^n_+}\Bigg(\int_{\{y\in\mathbb{R}^n_+:\ |y|<|x|/2\}}\big|f(y)\big|\ y_n^\gamma\,dy\Bigg)^p|x|^{-(n+\gamma)p}\omega_1(x)x_n^\gamma\,dx.\end{aligned}$$

Since $\mathcal{A} < \infty$, the Hardy inequality

$$\int\limits_{\mathbb{R}^n_+} \omega_1(x)|x|^{-(n+\gamma)p}\Bigg(\int\limits_{\{y\in\mathbb{R}^n_+:\ |y|<|x|/2\}} \big|f(y)\big|\ y_n^\gamma\,dy\Bigg)^p x_n^\gamma\,dx \le$$

$$\le C\int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p\omega(x)x_n^\gamma\,dx$$

holds and $C \le c'\mathcal{A}$, where c' depends only on n and p. In fact the condition $\mathcal{A} < \infty$ is necessary and sufficient for the validity of this inequality (see [2], [16]). Hence, we obtain

$$\int\limits_{\mathbb{R}^n_+} \big|T_1 f(x)\big|^p\omega_1(x)\ x_n^\gamma\,dx \le c_1\int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p\omega(x)\ x_n^\gamma\,dx. \tag{5}$$

where c_1 is independent of f.

Next we estimate $\|T_3 f\|_{L_{p,\omega_1,\gamma}}$. As it is easy to verify, for $x \in E_k$, $y \in E_{k,3}$ we have $|y| > 2|x|$ and $|x-y| \ge |y|/2$. Since $E_k \cap \operatorname{supp} f_{k,3} = \emptyset$, for $x \in E_k$ by (2) we obtain

$$T_3 f(x) \le c_0 \int\limits_{\{y\in\mathbb{R}^n_+:\ |y|>2|x|\}} T^y|x|^{-n-\gamma}\big|f(y)\big|\ y_n^\gamma\,dy \le$$

$$\le 2^{n+\gamma}c_0 \int\limits_{\{y\in\mathbb{R}^n_+:\ |y|>2|x|\}} \big|f(y)\big|\big|x-y\big|^{-n-\gamma}\ y_n^\gamma\,dy \le$$

$$\le 2^{n+\gamma}c_0 \int\limits_{\{y\in\mathbb{R}^n_+:\ |y|>2|x|\}} \big|f(y)\big|\big|y\big|^{-n-\gamma}\ y_n^\gamma\,dy.$$

Hence, taking into account the latter estimates we have

$$\int\limits_{\mathbb{R}^n_+} \big|T_3 f(x)\big|^p\omega_1(x)\ x_n^\gamma\,dx \le$$

$$\le \big(2^{n+\gamma}c_0\big)^p\int\limits_{\mathbb{R}^n_+}\Bigg(\int\limits_{\{y\in\mathbb{R}^n_+:\ |y|>|2x|\}} \big|f(y)\big|\big|y\big|^{-n-\gamma}\ y_n^\gamma\,dy\Bigg)^p\omega_1(x)x_n^\gamma\,dx.$$

Since $\mathcal{B} < \infty$, the Hardy inequality

$$\int\limits_{\mathbb{R}^n_+} \omega_1(x)\Bigg(\int\limits_{\{y\in\mathbb{R}^n_+:\ |y|>|2x|\}} \big|f(y)\big|\big|y\big|^{-n-\gamma}y_n^\gamma\,dy\Bigg)^p x_n^\gamma\,dx \le$$

$$\le C\int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p\omega(x)x_n^\gamma\,dx$$

holds and $C \le c'\mathcal{B}$, where c' depends only on n and p. In fact the condition $\mathcal{B} < \infty$ is necessary and sufficient for the validity of this inequality (see [2], [16]). Hence, we conclude that

$$\int\limits_{\mathbb{R}^n_+} \big|T_3 f(x)\big|^p \omega_1(x)\ x_n^\gamma\, dx \le c_2 \int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p \omega(x)\ x_n^\gamma\, dx, \qquad (6)$$

where c_2 is independent of f.

Finally, we estimate $\|T_2 f\|_{L_{p,\omega_1,\gamma}}$. By the $L_{p,\gamma}(\mathbb{R}^n_+)$ boundedness of T and condition (a) we have

$$\int\limits_{\mathbb{R}^n_+} \big|T_2 f(x)\big|^p \omega_1(x) x_n^\gamma\, dx = \int\limits_{\mathbb{R}^n_+} \Big(\sum_{k\in Z} \big|Tf_{k,2}(x)\big|\chi_{E_k}(x)\Big)^p \omega_1(x) x_n^\gamma\, dx =$$

$$= \int\limits_{\mathbb{R}^n_+} \Big(\sum_{k\in Z} \big|Tf_{k,2}(x)\big|^p \chi_{E_k}(x)\Big) \omega_1(x) x_n^\gamma\, dx = \sum_{k\in Z} \int\limits_{E_k} \big|Tf_{k,2}(x)\big|^p \omega_1(x)\ x_n^\gamma\, dx \le$$

$$\le \sum_{k\in Z} \sup_{x\in E_k} \omega_1(x) \int\limits_{\mathbb{R}^n_+} \big|Tf_{k,2}(x)\big|^p x_n^\gamma dx \le \big\|T\big\|^p \sum_{k\in Z} \sup_{x\in E_k} \omega_1(x) \int\limits_{\mathbb{R}^n_+} \big|f_{k,2}(x)\big|^p x_n^\gamma\, dx =$$

$$= \big\|T\big\|^p \sum_{k\in Z} \sup_{y\in E_k} \omega_1(y) \int\limits_{E_{k,2}} \big|f(x)\big|^p x_n^\gamma\, dx,$$

where $\|T\| \equiv \|T\|_{L_{p,\gamma}(\mathbb{R}^n_+)\to L_{p,\gamma}(\mathbb{R}^n_+)}$. Since $2^{k-1} < |x| \le 2^{k+2}$ for $x \in E_{k,2}$, by condition (a) we have

$$\sup_{y\in E_k} \omega_1(y) = \sup_{2^{k-1}<|y|\le 2^{k+2}} \omega_1(y) \le \sup_{|x|/4<|y|\le 4|x|} \omega_1(y) \le b\,\omega(x)$$

for almost all $x \in E_{k,2}$. Therefore

$$\int\limits_{\mathbb{R}^n_+} \big|T_2 f(x)\big|^p \omega_1(x) x_n^\gamma\, dx \le$$

$$\le \big\|T\big\|^p b \sum_{k\in Z} \int\limits_{E_{k,2}} \big|f(x)\big|^p \omega(x) x_n^\gamma\, dx \le c_3 \int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p \omega(x) x_n^\gamma\, dx, \qquad (7)$$

where $c_3 = 3\|T\|^p b$, since the multiplicity of covering $\{E_{k,2}\}_{k\in Z}$ is equal to 3.

Inequalities (2), (5), (6), (2) imply (3) which completes the proof. □

Similarly, we can prove the weak variant of Theorem 1.

Theorem 2. *Let $p \in [1,\infty)$. Suppose that T is a sublinear bounded operator from $L_{p,\gamma}(\mathbb{R}^n_+)$ to $WL_{p,\gamma}(\mathbb{R}^n_+)$, i.e.,*

$$\int\limits_{\{x\in\mathbb{R}^n_+ : |Tf(x)|>\lambda\}} x_n^\gamma\, dx \le \frac{c}{\lambda^p}\int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p x_n^\gamma\, dx$$

and satisfies (2)*. Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n_+$ and conditions* (*a*), (*b*), (*c*) *be satisfied.*

Then there exists a constant c, independent of f, such that for all $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_+)$

$$\int\limits_{\{x\in\mathbb{R}^n_+ : |Tf(x)|>\lambda\}} \omega_1(x) x_n^\gamma dx \le \frac{c}{\lambda^p}\int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p \omega(x) x_n^\gamma\, dx. \tag{8}$$

Let K is a B_n singular kernel and T be the B_n singular integral operator

$$Tf(x) = p.v.\int\limits_{\mathbb{R}^n_+} T^y K(x) f(y) y_n^\gamma\, dy.$$

Then T satisfies the condition (2).

Thus, we have

Corollary 1 ([7])**.** *Let $p \in (1,\infty)$, T be the B_n singular integral operator. Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n_+$ and conditions* (*a*), (*b*), (*c*) *be satisfied. Then inequality* (3) *is valid.*

Corollary 2. *Let $p \in [1,\infty)$, T be the B_n singular integral operator. Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n_+$ and conditions* (*a*), (*b*), (*c*) *be satisfied. Then inequality* (8) *is valid.*

Theorem 3. *Let $p \in (1,\infty)$, T be a sublinear bounded operator from $L_{p,\gamma}(\mathbb{R}^n_+)$ to $L_{p,\gamma}(\mathbb{R}^n_+)$ satisfying* (2)*.*

Moreover, let $\omega(x_n)$, $\omega_1(x_n)$ be a weight functions on $\mathbb{R}_+$ and the following three conditions be satisfied:

(a_1) *there exists a constant $b > 0$ such that*

$$\sup_{x_n/4<y_n<4x_n} \omega_1(y_n) \le b\omega(x_n) \quad \text{for a.e. } x_n > 0,$$

$$(b_1) \quad \mathcal{A}_1 \equiv \sup_{r>0}\left(\int\limits_{2r}^{\infty} \omega_1(x_n) x_n^{-(1+\gamma)p+\gamma}\, dx_n\right)\left(\int\limits_0^r \omega^{1-p'}(x_n) x_n^\gamma dx_n\right)^{p-1} < \infty,$$

$$(c_1) \quad \mathcal{B}_1 \equiv \sup_{r>0}\left(\int\limits_0^r \omega_1(x_n) x_n^\gamma dx_n\right)\left(\int\limits_{2r}^{\infty} \omega^{1-p'}(x_n) x_n^{-(1+\gamma)p'+\gamma} dx_n\right)^{p-1} < \infty.$$

Then there exists a constant c, independent of f, such that for all $f \in L_{p,\omega}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n_+} \big|Tf(x)\big|^p \omega_1(x_n) x_n^\gamma \, dx \le c \int_{\mathbb{R}^n_+} \big|f(x)\big|^p \omega(x_n) x_n^\gamma \, dx. \tag{9}$$

Moreover, condition (a) can be replaced by the condition
(a_1^*) *there exists a constant $b > 0$ such that*

$$\omega_1(x_n)\Big(\sup_{x_n/4<y_n<4x_n} \frac{1}{\omega(y_n)}\Big) \le b \quad \textit{for a.e.} \quad x_n > 0.$$

Proof. For $k \in Z$ we define $E_k = \{x \in \mathbb{R}^n_+ : 2^k < x_n \le 2^{k+1}\}$, $E_{k,1} = \{x \in \mathbb{R}^n_+ : x_n \le 2^{k-1}\}$, $E_{k,2} = \{x \in \mathbb{R}^n_+ : 2^{k-1} < x_n \le 2^{k+2}\}$, $E_{k,3} = \{x \in \mathbb{R}^n_+ : x_n > 2^{k+2}\}$. Then $E_{k,2} = E_{k-1} \cup E_k \cup E_{k+1}$ and the multiplicity of the covering $\{E_{k,2}\}_{k\in Z}$ is equal to 3.

Given $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_+)$, we write

$$\begin{aligned} \big|Tf(x)\big| = \sum_{k\in Z} \big|Tf(x)\big|\chi_{E_k}(x) \le \sum_{k\in Z} \big|Tf_{k,1}(x)\big|\chi_{E_k}(x) + \\ + \sum_{k\in Z} \big|Tf_{k,2}(x)\big|\chi_{E_k}(x) + \sum_{k\in Z} \big|Tf_{k,3}(x)\big|\chi_{E_k}(x) \equiv \\ \equiv T_1 f(x) + T_2 f(x) + T_3 f(x), \end{aligned} \tag{10}$$

where χ_{E_k} is the characteristic function of the set E_k, $f_{k,i} = f\chi_{E_{k,i}}$, $i = 1,2,3$. We shall estimate $\|T_1 f\|_{L_{p,\omega_1,\gamma}}$. Note that for $x \in E_k$, $y \in E_{k,1}$ we have $y_n \le 2^{k-1} \le x_n/2$. Moreover, $E_k \cap \operatorname{supp} f_{k,1} = \emptyset$ and $|x_n - y_n| \ge x_n/2$. Hence, by (2)

$$\begin{aligned} T_1 f(x) \le c_4 \sum_{k\in Z} \bigg(\int_{\mathbb{R}^n_+} \big|f_{k,1}(y)\big| T^y \big|x\big|^{-n-\gamma} \, dy\bigg)\chi_{E_k} \le \\ \le c_4 \int_{\mathbb{R}^{n-1}} \int_0^{x_n/2} T^y \big|x\big|^{-n-\gamma} \big|f(y)\big| y_n^\gamma \, dy \le \\ \le c_5 \int_{\mathbb{R}^{n-1}} \int_0^{x_n/2} \big(x_n + |x' - y'|\big)^{-n-\gamma} \big|f(y)\big| y_n^\gamma \, dy_n \, dy' \end{aligned}$$

for any $x \in E_k$. Using this last inequality we have

$$\int\limits_{\mathbb{R}^n_+} \big|T_1 f(x)\big|^p \omega_1(x_n) x_n^\gamma\, dx \leq$$

$$\leq c_5^p \int\limits_{\mathbb{R}^n_+} \Bigg(\int\limits_{\mathbb{R}^{n-1}} \int\limits_0^{x_n/2} \big(x_n + |x' - y'|\big)^{-n-\gamma} \big|f(y)\big| y_n^\gamma\, dy_n\, dy' \Bigg)^p \omega_1(x_n) x_n^\gamma\, dx.$$

For $x = (x', x_n) \in \mathbb{R}^n$ let

$$I(x_n) = \int\limits_{\mathbb{R}^{n-1}} \Bigg(\int\limits_{\mathbb{R}^{n-1}} \int\limits_0^{x_n/2} \big(x_n + |x' - y'|\big)^{-n-\gamma} \big|f(y', y_n)\big| y_n^\gamma\, dy_n\, dy' \Bigg)^p dx' =$$

$$= \int\limits_{\mathbb{R}^{n-1}} \Bigg(\int\limits_0^{x_n/2} \Bigg(\int\limits_{\mathbb{R}^{n-1}} \big(x_n + |x' - y'|\big)^{-n-\gamma} \big|f(y', y_n)\big|\, dy' \Bigg) y_n^\gamma\, dy_n \Bigg)^p dx'.$$

Using the Minkowski and Young inequalities we obtain

$$I(x_n) \leq \Bigg[\int\limits_0^{x_n/2} \Bigg(\int\limits_{\mathbb{R}^{n-1}} \big|f(y', y_n)\big|^p\, dy' \Bigg)^{1/p} \Bigg(\int\limits_{\mathbb{R}^{n-1}} \frac{dx'}{(x_n + |x'|)^{n+\gamma}} \Bigg) y_n^\gamma\, dy_n \Bigg]^p =$$

$$= \Bigg(\int\limits_0^{x_n/2} \big\|f(\cdot, y_n)\big\|_{p, \mathbb{R}^{n-1}} y_n^\gamma\, dy_n \Bigg)^p \Bigg(\int\limits_{\mathbb{R}^{n-1}} \frac{dx'}{(x_n + |x'|)^{n+\gamma}} \Bigg)^p =$$

$$= x_n{}^{-(1+\gamma)p} \Bigg(\int\limits_0^{x_n/2} \big\|f(\cdot, y_n)\big\|_{p, \mathbb{R}^{n-1}} y_n^\gamma\, dy_n \Bigg)^p \Bigg(\int\limits_{\mathbb{R}^{n-1}} \frac{dx'}{(|x'| + 1)^{n+\gamma}} \Bigg)^p =$$

$$= c_6 x_n{}^{-(1+\gamma)p} \Bigg(\int\limits_0^{x_n/2} \big\|f(\cdot, y_n)\big\|_{p, \mathbb{R}^{n-1}} y_n^\gamma\, dy_n \Bigg)^p.$$

Taking into account the latter estimates and integrating over $\mathbb{R}_+$ we get

$$\int\limits_{\mathbb{R}^n_+} \big|T_1 f(x)\big|^p \omega_1(x_n) x_n^\gamma\, dx \leq$$

$$\leq c_7 \int\limits_{\mathbb{R}_+} \omega_1(x_n) x_n^{-(1+\gamma)p} \Bigg(\int\limits_0^{x_n/2} \big\|f(\cdot, y_n)\big\|_{p, \mathbb{R}^{n-1}} y_n^\gamma\, dy_n \Bigg)^p x_n^\gamma\, dx_n.$$

Since $\mathcal{A}_1 < \infty$, the Hardy inequality

$$\int\limits_{\mathbb{R}_+} \omega_1(x_n) x_n^{-(1+\gamma)p} \Bigg(\int\limits_0^{x_n/2} \big\|f(\cdot, y_n)\big\|_{p,\mathbb{R}^{n-1}} y_n^\gamma \, dy_n \Bigg)^p x_n^\gamma \, dx_n \le$$

$$\le C \int\limits_{\mathbb{R}_+} \big\|f(\cdot, x_n)\big\|_{p,\mathbb{R}^{n-1}}^p \omega(x_n) x_n^\gamma \, dx_n$$

holds and $C \le c' \mathcal{A}_1$, where c' depends only on n and p. In fact the condition $\mathcal{A}_1 < \infty$ is necessary and sufficient for the validity of this inequality (see [8], [17]). Hence, we obtain

$$\int\limits_{\mathbb{R}^n_+} \big|T_1 f(x)\big|^p \omega_1(x_n) x_n^\gamma \, dx \le c_9 \int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p \omega(x_n) x_n^\gamma \, dx. \tag{11}$$

Let us estimate $\|T_3 f\|_{L_{p,\omega_1,\gamma}}$. Further, it is easy to verify that $y_n > 2x_n$ and $|x_n - y_n| \ge y_n/2$ for $x \in E_k$, $y \in E_{k,3}$. Since $E_k \cap \operatorname{supp} f_{k,3} = \emptyset$, for $x \in E_k$ by (2) we obtain

$$T_3 f(x) \le c_5 \int\limits_{\mathbb{R}^{n-1}} \int\limits_{2x_n}^{\infty} \big|f(y)\big| \big(y_n + |x' - y'|\big)^{-n-\gamma} y_n^\gamma \, dy_n \, dy'.$$

Using this last inequality we have

$$\int\limits_{\mathbb{R}^n_+} \big|T_3 f(x)\big|^p \omega_1(x_n) x_n^\gamma \, dx \le$$

$$\le c_5^p \int\limits_{\mathbb{R}^n_+} \Bigg(\int\limits_{\mathbb{R}^{n-1}} \int\limits_{2x_n}^{\infty} \big|f(y)\big| \big(y_n + |x' - y'|\big)^{-n-\gamma} y_n^\gamma \, dy_n \, dy' \Bigg)^p \omega_1(x_n) x_n^\gamma \, dx.$$

Let

$$I_1(x_n) = \int\limits_{\mathbb{R}^{n-1}} \Bigg(\int\limits_{2x_n}^{\infty} \int\limits_{\mathbb{R}^{n-1}} \big|f(y)\big| \big(y_n + |x' - y'|\big)^{-n-\gamma} y_n^\gamma \, dy_n dy' \Bigg)^p x_n^\gamma \, dx'$$

for $x = (x', x_n) \in \mathbb{R}^n$ Using the Minkowski and Young inequalities we obtain

$$I_1(x_n) \le \Bigg[\int\limits_{2x_n}^{\infty} \bigg(\int\limits_{\mathbb{R}^{n-1}} \big|f(y', y_n)\big|^p dy' \bigg)^{1/p} \bigg(\int\limits_{\mathbb{R}^{n-1}} \frac{dy'}{(y_n + |y'|)^{n+\gamma}} \bigg) y_n^{\gamma} \, dy_n \Bigg]^p =$$

$$= c_6 \bigg(\int\limits_{2x_n}^{\infty} y_n^{-1-\gamma} \big\|f(\cdot, y_n)\big\|_{p,\mathbb{R}^{n-1}} y_n^{\gamma} \, dy_n \bigg)^p \bigg(\int\limits_{\mathbb{R}^{n-1}} \frac{dy'}{(|y'| + 1)^{n+\gamma}} \bigg)^p =$$

$$= c_7 \bigg(\int\limits_{2x_n}^{\infty} y_n^{-1-\gamma} \big\|f(\cdot, y_n)\big\|_{p,\mathbb{R}^{n-1}} y_n^{\gamma} \, dy_n \bigg)^p .$$

Integrating over $\mathbb{R}_+$ we get

$$\int\limits_{\mathbb{R}^n_+} \big|T_3 f(x)\big|^p \omega_1(x_n) x_n^{\gamma} \, dx \le$$

$$\le c_8 \int\limits_{\mathbb{R}_+} \bigg(\int\limits_{2x_n}^{\infty} y_n^{-1-\gamma} \big\|f(\cdot, y_n)\big\|_{p,\mathbb{R}^{n-1}} y_n^{\gamma} \, dy_n \bigg)^p \omega_1(x_n) x_n^{\gamma} \, dx_n .$$

Since $\mathcal{B}_1 < \infty$, the Hardy inequality

$$\int\limits_{\mathbb{R}_+} \omega_1(x_n) \bigg(\int\limits_{2x_n}^{\infty} y_n^{-1-\gamma} \big\|f(\cdot, y_n)\big\|_{p,\mathbb{R}^{n-1}} y_n^{\gamma} \, dy_n \bigg)^p x_n^{\gamma} \, dx_n \le$$

$$\le C \int\limits_{\mathbb{R}_+} \big\|f(\cdot, x_n)\big\|_{p,\mathbb{R}^{n-1}}^p x_n^{-(1+\gamma)p} \omega(x_n) x_n^{(1+\gamma)p} x_n^{\gamma} \, dx_n =$$

$$= C \int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p \omega(x_n) x_n^{\gamma} \, dx$$

holds and, moreover, $C \le c' \mathcal{B}_1$, where c' depends only on n, γ and p. In fact the condition $\mathcal{B}_1 < \infty$ is necessary and sufficient for the validity of this inequality (see [8], [17]). Hence, we obtain

$$\int\limits_{\mathbb{R}^n_+} \big|T_3 f(x)\big|^p \omega_1(x_n) x_n^{\gamma} \, dx \le c_{10} \int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p \omega(x_n) x_n^{\gamma} \, dx. \tag{12}$$

Finally, we estimate $\|T_2 f\|_{L_{p,\omega_1,\gamma}}$. By the $L_{p,\gamma}(\mathbb{R}^n_+)$ boundedness of T and condition (a_1) we have

$$\int\limits_{\mathbb{R}^n_+} \big|T_2 f(x)\big|^p \omega_1(x_n) x_n^\gamma \, dx = \int\limits_{\mathbb{R}^n_+} \Big(\sum_{k\in Z} \big|T f_{k,2}(x)\big| \chi_{E_k}(x_n) \Big)^p \omega_1(x_n) x_n^\gamma \, dx =$$

$$= \int\limits_{\mathbb{R}^n_+} \Big(\sum_{k\in Z} \big|T f_{k,2}(x)\big|^p \chi_{E_k}(x_n) \Big) \omega_1(x_n) x_n^\gamma dx = \sum_{k\in Z} \int\limits_{E_k} \big|T f_{k,2}(x)\big|^p \omega_1(x_n) x_n^\gamma dx \le$$

$$\le \sum_{k\in Z} \sup_{x\in E_k} \omega_1(x_n) \int\limits_{\mathbb{R}^n} \big|T f_{k,2}(x)\big|^p x_n^\gamma dx \le \big\|T\big\|^p \sum_{k\in Z} \sup_{x\in E_k} \omega_1(x_n) \int\limits_{\mathbb{R}^n} \big|f_{k,2}(x)\big|^p x_n^\gamma dx =$$

$$= \big\|T\big\|^p \sum_{k\in Z} \sup_{y\in E_k} \omega_1(y_n) \int\limits_{E_{k,2}} \big|f(x)\big|^p x_n^\gamma \, dx,$$

where $\|T\| \equiv \|T\|_{L_{p,\gamma}(\mathbb{R}^n_+) \to L_{p,\gamma}(\mathbb{R}^n_+)}$. Since $2^{k-1} < x_n \le 2^{k+2}$ for $x \in E_{k,2}$, by condition (a_1) we have

$$\sup_{y\in E_k} \omega_1(y_n) = \sup_{2^{k-1} < y_n \le 2^{k+2}} \omega_1(y_n) \le \sup_{x_n/4 < y_n < 4x_n} \omega_1(y_n) \le b\omega(x_n)$$

for almost all $x \in E_{k,2}$. Therefore

$$\int\limits_{\mathbb{R}^n_+} \big|T_2 f(x)\big|^p \omega_1(x_n) x_n^\gamma \, dx \le$$

$$\le \big\|T\big\|^p b \sum_{k\in Z} \int\limits_{E_{k,2}} \big|f(x)\big|^p \omega(x_n) \, dx \le c_{11} \int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p \omega(x_n) x_n^\gamma \, dx, \qquad (13)$$

where $c_{11} = 3\|T\|^p b$.

Inequalities (2), (11), (12), (2) imply (9) which completes the proof. □

Similarly we have weak variant of Theorem 3.

Theorem 4. *Let $p \in [1,\infty)$. Suppose that T is a sublinear bounded operator from $L_{p,\gamma}(\mathbb{R}^n_+)$ to $WL_{p,\gamma}(\mathbb{R}^n_+)$ and (2) is satisfied. Moreover, let $\omega(x_n)$, $\omega_1(x_n)$ be weight functions on $\mathbb{R}_+$ and conditions (a_1), (b_1), (c_1) be satisfied.*

Then there exists a constant c, independent of f, such that for all $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_+)$

$$\int\limits_{\{x\in\mathbb{R}^n_+ : |Tf(x)|>\lambda\}} \omega_1(x_n) x_n^\gamma \, dx \le \frac{c}{\lambda^p} \int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p \omega(x_n) x_n^\gamma \, dx. \qquad (14)$$

Corollary 3. *Let $p \in (1,\infty)$ and let T be the B_n singular integral operator. Moreover, let $\omega(x_n)$, $\omega_1(x_n)$ be weight functions on $(0,\infty)$ and conditions (a_1), (b_1), (c_1) be satisfied. Then inequality (9) is valid.*

Corollary 4. *Let $p \in [1, \infty)$ and let T be the B_n singular integral operator. Moreover, let $\omega(x_n)$, $\omega_1(x_n)$ be weight functions on $(0, \infty)$ and conditions (a_1), (b_1), (c_1) be satisfied. Then inequality (14) is valid.*

Remark 1. Note that, if instead of $\omega(x)$, $\omega_1(x)$ respectively put $\omega(x_n)$, $\omega_1(x_n)$, then from conditions (a), (b), (c) will not follows conditions (a_1), (b_1), (c_1) respectively.

Theorem 5. *Let $p \in (1, \infty)$. Assume that T is a sublinear bounded operator from $L_{p,\gamma}(\mathbb{R}^n_+)$ to $L_{p,\gamma}(\mathbb{R}^n_+)$ and (2) is satisfied. Moreover, let $\omega(t)$ be a weight function on $(0, \infty)$, $\omega_1(t)$ be a positive increasing function on $(0, \infty)$ and the weighted pair $(\omega(|x|), \omega_1(|x|))$ satisfies conditions (a), (b).*

Then there exists a constant $c > 0$, such that for all $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_+)$

$$\int\limits_{\mathbb{R}^n_+} \big|Tf(x)\big|^p \omega_1(|x|) x_n^\gamma \, dx \le c \int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p \omega(|x|) x_n^\gamma \, dx. \tag{15}$$

Proof. Suppose that $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_+)$ and ω_1 are positive increasing functions on $(0, \infty)$ and ω, ω_1 satisfied the conditions (a'), (b').

Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(0+) + \int\limits_0^t \psi(\lambda) d\lambda,$$

where $\omega_1(0+) = \lim_{t \to 0} \omega_1(t)$ and $\omega_1(t) \ge 0$ on $(0, \infty)$. In fact there exists a sequence of increasing absolutely continuous functions ϖ_n, such that $\varpi_n(t) \le \omega_1(t)$ and $\lim\limits_{n \to \infty} \varpi_n(t) = \omega_1(t)$ for any $t \in (0, \infty)$ (see [10], [11] for details).

We have

$$\int\limits_{\mathbb{R}^n_+} \big|Tf(x)\big|^p \omega_1(|x|) x_n^\gamma \, dx = \omega_1(0+) \int\limits_{\mathbb{R}^n_+} \big|Tf(x)\big|^p x_n^\gamma \, dx +$$

$$+ \int\limits_{\mathbb{R}^n_+} \big|Tf(x)\big|^p \bigg(\int\limits_0^{|x|} \psi(\lambda) d\lambda \bigg) x_n^\gamma \, dx = J_1 + J_2.$$

If $\omega_1(0+) = 0$, then $J_1 = 0$. If $\omega_1(0+) \ne 0$, then by the boundedness of T in $L_{p,\gamma}(\mathbb{R}^n_+)$ and (a) we conclude that

$$J_1 \le \big\|T\big\|^p \omega_1(0+) \int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p x_n^\gamma \, dx \le$$

$$\le \big\|T\big\|^p \int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p \omega_1(|x|) x_n^\gamma \, dx \le b \, \big\|T\big\|^p \int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p \omega(|x|) x_n^\gamma \, dx.$$

Changing the order of integration in J_2 we have

$$J_2 = \int_0^\infty \psi(\lambda)\Big(\int_{\{x\in\mathbb{R}^n_+:\ |x|>\lambda\}} \big|Tf(x)\big|^p x_n^\gamma\, dx\Big) d\lambda \le$$

$$\le 2^{p-1}\int_0^\infty \psi(\lambda)\Big(\int_{\{x\in\mathbb{R}^n_+:\ |x|>\lambda\}} \big|T(f\chi_{\{|x|>\lambda/2\}})(x)\big|^p x_n^\gamma\, dx +$$

$$+ \int_{\{x\in\mathbb{R}^n_+:\ |x|>\lambda\}} \big|T(f\chi_{\{|x|\le\lambda/2\}})(x)\big|^p x_n^\gamma\, dx\Big) d\lambda = J_{21} + J_{22}.$$

Using the boundeedness of T in $L_{p,\gamma}(\mathbb{R}^n_+)$ and condition (a) we obtain

$$J_{21} \le \big\|T\big\|^p \int_0^\infty \psi(t)\Big(\int_{\{y\in\mathbb{R}^n_+:\ |y|>\lambda/2\}} \big|f(y)\big|^p y_n^\gamma\, dy\Big)\, dt =$$

$$= \big\|T\big\|^p \int_{\mathbb{R}^n_+} \big|f(y)\big|^p \Big(\int_0^{2|y|} \psi(\lambda) d\lambda\Big) y_n^\gamma\, dy \le$$

$$\le \big\|T\big\|^p \int_{\mathbb{R}^n_+} \big|f(y)\big|^p \omega_1(2|y|) y_n^\gamma dy \le b\,\big\|T\big\|^p \int_{\mathbb{R}^n_+} \big|f(y)\big|^p \omega(|y|) y_n^\gamma\, dy.$$

Let us estimate J_{22}. For $|x|>\lambda$ and $|y|\le\lambda/2$ we have $|x|/2\le|x-y|\le 3|x|/2$, and consequently

$$J_{22} \le c_4 \int_0^\infty \psi(\lambda)\Big(\int_{\{x\in\mathbb{R}^n_+:\ |x|>\lambda\}} \Big(\int_{\{y\in\mathbb{R}^n_+:\ |y|\le 2\lambda\}} T^y|x|^{-n-\gamma}|f(y)| y_n^\gamma dy\Big)^p x_n^\gamma dx\Big) d\lambda \le$$

$$\le c_5 \int_0^\infty \psi(\lambda)\Big(\int_{\{x\in\mathbb{R}^n_+:\ |x|>\lambda\}} \Big(\int_{\{y\in\mathbb{R}^n_+:\ |y|\le 2\lambda\}} \big|f(y)\big| y_n^\gamma dy\Big)^p |x|^{-(n+\gamma)p} x_n^\gamma dx\Big) d\lambda =$$

$$= c_6 \int_0^\infty \psi(\lambda)\lambda^{-(n+\gamma)(p-1)} \Big(\int_{\{y\in\mathbb{R}^n_+:\ |y|\le\lambda/2\}} \big|f(y)\big| y_n^\gamma\, dy\Big)^p d\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda)\lambda^{-(n+\gamma)(p-1)}\Bigg(\int_{\{y\in\mathbb{R}^n_+:\ |y|\le\lambda/2\}} \big|f(y)\big|y_n^\gamma\,dy\Bigg)^p d\lambda \le$$

$$\le C\int_{\mathbb{R}^n_+} \big|f(y)\big|^p\omega(|y|)y_n^\gamma dy$$

for $p\in(1,\infty)$ is characterized by the condition $C\le c'\mathcal{A}'$ (see [8], [17]), where

$$\mathcal{A}'\equiv\sup_{\tau>0}\Bigg(\int_{2\tau}^\infty \psi(t)t^{-(n+\gamma)(p-1)}\,d\tau\Bigg)\Bigg(\int_{E(0,\tau)} \omega^{1-p'}(|y|)y_n^\gamma\,dy\Bigg)^{p-1}<\infty.$$

Note that

$$\int_{2t}^\infty \psi(\tau)\tau^{-(n+\gamma)(p-1)}d\tau=$$

$$=(n+\gamma)(p-1)\int_{2t}^\infty \psi(\tau)d\tau\int_\tau^\infty \lambda^{-1-(n+\gamma)(p-1)}d\lambda=$$

$$=(n+\gamma)(p-1)\int_{2t}^\infty \lambda^{-1-(n+\gamma)(p-1)}d\lambda\int_{2t}^\lambda \psi(\tau)d\tau\le$$

$$\le(n+\gamma)(p-1)\int_{2t}^\infty \lambda^{-1-(n+\gamma)(p-1)}\omega_1(\lambda)d\lambda=$$

$$=\frac{(n+\gamma)(p-1)}{\omega(n,\gamma)}\int_{\mathbb{R}^n_+\setminus E(0,2t)} \omega_1(|y|)|y|^{-(n+\gamma)p}y_n^\gamma\,dy.$$

Condition (b) of the theorem guarantees that $\mathcal{A}'\le\frac{(n+\gamma)(p-1)}{\omega(n,\gamma)}\mathcal{A}<\infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22}\le c_7\int_{\mathbb{R}^n_+} \big|f(x)\big|^p\omega(|x|)x_n^\gamma\,dx.$$

Combining the estimates for J_1 and J_2, we arrive at (15) for $\omega_1(t)=\omega_1(0+)+\int_0^t\psi(\tau)d\tau$. By Fatou's theorem we finally have the desired result. □

Corollary 5 ([5], [7])**.** *Let $p\in(1,\infty)$, K be a B_n singular kernel and T be the corresponding operator. Moreover, let $\omega(t)$ be a weight function on*

$(0,\infty)$, $\omega_1(t)$ be a positive increasing function on $(0,\infty)$ and the weighted pair $(\omega(|x|),\,\omega_1(|x|))$ satisfies conditions (a), (b). Then inequality (15) *is valid.*

Example 1. Let

$$\omega(t)=\begin{cases} t^{(n+\gamma)(p-1)}\ln^p\frac{1}{t}, & \text{for } t\in(0,\frac12)\\ (2^{\beta-p+1}\ln^p 2)t^{\beta}, & \text{for } t\in[\frac12,\infty)\end{cases}$$

$$\omega_1(t)=\begin{cases} t^{(n+\gamma)(p-1)}, & \text{for } t\in(0,\frac12)\\ 2^{\alpha-p+1}t^{\alpha}, & \text{for } t\in[\frac12,\infty)\end{cases}$$

where $0<\alpha\le\beta<(n+\gamma)(p-1)$. Then the weighted pair $(\omega(|x|),\,\omega_1(|x|))$ satisfies the condition of Theorem 5.

Theorem 6. *Let $p\in(1,\infty)$. Suppose that T is a sublinear bounded operator from $L_{p,\gamma}(\mathbb{R}^n_+)$ to $L_{p,\gamma}(\mathbb{R}^n_+)$ satisfying* (2). *Moreover, let $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$ and the weighted pair $(\omega(|x|),\,\omega_1(|x|))$ satisfies conditions (a), (c). Then inequality* (15) *is valid.*

Proof. Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t)=\omega_1(+\infty)+\int\limits_t^{\infty}\psi(\tau)d\tau,$$

where $\omega_1(+\infty)=\lim\limits_{t\to\infty}\omega_1(t)$ and $\omega_1(t)\ge 0$ on $(0,\infty)$. In fact there exists a sequence of decreasing absolutely continuous fuctions ϖ_n such that $\varpi_n(t)\le\omega_1(t)$ and $\lim_{n\to\infty}\varpi_n(t)=\omega_1(t)$ for any $t\in(0,\infty)$(see [10], [11] for details).

We have

$$\int\limits_{\mathbb{R}^n_+}\big|Tf(x)\big|^p\omega_1(|x|)x_n^{\gamma}\,dx=\omega_1(+\infty)\int\limits_{\mathbb{R}^n_+}\big|Tf(x)\big|^p x_n^{\gamma}\,dx+$$

$$+\int\limits_{\mathbb{R}^n_+}\big|Tf(x)\big|^p\bigg(\int\limits_{|x|}^{\infty}\psi(\tau)d\tau\bigg)x_n^{\gamma}\,dx=I_1+I_2.$$

If $\omega_1(+\infty)=0$, then $I_1=0$. Further, if $\omega_1(+\infty)\ne 0$, then by the boundedness of T in $L_{p,\gamma}(\mathbb{R}^n_+)$ and condition (a) we have

$$J_1\le\big\|T\big\|\omega_1(+\infty)\int\limits_{\mathbb{R}^n_+}\big|f(x)\big|^p x_n^{\gamma}\,dx\le$$

$$\le\big\|T\big\|\int\limits_{\mathbb{R}^n_+}\big|f(x)\big|^p\omega_1(|x|)x_n^{\gamma}\,dx\le b\,\big\|T\big\|\int\limits_{\mathbb{R}^n_+}\big|f(x)\big|^p\omega(|x|)x_n^{\gamma}\,dx.$$

Changing the order of integration in J_2 we have

$$J_2 = \int_0^\infty \psi(\lambda)\Big(\int_{\{x\in\mathbb{R}^n_+:\ |x|<\lambda\}} \big|Tf(x)\big|^p x_n^\gamma\,dx\Big)d\lambda \le$$

$$\le 2^{p-1}\int_0^\infty \psi(\lambda)\Big(\int_{\{x\in\mathbb{R}^n_+:\ |x|<\lambda\}} \big|T(f\chi_{\{|x|<2\lambda\}})(x)\big|^p x_n^\gamma\,dx +$$

$$+ \int_{\{x\in\mathbb{R}^n_+:\ |x|<\lambda\}} \big|T(f\chi_{\{|x|\ge 2\lambda\}})(x)\big|^p x_n^\gamma\,dx\Big)d\lambda = J_{21}+J_{22}.$$

Using the boundeedness of T in $L_p(\mathbb{R}^n)$ and condition (a) we obtain

$$J_{21} \le \|T\|\int_0^\infty \psi(t)\Big(\int_{|y|<2\lambda} \big|f(y)\big|^p y_n^\gamma\,dy\Big)\,dt =$$

$$= \|T\|\int_{\mathbb{R}^n_+} \big|f(y)\big|^p\Big(\int_{|y|/2}^\infty \psi(\lambda)d\lambda\Big) y_n^\gamma\,dy \le$$

$$\le \|T\|\int_{\mathbb{R}^n_+} \big|f(y)\big|^p \omega_1(|y|/2) y_n^\gamma\,dy \le b\|T\|\int_{\mathbb{R}^n_+} \big|f(y)\big|^p \omega(|y|) y_n^\gamma\,dy.$$

Let us estimate J_{22}. For $|x|<\lambda$ and $|y|\ge 2\lambda$ we have $|y|/2 \le |x-y| \le 3|y|/2$, and consequently

$$J_{22} \le c_8\int_0^\infty \psi(\lambda)\Big(\int_{\{x\in\mathbb{R}^n_+:\ |x|<\lambda\}}\Big(\int_{\{y\in\mathbb{R}^n_+:\ |y|\ge 2\lambda\}} T^y\big|x\big|^{-n-\gamma}\big|f(y)\big| y_n^\gamma dy\Big)^p x_n^\gamma dx\Big)d\lambda \le$$

$$\le 2^n c_8\int_0^\infty \psi(\lambda)\Big(\int_{\{x\in\mathbb{R}^n_+:\ |x|<\lambda\}}\Big(\int_{\{y\in\mathbb{R}^n_+:\ |y|\ge 2\lambda\}} \big|y\big|^{-n-\gamma}\big|f(y)\big| y_n^\gamma dy\Big)^p x_n^\gamma dx\Big)d\lambda =$$

$$= c_9\int_0^\infty \psi(\lambda)\lambda^{n+\gamma}\Big(\int_{\{y\in\mathbb{R}^n_+:\ |y|\ge 2\lambda\}} \big|y\big|^{-n-\gamma}\big|f(y)\big| y_n^\gamma\,dy\Big)^p d\lambda.$$

Further, the Hardy inequality

$$\int_0^\infty \psi(\lambda)\lambda^{n+\gamma}\Bigg(\int_{\{y\in\mathbb{R}^n_+:\ |y|\geq 2\lambda\}} |y|^{-n-\gamma}|f(y)|y_n^\gamma\,dy\Bigg)^p d\lambda \leq$$

$$\leq C\int_{\mathbb{R}^n_+} |f(y)|^p|y|^{-(n+\gamma)p}|y|^{(n+\gamma)p}\omega(|y|)y_n^\gamma\,dy = C\int_{\mathbb{R}^n_+}|f(y)|^p\omega(|y|)y_n^\gamma\,dy$$

for $p\in(1,\infty)$ is characterized by the condition $C\leq c\mathcal{B}'$ (see [8], [17]), where

$$\mathcal{B}'\equiv\sup_{\tau>0}\Bigg(\int_0^\tau \psi(t)t^{n+\gamma}\,d\tau\Bigg)\Bigg(\int_{\mathbb{R}^n_+\setminus E(0,2\tau)}\omega^{1-p'}\big(|y|\big)|y|^{-(n+\gamma)p'}y_n^\gamma\,dy\Bigg)^{p-1}<\infty.$$

Note that

$$\int_0^\tau \psi(t)t^{n+\gamma}\,dt = (n+\gamma)\int_0^\tau \psi(t)\,dt\int_0^t \lambda^{n+\gamma-1}d\lambda =$$

$$= (n+\gamma)\int_0^\tau \lambda^{n+\gamma-1}\,d\lambda\int_\lambda^t \psi(\tau)\,d\tau \leq$$

$$\leq (n+\gamma)\int_0^\tau \lambda^{n+\gamma-1}\omega_1(\lambda)\,d\lambda = \frac{n+\gamma}{\omega(n,\gamma)}\int_{E(0,r)}\omega_1\big(|x|\big)x_n^\gamma\,dx.$$

Condition (c) of the theorem guarantees that $\mathcal{B}'\leq\frac{n+\gamma}{\omega(n,\gamma)}\mathcal{B}<\infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22}\leq c_{10}\int_{\mathbb{R}^n_+}|f(x)|^p\omega\big(|x|\big)x_n^\gamma\,dx.$$

Combining the estimates for J_1 and J_2, we get (15) for $\omega_1(t)=\omega_1(+\infty)+\int_t^\infty\psi(\tau)d\tau$. Fatou's theorem completes the proof. □

Corollary 6 ([5], [7]). *Let $p\in(1,\infty)$, K be a B_n singular kernel and T be the corresponding operator. Moreover, let $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$ and the weighted pair $(\omega(|x|),\omega_1(|x|))$ satisfies conditions (a), (c). Then inequality (15) is valid.*

Example 2. Let

$$\omega(t) = \begin{cases} \frac{1}{t^{n+\gamma}} \ln^{\nu} \frac{1}{t}, & \text{for } t < d \\ (d^{-n-\gamma-\alpha} \ln^{\nu} \frac{1}{d}) t^{\alpha}, & \text{for } t \geq d, \end{cases}$$

$$\omega_1(t) = \begin{cases} \frac{1}{t^{n+\gamma}} \ln^{\beta} \frac{1}{t}, & \text{for } t < d \\ \left(d^{-n-\gamma-\lambda} \ln^{\beta} \frac{1}{d}\right) t^{\lambda}, & \text{for } t \geq d \end{cases}$$

where $\beta < \nu \leq 0$, $-n-\gamma < \lambda < \alpha < 0$, $d = e^{\frac{\beta}{n+\gamma}}$. Then the weighted pair $(\omega(|x|), \omega_1(|x|))$ satisfies the condition of Theorem 6.

Theorem 7. *Let $p \in (1, \infty)$. Suppose that T is a sublinear bounded operator from $L_{p,\gamma}(\mathbb{R}_+^n)$ to $L_{p,\gamma}(\mathbb{R}_+^n)$ and and satisfies (2). Moreover, let $\omega(t)$ be a weight function on $(0, \infty)$, $\omega_1(t)$ be a positive increasing function on $(0, \infty)$ and $\omega(x_n)$, $\omega_1(x_n)$ be satisfied the conditions (a_1), (b_1). Then inequality (9) is valid.*

Proof. Suppose that $f \in L_{p,\omega,\gamma}(\mathbb{R}_+^n)$, ω_1 are positive increasing functions on $(0, \infty)$ and $\omega(t)$, $\omega_1(t)$ satisfied the conditions (a_1), (b_1).

Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(0+) + \int_0^t \psi(\lambda)\, d\lambda,$$

where $\omega_1(0+) = \lim\limits_{t \to 0} \omega_1(t)$ and $\omega_1(t) \geq 0$ on $(0, \infty)$. In fact there exists a seqence of increasing absolutely continuous functions ϖ_n such that $\varpi_n(t) \leq \omega_1(t)$ and $\lim\limits_{n \to \infty} \varpi_n(t) = \omega_1(t)$ for any $t \in (0, \infty)$ (see [10], [11] for details).

We have

$$\int_{\mathbb{R}_+^n} \left|Tf(x)\right|^p \omega_1(x_n) x_n^{\gamma}\, dx = \omega_1(0+) \int_{\mathbb{R}_+^n} \left|Tf(x)\right|^p x_n^{\gamma}\, dx +$$

$$+ \int_{\mathbb{R}_+^n} \left|Tf(x)\right|^p \left(\int_0^{x_n} \psi(\lambda) d\lambda \right) x_n^{\gamma}\, dx = J_1 + J_2.$$

If $\omega_1(0+) = 0$, then $J_1 = 0$. If $\omega_1(0+) \neq 0$, then by the boundedness of T in $L_{p,\gamma}(\mathbb{R}_+^n)$ and (a) we obtain

$$J_1 \leq \left\|T\right\|^p \omega_1(0+) \int_{\mathbb{R}_+^n} \left|f(x)\right|^p x_n^{\gamma} dx \leq$$

$$\leq \left\|T\right\|^p \int_{\mathbb{R}_+^n} \left|f(x)\right|^p \omega_1(x_n) x_n^{\gamma}\, dx \leq \left\|T\right\|^p b \int_{\mathbb{R}_+^n} \left|f(x)\right|^p \omega(x_n) x_n^{\gamma}\, dx.$$

Changing the order of integration in J_2 we have

$$J_2 = \int_0^\infty \psi(\lambda)\Big(\int_{\mathbb{R}^{n-1}_+} \int_\lambda^\infty \big|Tf(x)\big|^p x_n^\gamma \, dx\Big)\, d\lambda \le$$

$$\le 2^{p-1} \int_0^\infty \psi(\lambda)\Big(\int_{\mathbb{R}^{n-1}} \int_\lambda^\infty \big|T(f\chi_{\{x_n>\lambda/2\}})(x)\big|^p x_n^\gamma \, dx +$$

$$+ \int_{\mathbb{R}^{n-1}} \int_\lambda^\infty \big|T(f\chi_{\{x_n\le\lambda/2\}})(x)\big|^p x_n^\gamma \, dx\Big)\, d\lambda = J_{21} + J_{22}.$$

Using the boundeedness of T in $L_{p,\gamma}(\mathbb{R}^n_+)$ we obtain

$$J_{21} \le \big\|T\big\|^p \int_0^\infty \psi(t)\Big(\int_{\mathbb{R}^{n-1}} \int_{\lambda/2}^\infty \big|f(y)\big|^p y_n^\gamma \, dy\Big)\, dt =$$

$$= \big\|T\big\|^p \int_0^\infty \psi(t)\Big(\int_{\lambda/2}^\infty \big\|f(\cdot, y_n)\big\|^p_{p,\mathbb{R}^{n-1}} y_n^\gamma \, dy_n\Big)\, dt =$$

$$= \big\|T\big\|^p \int_{\mathbb{R}^n_+} \big\|f(\cdot, y_n)\big\|^p_{p,\mathbb{R}^{n-1}} \Big(\int_0^{2y_n} \psi(\lambda) d\lambda\Big) y_n^\gamma \, dy_n \le$$

$$\le \big\|T\big\|^p \int_{\mathbb{R}^n_+} \big\|f(\cdot, y_n)\big\|^p_{p,\mathbb{R}^{n-1}} \omega_1(2y_n) y_n^\gamma \, dy_n \le$$

$$\le b\big\|T\big\|^p \int_{\mathbb{R}^n_+} \big|f(y)\big|^p \omega(y_n) x_n^\gamma \, dy.$$

Let us estimate J_{22}. For $x_n > \lambda$ and $y_n \le \lambda/2$ we have $x_n/2 \le |x_n - y_n| \le 3x_n/2$, and consequently

$$J_{22} \le c_9 \int_0^\infty \psi(\lambda)\Big(\int_{\mathbb{R}^{n-1}} \int_\lambda^\infty \Big(\int_{\mathbb{R}^{n-1}} \int_0^{\lambda/2} \frac{|f(y)|}{|x-y|^{n+\gamma}} \, dy\Big)^p x_n^\gamma \, dx\Big)\, d\lambda \le$$

$$\le c_{10} \int_0^\infty \psi(\lambda)\Big(\int_\lambda^\infty \int_{\mathbb{R}^{n-1}} \Big(\int_0^{\lambda/2} \int_{\mathbb{R}^{n-1}} \frac{|f(y)|}{(x_n + |x' - y'|)^{n+\gamma}} y_n^\gamma \, dy\Big)^p x_n^\gamma \, dx\Big)\, d\lambda.$$

For $x=(x',x_n)\in\mathbb{R}^n_+$ let

$$J(x_n,\lambda)=\int\limits_{\mathbb{R}^{n-1}}\bigg(\int\limits_0^{\lambda/2}\int\limits_{\mathbb{R}^{n-1}}\frac{|f(y)|}{(x_n+|x'-y'|)^{n+\gamma}}y_n^\gamma\,dy\bigg)^p dx'.$$

Using the Minkowski and Young inequalities we obtain

$$J(x_n,\lambda)\le\bigg[\int\limits_0^{\lambda/2}\bigg(\int\limits_{\mathbb{R}^{n-1}}|f(y)|^p\,dy'\bigg)^{1/p}\bigg(\int\limits_{\mathbb{R}^{n-1}}\frac{dy'}{(|y'|+x_n)^{n+\gamma}}\bigg)y_n^\gamma\,dy_n\bigg]^p=$$

$$\le\bigg(\int\limits_0^{\lambda/2}\big\|f(\cdot,y_n)\big\|_{p,\mathbb{R}^{n-1}}y_n^\gamma\,dy_n\bigg)^p\bigg(\int\limits_{\mathbb{R}^{n-1}}\frac{dy'}{(|y'|+x_n)^{n+\gamma}}\bigg)^p=$$

$$=c_3{x_n}^{-(1+\gamma)p}\bigg(\int\limits_0^{\lambda/2}\big\|f(\cdot,y_n)\big\|_{p,\mathbb{R}^{n-1}}y_n^\gamma\,dy_n\bigg)^p\bigg(\int\limits_{\mathbb{R}^{n-1}}\frac{dy'}{(1+|y'|)^{n+\gamma}}\bigg)^p=$$

$$=c_4{x_n}^{-(1+\gamma)p}\bigg(\int\limits_0^{\lambda/2}\big\|f(\cdot,y_n)\big\|_{p,\mathbb{R}^{n-1}}y_n^\gamma\,dy_n\bigg)^p.$$

Taking into account the latter estimates and integrating over $(0,\infty)\times(\lambda,\infty)$ we get

$$J_{22}\le c_5\int\limits_0^\infty\psi(\lambda)\bigg(\int\limits_\lambda^\infty\bigg(\int\limits_0^{\lambda/2}\big\|f(\cdot,y_n)\big\|_{p,\mathbb{R}^{n-1}}y_n^\gamma\,dy_n\bigg)^p x_n^{-(1+\gamma)p}x_n^\gamma\,dx\bigg)d\lambda=$$

$$=\frac{2c_5}{p-1}\int\limits_0^\infty\psi(\lambda)\lambda^{-(1+\gamma)p+\gamma+1}\bigg(\int\limits_0^{\lambda/2}\big\|f(\cdot,y_n)\big\|_{p,\mathbb{R}^{n-1}}y_n^\gamma\,dy_n\bigg)^p d\lambda.$$

Further, the Hardy inequality

$$\int\limits_0^\infty\psi(\lambda)\lambda^{-(1+\gamma)p+\gamma+1}\bigg(\int\limits_0^{\lambda/2}\big\|f(\cdot,y_n)\big\|_{p,\mathbb{R}^{n-1}}y_n^\gamma\,dy\bigg)^p d\lambda\le$$

$$\le C\int\limits_{\mathbb{R}_+}\big\|f(\cdot,y_n)\big\|^p_{p,\mathbb{R}^{n-1}}\omega(y_n)y_n^\gamma\,dy_n=C\int\limits_{\mathbb{R}^n_+}|f(y)|^p\omega(y_n)y_n^\gamma\,dy,$$

for $p\in(1,\infty)$ is characterized by the condition $C\le c'\mathcal{A}''$, where

$$\mathcal{A}''\equiv\sup_{\tau>0}\bigg(\int\limits_{2\tau}^\infty\psi(t)t^{-(1+\gamma)p+\gamma+1}\,d\tau\bigg)\bigg(\int\limits_0^\tau\omega^{1-p'}(t)t^\gamma\,dt\bigg)^{p-1}<\infty.$$

Note that

$$\int_{2t}^{\infty} \psi(\tau)\tau^{-(1+\gamma)p+\gamma+1}\,d\tau = (1+\gamma)(p-1)\int_{2t}^{\infty}\psi(\tau)\,d\tau\int_{\tau}^{\infty}\lambda^{-(1+\gamma)p+\gamma}\,d\lambda =$$

$$= (1+\gamma)(p-1)\int_{2t}^{\infty}\lambda^{-(1+\gamma)p+\gamma}d\lambda\int_{2t}^{\lambda}\psi(\tau)\,d\tau$$

$$\leq (1+\gamma)(p-1)\int_{2t}^{\infty}\lambda^{-(1+\gamma)p+\gamma}\omega_1(\lambda)\,d\lambda.$$

Condition (b_1) of the theorem guarantees that $\mathcal{A}'' \leq (1+\gamma)(p-1)\mathcal{A}_1 < \infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22} \leq c_{11}\int_{\mathbb{R}^n_+}\left|f(x)\right|^p\omega(x_n)x_n^{\gamma}\,dx.$$

Combining the estimates for J_1 and J_2, we get (15) for $\omega_1(t) = \omega_1(0+) + \int_0^t\psi(\tau)\,d\tau$. Fatou's theorem completes the proof of the theorem. ☐

Example 3. Let

$$\omega(t) = \begin{cases} t^{p-1}\ln^p\frac{1}{t}, & \text{for } t\in(0,\frac{1}{2}) \\ (2^{\beta-p+1}\ln^p 2)t^{\beta}, & \text{for } t\in[\frac{1}{2},\infty), \end{cases}$$

$$\omega_1(t) = \begin{cases} t^{p-1}, & \text{for } t\in(0,\frac{1}{2}) \\ 2^{\alpha-p+1}t^{\alpha}, & \text{for } t\in[\frac{1}{2},\infty), \end{cases}$$

where $0<\alpha,\, \beta<p-1$. Then the pair $(\omega(x_n),\omega_1(x_n))$ satisfies the condition of Theorem 7.

Corollary 7. *Let $p\in(1,\infty)$, K be a B_n singular kernel and T be the corresponding operator. Moreover, let $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive increasing function on $(0,\infty)$ and $\omega(x_n)$, $\omega_1(x_n)$ be satisfied the conditions (a_1), (b_1). Then inequality (9) is valid.*

Theorem 8. *Let $p\in(1,\infty)$. Assume that T be a sublinear bounded operator from $L_{p,\gamma}(\mathbb{R}^n_+)$ to $L_{p,\gamma}(\mathbb{R}^n_+)$ satisfying (2). Moreover, let $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$ and $\omega(x_n)$, $\omega_1(x_n)$ be satisfied the conditions (a_1), (c_1). Then inequality (9) holds.*

Proof. Without loss of generality we can assume that ω_1 may be represented as follows:

$$\omega_1(t) = \omega_1(+\infty) + \int_t^{\infty}\psi(\tau)\,d\tau,$$

where $\omega_1(+\infty) = \lim\limits_{t\to\infty} \omega_1(t)$ and $\omega_1(t) \geq 0$ on $(0, \infty)$. In fact there exists a sequence of decreasing absolutely continuous fuctions ϖ_n such that $\varpi_n(t) \leq \omega_1(t)$ and $\lim_{n\to\infty} \varpi_n(t) = \omega_1(t)$ for any $t \in (0, \infty)$ (see [10], [11] for details).

We have

$$\int\limits_{\mathbb{R}^n_+} \big|Tf(x)\big|^p \omega_1(x_n) x_n^\gamma\, dx = \omega_1(+\infty) \int\limits_{\mathbb{R}^n_+} \big|Tf(x)\big|^p x_n^\gamma\, dx +$$

$$+ \int\limits_{\mathbb{R}^n_+} \big|Tf(x)\big|^p \bigg(\int\limits_{x_n}^{\infty} \psi(\tau) d\tau \bigg) x_n^\gamma\, dx = I_1 + I_2.$$

If $\omega_1(+\infty) = 0$, then $I_1 = 0$. If $\omega_1(+\infty) \neq 0$, then by the boundedness of T in $L_{p,\gamma}(\mathbb{R}^n_+)$ we obtain

$$J_1 \leq \big\|T\big\|^p \omega_1(+\infty) \int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p x_n^\gamma\, dx \leq$$

$$\leq \big\|T\big\|^p \int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p \omega_1(x_n) x_n^\gamma\, dx \leq b \big\|T\big\|^p \int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p \omega(x_n)\, x_n^\gamma\, dx.$$

Changing the order of integration in J_2 we conclude that

$$J_2 = \int\limits_0^\infty \psi(\lambda) \bigg(\int\limits_{\mathbb{R}^{n-1}} \int\limits_0^\lambda \big|Tf(x)\big|^p x_n^\gamma\, dx \bigg) d\lambda \leq$$

$$\leq 2^{p-1} \int\limits_0^\infty \psi(\lambda) \bigg(\int\limits_{\mathbb{R}^{n-1}} \int\limits_0^\lambda \big|T(f\chi_{\{|x_n|<2\lambda\}})(x)\big|^p x_n^\gamma\, dx +$$

$$+ \int\limits_{\mathbb{R}^{n-1}} \int\limits_0^\lambda \big|T(f\chi_{\{x_n \geq 2\lambda\}})(x)\big|^p x_n^\gamma\, dx \bigg) d\lambda = J_{21} + J_{22}.$$

Using the boundeedness of T in $L_{p,\gamma}(\mathbb{R}^n_+)$ we obtain

$$J_{21} \leq \big\|T\big\|^p \int\limits_0^\infty \psi(t) \bigg(\int\limits_{\mathbb{R}^{n-1}} \int\limits_0^{2\lambda} \big|f(y)\big|^p y_n^\gamma\, dy \bigg) dt =$$

$$= \big\|T\big\|^p \int\limits_{\mathbb{R}^n} \big|f(y)\big|^p \bigg(\int\limits_{y_n/2}^\infty \psi(\lambda) d\lambda \bigg) y_n^\gamma\, dy \leq \big\|T\big\|^p \int\limits_{\mathbb{R}^n_+} \big|f(y)\big|^p \omega_1(y_n/2) y_n^\gamma\, dy \leq$$

$$\leq b \big\|T\big\|^p \int\limits_{\mathbb{R}^n_+} \big|f(y)\big|^p \omega(y_n) y_n^\gamma\, dy.$$

Let us estimate J_{22}. For $x_n < \lambda$ and $y_n \geq 2\lambda$ we have $y_n/2 \leq |x_n - y_n| \leq 3y_n/2$. Hence,

$$J_{22} \leq c_{12} \int_0^\infty \psi(\lambda) \Big(\int_{\mathbb{R}^{n-1}} \int_0^\lambda \Big(\int_{\mathbb{R}^{n-1}} \int_{2\lambda}^\infty \frac{|f(y)|}{(|x'-y'|+|x_n-y_n|)^{n+\gamma}} y_n^\gamma dy \Big)^p x_n^\gamma dx \Big) d\lambda \leq$$

$$\leq 2^n c_{12} \int_0^\infty \psi(\lambda) \Big(\int_{\mathbb{R}^{n-1}} \int_0^\lambda \Big(\int_{\mathbb{R}^{n-1}} \int_{2\lambda}^\infty \frac{|f(y)|}{(|x'-y'|+y_n)^{n+\gamma}} y_n^\gamma \, dy \Big)^p x_n^\gamma \, dx \Big) d\lambda.$$

For $x = (x', x_n) \in \mathbb{R}^n$ let

$$J_1(x_n, \lambda) = \int_{\mathbb{R}^{n-1}} \Big(\int_{2\lambda}^\infty \int_{\mathbb{R}^{n-1}} \frac{|f(y)|}{(|x'-y'|+y_n)^{n+\gamma}} y_n^\gamma \, dy \Big)^p dx'.$$

Using the Minkowski and Young inequalities we obtain

$$J_1(x_n, \lambda) \leq \Big[\int_{2\lambda}^\infty \Big(\int_{\mathbb{R}^{n-1}} |f(y)|^p dy' \Big)^{1/p} \Big(\int_{\mathbb{R}^{n-1}} \frac{dy'}{(|y'|+y_n)^{n+\gamma}} \Big) y_n^\gamma \, dy_n \Big]^p =$$

$$\leq \Big(\int_{2\lambda}^\infty \|f(\cdot, y_n)\|_{p,\mathbb{R}^{n-1}} y_n^\gamma \, dy_n \Big)^p \Big(\int_{\mathbb{R}^{n-1}} \frac{dy'}{(|y'|+y_n)^{n+\gamma}} \Big)^p =$$

$$= c_3 \Big(\int_{2\lambda}^\infty \|f(\cdot, y_n)\|_{p,\mathbb{R}^{n-1}} {y_n}^{-1-\gamma} y_n^\gamma \, dy_n \Big)^p \Big(\int_{\mathbb{R}^{n-1}} \frac{dy'}{(1+|y'|)^{n+\gamma}} \, dy' \Big)^p =$$

$$= c_4 \Big(\int_{2\lambda}^\infty \|f(\cdot, y_n)\|_{p,\mathbb{R}^{n-1}} y_n^{-1-\gamma} y_n^\gamma \, dy_n \Big)^p.$$

Integrating over $(0, \infty) \times (0, \lambda)$ we get

$$J_{22} \leq c_5 \int_0^\infty \psi(\lambda) \Big(\int_0^\lambda \Big(\int_{2\lambda}^\infty \|f(\cdot, y_n)\|_{p,\mathbb{R}^{n-1}} y_n^{-1-\gamma} y_n^\gamma \, dy_n \Big)^p x_n^\gamma \, dx_n \Big) d\lambda =$$

$$= 2c_5 \int_0^\infty \psi(\lambda) \lambda^{1+\gamma} \Big(\int_{2\lambda}^\infty \|f(\cdot, y_n)\|_{p,\mathbb{R}^{n-1}} y_n^{-1-\gamma} y_n^\gamma \, dy_n \Big)^p d\lambda.$$

Further, the Hardy inequality

$$\int_0^\infty \psi(\lambda)\lambda^{1+\gamma}\Big(\int_{2\lambda}^\infty \big\|f(\cdot,y_n)\big\|_{p,\mathbb{R}^{n-1}} y_n^{-1-\gamma} y_n^\gamma\,dy\Big)^p d\lambda \le$$

$$\le C\int_{\mathbb{R}_+} \big\|f(\cdot,x_n)\big\|^p_{p,\mathbb{R}^{n-1}}\omega(x_n)x_n^\gamma\,dx_n = C\int_{\mathbb{R}^n_+}|f(y)|^p\omega(y_n)y_n^\gamma\,dy,$$

for $p\in(1,\infty)$ is characterized by the condition $C\le c'\mathcal{B}''$, where

$$\mathcal{B}''\equiv\sup_{\tau>0}\Big(\int_0^\tau \psi(t)t^{1+\gamma}\,d\tau\Big)\Big(\int_{2\tau}^\infty \omega^{1-p'}(t)t^{-(1+\gamma)p'}t^\gamma\,dt\Big)^{p-1}<\infty.$$

Note that

$$\int_0^\tau \psi(t)t^{1+\gamma}\,dt=(1+\gamma)\int_0^\tau\psi(t)\,dt\int_0^t\lambda^\gamma\,d\lambda=$$

$$=(1+\gamma)\int_0^\tau\lambda^\gamma\,d\lambda\int_\lambda^t\psi(\tau)\,d\tau\le(1+\gamma)\int_0^\tau\omega(\lambda)\lambda^\gamma\,d\lambda.$$

Condition (c_1) of the theorem guarantees that $\mathcal{B}''\le\mathcal{B}_1<\infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22}\le c\int_{\mathbb{R}^n_+}\big|f(x)\big|^p\omega(x_n)x_n^\gamma\,dx.$$

Combining the estimates for J_1 and J_2, we get (15) for $\omega_1(t)=\omega_1(+\infty)+\int_t^\infty\psi(\tau)\,d\tau$. Fatou's theorem iplies (9). The theorem has been proved. □

Corollary 8. *Let $p\in(1,\infty)$, K be a B_n singular kernel and T be the corresponding operator. Moreover, let $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$ and $\omega(x_n)$, $\omega_1(x_n)$ be satisfied the conditions (a_1), (c_1). Then inequality (9) is valid.*

Example 4. Let

$$\omega(t)=\begin{cases}\frac{1}{t}\ln^\nu\frac{1}{t}, & \text{for } t<d\\ (d^{-1-\alpha}\ln^\nu\frac{1}{d})t^\alpha, & \text{for } t\ge d,\end{cases}$$

$$\omega_1(t)=\begin{cases}\frac{1}{t}\ln^\beta\frac{1}{t}, & \text{for } t<d\\ (d^{-1-\lambda}\ln^\beta\frac{1}{d})t^\lambda, & \text{for } t\ge d,\end{cases}$$

where $\beta<\nu\le0$, $-1<\lambda<\alpha<0$, $d=e^\beta$. Then the pair $(\omega(x_n),\omega_1(x_n))$ satisfies the condition of Theorem 8.

Acknowledgements

The author would like to express their thanks to her supervisor Academician. A. D. Gadjiev for the statement of the problem and many helpful discussions about this subject.

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(Received 01.07.2005)

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