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PROFESSOR ARCHIL KHARADZE

ON THE OCCASION OF HIS 110 ANNIVERSARY

Professor Archil Kharadze - prominent mathematician, devoted teacher and distinguished personality - was born in April 21, 1895, in a village of Western Georgia. He attended his elementary school in the same rural area, and then in 1912 successfully finished, being awarded the Silver medal, his middle school education at a gymnasium (grammar school) in Tbilisi. His father Kirile Kharadze who had a strong appreciation towards education but did not have enough funds, still managed to send his gifted son to Moscow for higher education, and in the same year of 1912 Archil Kharadze became a student of Department of Physics and Mathematics of the famous MGU, Moscow State University. That period, like probably all other periods for Moscow State University, was excellent for the history of this university. Mathematics courses have been conducted by famous mathematicians including D. Egorov, B. Mlodzeevski, L. Larkin, N. Luzin. Very soon young A. Kharadze became one of the advanced students at the department. Being in his third year, in 1915, he made his first research and was marked by the university with official Certificate of Approval for it. He has finished university education by the end of 1916, and was officially graduated, after passing State examinations, in March of 1917 with first grade diploma and an official offer (by recommendation of Prof. D. Egorov) to remain at the university for preparing to professorship. However, because of the financial shortage, this offer was not realized, and young mathematician A. Kharadze returned to Tbilisi by fall of the same year 1917. After a few months, in May of 1918, at the age of 23, he started to work at Tbilisi University which was officially inaugurated just a few months before he became a university lecturer. In 1930 he was appointed a university professor and the head of Chair of mathematical analysis. In 1975, after several persistent applications made by him to the rector of the university, he left the position of the head of chair.

During the long-lasting pedagogical and research career, Prof. A. Kharadze had a considerable influence on Georgian mathematicians and mathematics due to his excellent many-sided research in mathematics, his devotion to teaching mathematics, his exemplary personal properties and the general attitude in diverse problems usually arising in the social life of any community of people.

Starting to teach at the university, A. Kharadze immediately faced two major problems: lack (or full absence) of mathematical terminology in the Georgian language and full absence of Georgian text-books in basic mathematical disciplines. Both of these problems caused serious difficulties in teaching, and required the urgent care. The full responsibility for this direction, as well as for many others which usually appear in any pioneering undertaking, fell naturally on the "magnificent four" of the Georgian mathematicians of the first generation Georgi Nikoladze (born in 1888, graduated from Technological Institute of St. Petersburg in 1913), Andria Razmadze (1889, Moscow State University, 1910), Nikoloz Muskhelishvili (1891, St. Petersburg University, 1915) and the youngest of them Archil Kharadze (1895, Moscow State University, 1917). They, these four, made the principal contribution to the establishment of Georgian mathematical terminology significantly improving the possibility to teach and write mathematics in Georgian. Of course, this job could not be, and by no means was, a single work done by one attempt. It was a constant care of the just mentioned founders of this initiative as well as of their followers in the next generations. And I want to mention in this respect the name of Prof. G. Chogoshvili. The work on terminology is, of course, closely connected with the writing of textbooks. One of the first Georgian mathematical textbooks was the manual "A theory of determinants" by A. Kharadze, first issued in 1920. In subsequent years two more editions of this book and also several editions of the two large textbooks in mathematical analysis and foundations of higher mathematics for non-mathematical specialities have been published by him and also in cooperation with Prof. A. Rukhadze. Later, in forties of the last century, Prof. A. Kharadze invited professors V. Chelidze, B. Khvedelidze and I. Kartsivadze to work together with him on the project of writing a fundamental course of mathematical analysis for mathematical specialities. This work lasted several years and was completed successfully in 1950. Since than this capital book in two volumes had four editions, and still remains a good source for mathematics students to go deep in the subject.

Now I want to sketch out Prof. A. Kharadze's mathematical inheritance. I want to give an idea about scope of his mathematical interest. I cannot speak about technical details, and will just try to show some areas of his research and give some of his results only for those particular cases which allow simple formulations.

It is my very pleasant obligation to say here that during the preparation of the mathematical part of this communication I was essentially using the very interesting small book written by Prof. I. Kartsivadze and Prof. B. Khvedelidze, and published by Tbilisi State University in 1985 on the occasion of Prof. A. Kharadze's 90 anniversary.

1. We start with a qualitative approach to explicit solutions of algebraic equations of third and fourth order based on what is known as the notion of circulant determinant. A circular determinant is the determinant of a circular matrix which means a matrix any of whose row, starting from the second is a circular permutation of the first one. It was noticed by Prof. A. Kharadze that any algebraic equation of order $k = 3$ and $k = 4$ can be written as a circulant determinant $\Delta_k(x)$, and moreover, the determinant $\Delta_k(x)$ can be expressed as the product of linear forms. These two arguments for the case $k = 4$ give

$$\Delta_4(x) = \begin{vmatrix} x, & a, & b, & c \\ c, & x, & a, & b \\ b, & c, & x, & a \\ a, & b, & c, & x \end{vmatrix}$$

and $\Delta_4(x) = (x + a + b + c)(x + ia - b - ic)(x - a + b - c)(x - ia - b + ic)$ with a, b, c depending on the coefficients of the equation. Therefore, the roots of the equation can be expressed in an explicit elementary form if this is the case for the dependence of a, b, c on the coefficients which happens for some classes of the equations. The analysis of reasons why this approach did not work for $k > 4$ was given as well.

2. We continue with the introduction of special numerical sequences and closely related with them polynomials which can be regarded as a generalization of some classical objects. These polynomials have an independent interest and, besides, they are used by A. Kharadze in other areas of his research, and we will be talking about that a bit later.

3. Among geometrical investigations of Prof. A. Kharadze we note a contribution to the theory of generalized evolutes and their applications. He gave an extension of the notion of pedal of plane curve to the case of two-dimensional surfaces in three dimensional space. These ideas and results he then successfully used to establish the from of the general solution of some partial differential equations. A particular case of such equations for three independent variables is the following one

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^3} + \frac{\partial^3 u}{\partial z^3} - 3\frac{\partial^3 u}{\partial x \partial y \partial z} = 0.$$

4. Many interesting and valuable results were obtained by Prof. A. Kharadze in the area of classical mathematical analysis. We mention a few of them starting with a simple elegant result concerning the generalization of the well-known Leibnitz's criterion for alternating signs series.

Let θ be a primitive root of the equation $x^k = 1$, and $(a_n)_{n\geq 1}$ be a decreasing sequence of positive numbers tending to zero. Then the series

$$\sum_n a_n \theta^n$$

is converging and the following inequality for its remainder is valid

$$|r_n| \leq \frac{a_n}{\sin\frac{\pi}{k}} \quad \text{if} \quad k \quad \text{is even}$$

and

$$|r_n| \leq \frac{a_n}{2\sin\frac{\pi}{2k}} \quad \text{if} \quad k \quad \text{is odd.}$$

5. Prof. A. Kharadze gave a condition for convergence of continuous functions which reminds a criterion for normal systems of holomorphic function. This could be considered as a result of his significant interest to the Montel's theory of normal families of functions.

6. It is particularly noteworthy a wide circle of papers by Prof. A. Kharadze on the localization of an intermediate point ξ in the intermediate value theorems (of Lagrange, Cauchy, Taylor, Rolley). On this subject the monograph entitled as "On application of intermediate value theorems for polynomials" was issued by him. This area is closely related with the theory of orthogonal polynomials, and Prof. A. Kharadze successfully used some generalizations of the sequences of classical orthogonal polynomials for finding the precise subintervals of localization of intermediate points ξ for diverse families of polynomials. In this directions some deep generalizations of Chakalov's and Favard's results were obtained. A very special case of an excellent theorem of Prof. A. Kharadze is Chakalov's theorem which can be formulated as follows: for the class of polynomials of degree $2n$ or $2n-1$ $(n = 2, 3, \dots)$ defined on the interval $(-1, 1)$, intermediate points for Lagrange theorem belong to the inner interval made up by the endroots of the Legendre polynomial of order n and this is the smallest interval with this property. For example, for any polynomial of the third order $(n = 2)$ these bounds are $-\frac{1}{\sqrt{3}} \leq \xi \leq \frac{1}{\sqrt{3}}$ (note that $\frac{1}{\sqrt{3}}$ and $-\frac{1}{\sqrt{3}}$ are the only two zeros and therefore are the endroots of the Legendre polynomial $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $-1 \leq x \leq 1$).

7. Another circle of papers by Prof. A. Kharadze deals with algebraic and analytical theory of polynomials in one and several variables. He studied new problems in the theory of classical orthogonal polynomials and also found some notable applications of results obtained by him in this area to diverse problems of mathematical analysis. We will mention here only two of them.

(i) It is known that every sequence of orthogonal polynomials is a Hamel basis in the linear space of polynomials. Choosing for Hamel basis generalized orthogonal polynomials introduced by Prof. A. Kharadze himself, he found areas for all zeros of polynomials in the complex plane. For example, if (H_n) denotes the sequence of generalized Hermitian polynomials, and polynomial of degree is represented as

$$\varphi(z) = \sum_{n=0}^{m} a_{nk} H_{nk}(z), \quad a_{mk} \neq 0,$$

then all zeros of the function φ are situated in the area of the complex plane described by the following inequality

$$r^k |\sin k\alpha| \leq b_k \Big(1 + \frac{M}{|a_{mk}|}\Big),$$

where $z = re^{ia}$, $M = \max |a_{nk}|$ $(n = 0, 1, \dots, m-1)$ and b_k are real (positive) numbers depending explicitly on the coefficients of the expansion of φ .

One of the consequences of the general theorem of Kharadze in this direction is the following result first proved by P. Turan: if is an even polynomial on the complex plane, and its representation by Hermitian polynomials is

$$f(z) = \sum_{k=0}^{m} c_{2k} H_{2k}(z), \quad c_{2m} \neq 0,$$

then all roots of f are situated in the area described by the inequality

$$|xy| \leq \frac{5}{4}\Big(1 + \frac{M}{|c_{2m}|}\Big), \quad M = \max|c_{2k}|, \quad k = 0, 1, \ldots, m-1, \quad x + iy = z.$$

(ii) In the analytic theory of polynomials it is known Grace phenomenon meaning the following: any linear relation between the coefficients of a polynomial characterizes in a certain sense the area of the complex plane where all zeros of the derivative of the polynomial are situated. Several refinements of this classical result belong to Prof. A. Kharadze. One of them is the following: if a polynomial of degree n satisfies the condition

$$f(i) - f(0) = i[f(-i) - f(0)],$$

then the derivative of f has at least one root in the circle with radius ctg $\frac{\pi}{4n}$ and center at zero.

Now let me finish my very schematic account on the mathematical inheritance of Prof. A. Kharadze, and go back to his personality. I could speak much about Archil Kharadze, his very non-trivial personality, his moral self-restriction (Solzhenitsin's expression). Yes, I could speak on this matter as much as my English would let me go on. But now I will be concise and try to express my attitude in short.

In some occasional cases professional communities have their spiritual leaders. Spiritual leaders usually do not have any official positions on top levels of the administrative staircase, neither are officially elected for the leadership. Unlike the case of official elections, when we sometimes make mistakes, in choosing the spiritual leaders mistakes are very rare, if at all. Spiritual leaders gain only new heavy duties and no benefits. And still, it is the highest moral position in the society. There is no sufficient condition for getting it but there are many necessary conditions that can be groupped around professional level, devotion and ability to serve public interests, spotless honesty. No meetings, no negotiations, no debates are needed to decide the choice. Guration date exists since no inauguration exists at all for such cases. The decision matures gradually, bit by bit to arrive to the full consensus. People believe that spiritual leaders are the conscience of their communities. Professor Archil Kharadze undoubtedly was the conscience and the honour of the Georgian mathematical community for many years. I am fully aware that no one from our mathematical community, in past years or afterwards, would question this statement.

N. Vakhania,

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