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ON LINEAR CONJUGATION PROBLEMS WITH THE DOUBLY PERIODIC JUMP LINE

N. KHATIASHVILI

Abstract. The linear conjugation problem for the class of linearly doubly quasi-periodic and exponentially doubly quasi-periodic functions is considered. The effective solutions are obtained by means of the Cauchy type integral with the Weierstrass kernel.

რეზიუმე. შესწავლილია წრფივი შეუღლების ამოცანა ორადპერიოდული ნახტომის წირით წრფივად კვაზიორადპერიოდულ და ექსპონენციალურად კვაზიორადპერიოდულ ფუნქციათა კლასებში.

Introduction

The linear conjugation problem for the different class of functions when the jump line consists of the countable number of contours where considered by numerous of authors [1], [3], [4], [5], [6], [8], [9], [12].

The Cauchy type integrals with the Weierstrass kernel was first considered by Sedov [12]. By means of this integrals he had successfully solved several doubly-periodic problems of hydrodynamics.

By means of the Cauchy type integrals with the Weierstrass kernel Chibricova [5] had solved the linear conjugation problem with the doubly-periodic jump line for the class of the doubly-periodic functions.

The linear conjugation problem on the Riemann surfaces for the class of the doubly-periodic functions was studied by Zverovich [14].

The effective solutions of the singular integral equation with the Weierstass kernel was obtained by the author in the case of segments [8].

In this paper are introduced the new class of functions: linearly doubly quasi-periodic and exponentially doubly quasi-periodic and the linear conjugation problem with the doubly-periodic jump line for this type of classes is studied. This new classes are more general and involve the class of the doubly-periodic functions. The problems considered in this work are solved by means of the Cauchy integral with the Weierstrass kernel. This case is closely connected with the various problems of the mathematical physics.

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1. The Doubly-Periodic and Linearly Doubly Quasi-Periodic Functions

Consider a complex z-plane, $z = x + iy$ and two complex numbers ω_1 and $i\omega_2$ satisfying the condition $\operatorname{Im} \frac{i\omega_2}{\omega_1} > 0$. Let us introduce some notations:

Definition 1.1. The line L is called *the doubly-periodic line* if it is a union of a countable number of smooth non-intersected contours L^j_{mn}, $j = 1, 2, \dots, k$; $m, n = 0, \pm 1, \dots$ doubly-periodically distributed with periods $2\omega_1$ and $2i\omega_2$ in the whole z-plane

$$L = \sum_{m,n=-\infty}^{\infty} L_{mn}, \quad L_{mn} = \sum_{j=1}^{k} L^j_{mn}, \quad L^{j_1}_{mn} \cap L^{j_2}_{mn} = \varnothing; \;\; j_1 \neq j_2. \tag{1.1}$$

If the contours L^j_{mn} are open, then by S we denote z-plane cut along L.

If the contours L^j_{mn} are closed,then the positive direction for every smooth closed contour is chosen counter-clockwise, the domain interior for every L^j_{mn} is denoted by S^+_{mnj}. The union of this domains is denoted by S^+, by S^- we denote the part of z-plane which is the complement of $S^+ + L$.

We now recall some definitions from the theory of an elliptic functions [2], [7], [9], [13].

Definition 1.2. The region D of z-plane is called *doubly-periodic region* if $z \in D$ implies $z + 2m\omega_1 + 2ni\omega_2 \in D$, $m, n = 0, \pm 1, \dots$.

The points z and $z + 2m\omega_1 + 2ni\omega_2 \in D$, $m, n = 0, \pm 1, \dots$ are called *the congruent points.*

For example, the areas S^+ and S^- defined above are the *doubly periodic* regions.

Definition 1.3. A function $F_0(z)$ defined in the doubly-periodic domain D (or on the doubly-periodic line) is called *doubly quasi-periodic* with the periods $2\omega_1$ and $2i\omega_2$ if the following condition is fulfilled

$$F_0(z + 2m\omega_1 + 2ni\omega_2) = F_0(z) + m\gamma_1 + n\gamma_2, \quad m, n = 0, \pm 1, \dots . \tag{1.2}$$

γ_1 and γ_2 are the definite constants, called the *addends* [9], [13].

If $\gamma_1 = \gamma_2 = 0$ the function $F_0(z)$ is called a *doubly-periodic function.*

Definition 1.4. The doubly-periodic meromorphic function is called *an elliptic function* [2], [7], [9], [13].

The parallelogram with the vertexes $0, 2\omega_1, 2\omega_1 + 2i\omega_2, 2i\omega_2$ is called the *fundamental parallelogram* (the sides $[2\omega_1, 2\omega_1 + 2i\omega_2]$ and $[2\omega_1 + 2i\omega_2, 2i\omega_2]$ are excluded), the interior domain of this parallelogram we denote by S_{00}.

The number of poles of an elliptic function in the fundamental parallelogram is called the order of this function.

Theorem 1.1 (Liuvill)**.** *The elliptic function holomorphic in every finite region of z-plane is a constant* [2], [7], [9], [13].

Theorem 1.2. *There not exists the non-zero elliptic function of the first order* [2], [7].

Theorem 1.3 (Liuvill). *Every elliptic function of the n-th order* $(n > 1)$ *riches each of its value in the fundamental parallelogram n times* [2], [7], [9], [13].

From this theorem follows that in the fundamental parallelogram the number of poles and zeros of an elliptic function are equal.

By the Definition 1.3 follows

Theorem 1.4. *If the doubly quasi-periodic function is differentiable in its domain of definition, then the first derivative of this function is the doubly-periodic function.*

By the Theorems 1.1 and 1.4 we immediately get

Theorem 1.5. *The doubly quasi-periodic function holomorphic in every finite region of z-plane is representable in the form*

$$F_0(z) = Az + B,$$

where A and B are an arbitrary constants, the addends of this function are $\gamma_1 = 2A\omega_1$ *and* $\gamma_2 = 2Ai\omega_2$. *as*

Definition 1.5. The function representable by the doubly series

$$\begin{aligned} &\zeta(z) = \frac{1}{z} + \sum_{m,n=-\infty}^{\infty} \Big(\frac{1}{z - T_{mn}} + \frac{1}{T_{mn}} + \frac{z}{T_{mn}^2}\Big), \quad |m| + |n| \neq 0, \\ &T_{mn} = 2m\omega_1 + 2ni\omega_2, \end{aligned} \tag{1.3}$$

is called the Weierstrass "ζ-function" for the periods $2\omega_1$ and $2i\omega_2$ [2], [7], [9], [13].

The series (1.3) is equiconvergent in every closed region of z-plane not containing the points $T_{mn} = 2m\omega_1 + 2ni\omega_2$; $m, n = 0, \pm 1, \dots$.

The Weierstrass "ζ- function" has the following properties:

1. It is meromorphic function with the simple poles T_{mn}, $m, n = 0, \pm 1, \dots$;
2. $\zeta(z)$ is doubly quasi-periodic,

$$\begin{aligned} &\zeta(z + 2\omega_1) = \zeta(z) + \delta_1, \quad \zeta(z + 2i\omega_2) = \zeta(z) + \delta_2, \\ &\delta_1 = 2\zeta(\omega_1), \quad \delta_2 = 2\zeta(i\omega_2), \quad i\omega_2\delta_1 - \omega_1\delta_2 = \pi i, \end{aligned} \tag{1.4}$$

where δ_1 and δ_2 are the addends of "ζ-function".

Definition 1.6. The function $\sigma(z)$ given by the infinite product

$$\sigma(z) = z \prod_{m,n=-\infty}^{\infty} \Big(1 - \frac{z}{T_{mn}}\Big) \exp\Big\{\frac{1}{T_{mn}} + \frac{z^2}{2T_{mn}^2}\Big\}, \quad |m| + |n| \neq 0,$$

is called the Weierstrass "σ-function" for the periods $2\omega_1$ and $2i\omega_2$ [2], [7], [9], [13].

σ function is the holomorphic function with simple zeros at the points $z = T_{mn}$ $(m, n = 0, \pm 1, \pm 2, \dots)$.

"σ-function" has the following properties

$$\begin{aligned} &1. \quad \sigma(z + 2\omega_1) = -\sigma(z)\exp(\delta_1 z + \delta_1\omega_1)\,, \\ &2. \quad \sigma(z + 2i\omega_2) = -\sigma(z)\exp(\delta_2 z + \delta_2 i\omega_2). \end{aligned} \tag{1.5}$$

"σ-function" is not doubly quasi-periodic, but by means of it any elliptic function can be constructed [2], [7], [9], [13].

Theorem 1.6. *Every elliptic function $F(z)$ of the n order $(n > 1)$ with zeros $\alpha_2, \dots, \alpha_n$ and poles $\beta_1, \beta_2, \dots, \beta_n$ in the fundamental parallelogram can be represented in the form*

$$F(z) = C\,\frac{\sigma(z-\alpha_1)\,\sigma(z-\alpha_2)\cdots\sigma(z-\alpha_n)}{\sigma(z-\beta_1)\,\sigma(z-\beta_2)\cdots\sigma(z-\beta_n)}\,,$$

where $\alpha_1 = (\beta_1 + \beta_2 + \dots + \beta_n) - (\alpha_2 + \dots + \alpha_n)$, C is an arbitrary chosen constant.

Let us introduce the new class of functions

Definition 1.7. A function $F(z)$ defined in the doubly-periodic domain D is called *polynomially doubly quasi-periodic of k-order* with the periods $2\omega_1$ and $2i\omega_2$ if the following conditions are fulfilled

$$F(z+2\omega_1) = F(z) + P_{k_1}(z), \quad F(z+2i\omega_2) = F(z) + Q_{k_2}(z), \quad z \in D,$$

where P_{k_1}, Q_{k_2}, are the definite polynomials of degree k_1 and k_2 respectively, $k = \max(k_1, k_2)$. This polynomials we call the *proper polynomials* of the function $F(z)$.

This class of functions we denote by $\mathcal{P}(k)$.

The class $\mathcal{P}(0)$ is the class of *doubly quasi-periodic* functions.

The sub-class of the class $\mathcal{P}(0)$ in the case $P_{k_1} = Q_{k_2} = 0$ we denote by $\mathcal{P}_0(0)$ and call the class of *doubly-periodic* functions.

The sub-class of the class of *polynomially doubly quasi-periodic* functions is the class of *linearly doubly quasi-periodic* functions $\mathcal{P}(1)$:

Definition 1.8. A function $F(z)$ defined in the doubly-periodic domain D is called *linearly doubly quasi-periodic* with the periods $2\omega_1$ and $2i\omega_2$ if the following conditions are fulfilled

$$F(z + 2\omega_1) = F(z) + A_1 z + C_1, \quad F(z + 2i\omega_2) = F(z) + A_2 z + C_2,$$

where A_1, C_1, A_2, C_2 are the definite constants.

In this case $P_{k_1} = A_1 z + C_1, Q_{k_2} = A_2 z + C_2$.

Example. Let a be any point of z-plane not belonging to the area S^+ and let us consider the branch of the function $\ln\sigma(z-a)$ holomorphic one-valued in S^+,satisfying the conditions

1. $\ln\sigma(z - a + 2\omega_1) = \ln\sigma(z-a) + \ln(-1) + \delta_1 z + \delta_1\omega_1$,
2. $\ln\sigma(z + 2i\omega_2) = \ln\sigma(z-a) + \ln(-1) + \delta_2 z + \delta_2 i\omega_2$.

Taking into account the properties of σ function (1.5) by direct verification we conclude that the function $\ln\sigma(z-a)$ is *linearly doubly quasi-periodic* with the proper polynomials

$$P_1(z)=\delta_1 z+\omega_1+\ln(-1), \quad Q_1(z)=\delta_2 z+i\omega_2+\ln(-1).$$

From the Definition 1.7 and the Theorem 1.1 immediately follows the following

Theorem 1.7. *If the polynomially doubly quasi-periodic function of k-order is differentiable k-times, then its derivative of k-order is the doubly quasi-periodic function.*

Theorem 1.8. *The polynomially doubly quasi-periodic holomorphic function of k-order holomorphic in every finite region of z-plane is the polynomial of $k+1$-order.*

For example the linearly doubly quasi-periodic holomorphic function is representable in the form

$$F(z)=Az^2+Bz+C,$$

where A, B, C are an arbitrary given constants and the proper polynomials of this function are

$$\begin{aligned} P_1(z)&=4A\omega_1 z+4A\omega_1^2+2B\omega_1,\\ Q_1(z)&=4Ai\omega_2 z-4A\omega_2^2+2Bi\omega_2. \end{aligned} \tag{1.6}$$

Definition 1.9. The function $\Phi(z)$ is called *sectionally holomorphic polynomially doubly quasi-periodic* with the jump line L, if it has the following properties:

1. It is holomorphic in each finite region not containing points of the line L;

2. $\Phi(z)$ is continuous on L from the left and from the right, with the possible exception of the points $c_1, c_2, \dots, c_q$, near which the following condition is fulfilled

$$|\Phi(z)|\le \frac{C}{|z-c|^{\alpha}}\,,$$

where c is one of the points $c_1, c_2, \dots, c_q$, and C and α are the certain real constants, $\alpha<1$;

3. The function $\Phi(z)$ is *polynomially doubly quasi-periodic* in S^+ and S^- i.e.,

$$\Phi(z+2\omega_1)=\Phi(z)+P_{k_1}(z), \quad \Phi(z+2i\omega_2)=\Phi(z)+Q_{k_2}(z), \quad z\in S^+,$$
$$\Phi(z+2\omega_1)=\Phi(z)+P_{k_3}(z), \quad \Phi(z+2i\omega_2)=\Phi(z)+Q_{k_4}(z), \quad z\in S^-,$$

where P_{k_1}, Q_{k_2}, P_{k_3}, Q_{k_4} are the polynomials of degree k_1, k_2, k_3, k_4 respectively.

Definition 1.10. The function $\Phi(z)$ defined in the doubly-periodic domain is called exponentially doubly quasi-periodic if the following conditions are fulfilled

1. $\Phi(z+2\omega_1) = \Phi(z)\exp(P_{k_1}(z))$,
2. $\Phi(z+2i\omega_2) = \Phi(z)\exp(Q_{k_2}(z))$,

where P_{k_1} and Q_{k_2} are the definite polynomials of the k_1 and k_2 orders respectively, $k = \max(k_1, k_2)$.

This class of functions we denote by $\mathcal{P}_e(k)$.

2. The Cauchy Type Integral With the Weierstrass Kernel

Now let us consider the function

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{L_{00}} \varphi(t)\,\zeta(t-z)\,dt, \quad z \notin L, \tag{2.1}$$

where ζ is the Weierstrass "ζ-function" for the periods $2\omega_1, 2i\omega_2$, $\varphi(t)$ is the given doubly-periodic function of Muskhelishvili H^* class on L_{00} [10]:

Definition 2.1. If the function $\varphi(t)$, given on L, satisfies the Holder condition on every closed part of L_{00} not containing the finite number of points points c_i $(i = 1, \dots, p)$; of L_{00}, and if at this points it is of the form

$$\varphi(t) = \frac{\varphi_i^*(t)}{(t-c_i)^{\alpha}}, \quad 0 < \alpha < 1,$$

where $\varphi_i^*(t)$ belongs to the class H on L_{00}, then $\varphi(t)$ will be said to belong to the class H^* on L_{00}.

Integrals of this type was first considered by Sedov [12].

The integral given by (2.1) is the Cauchy type integral and by (1.4) represents the doubly-quasi periodic function with the addends

$$\gamma_1 = -\frac{\delta_1}{2\pi i} \int\limits_{L_{00}} \varphi(t)\,dt, \quad \gamma_2 = -\frac{\delta_2}{2\pi i} \int\limits_{L_{00}} \varphi(t)\,dt.$$

Taking into account the representation (1.3) we have

$$\Phi^{\pm}(t_0) = \pm\frac{\varphi(t_0)}{2} + \frac{1}{2\pi i} \int\limits_{L_{00}} \varphi(t)\,\zeta(t-t_0)\,dt, \ \ t_0 \in L, \tag{2.2}$$

where by $\Phi^+(t_0)$ and $\Phi^-(t_0)$ are denoted the boundary limits from the left and from the right of L respectively.

The integral (2.1) is called the Cauchy type integral with the Weierstrass kernel.

The formulas (1.3), (1.4) and (2.1) implies the following theorem [5]

Theorem 2.1. *The function*

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{L_{00}} \varphi(t)\,\zeta(t-z)\,dt + Az + B, \tag{2.3}$$

where A and B are an arbitrary fixed constants, represents the doubly quasi-periodic function with the jump line L and with the addends

$$\gamma_1 = -\frac{\delta_1}{2\pi i} \int\limits_{L_{00}} \varphi(t)\,dt + 2A\omega_1, \quad \gamma_2 = -\frac{\delta_2}{2\pi i} \int\limits_{L_{00}} \varphi(t)\,dt + 2Ai\omega_2.$$

The function (2.3) *is doubly-periodic if and only if $A = 0$ and*

$$\int\limits_{L_{00}} \varphi(t)\,dt = 0.$$

From the Theorems 2.1 and 1.5 follows that the function (2.3), is the the general solution of the following boundary value problem

Problem 2.1. Find sectionally holomorphic doubly quasi-periodic function $\Phi(z)$ with the jump line L satisfying the boundary condition

$$\Phi^+(t_0) - \Phi^-(t_0) = \varphi(t_0), \quad t_0 \in L,$$

where $\varphi(t)$ is the given doubly periodic function on L, belonging to the Muskhelishvili H^* class on L_{00} [10].

From the Theorems 2.1 and 1.8 follows that the function

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{L_{00}} \varphi(t)\,\zeta(t-z)\,dt + P_{k+1}(z), \tag{2.4}$$

where $P_{k+1}(z)$ is an arbitrary polynomial of degree $k+1$, is the general solution of the following boundary value problem

Problem 2.2. In the class of functions $\mathcal{P}(k)$ find the sectionally-holomorphic function $\Phi(z)$ with the jump line L, satisfying the boundary condition

$$\Phi^+(t_0) - \Phi^-(t_0) = \varphi(t_0), \quad t_0 \in L,$$

where $\varphi(t_0)$ is the given doubly periodic function on L of Muskhelishvili H^* class on L_{00}.

In the sequel we will use the solutions of the following auxiliary problem

Problem 2.3. Find doubly-periodic function $\Phi(z)$ with the jump line L sectionally holomorphic everywhere with the possible exception of the points $\beta_1 + 2m\omega_1 + 2ni\omega_2, \beta_2 + 2m\omega_1 + 2ni\omega_2, \dots, \beta_q + 2m\omega_1 + 2ni\omega_2$; $m, n = 0, \pm 1, \dots$; $\beta_1, \beta_2, \dots, \beta_q \in S_{00} - L_{00}$, $\beta_i \neq \beta_j$, $i \neq j$, $i, j = 1, \dots, q$, where it may has the simple poles, also the following boundary condition is satisfied

$$\Phi^+(t_0) - \Phi^-(t_0) = \varphi(t_0), \quad t_0 \in L,$$

where $\varphi(t_0)$ is the given doubly periodic function on L of Muskhelishvili H^* class on L_{00}.

At first we will find the solution with the simple poles at the points $\beta_1 + 2m\omega_1 + 2ni\omega_2$, $m, n = 0, \pm 1, \dots$. By the Theorem 2.1 and the formula (1.4) we obtain

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{L_{00}} \varphi(t) \left[\zeta(t-z) - \zeta(\beta_1 - z)\right] dt + C, \tag{2.5}$$

where C is an arbitrary constant, is the solution of the Problem 2.3.

Let us check the uniqueness. If $\Phi_1(z)$ and $\Phi_2(z)$ are two possible solutions, then $(\Phi_1(t) - \Phi_2(t))^+ = (\Phi_1(t) - \Phi_2(t))^-$, $t \in L$, so the function $\Phi_1(z) - \Phi_2(z)$ is doubly-periodic holomorphic in every finite region of z-plane with the simple poles at the points $\beta_1 + 2m\omega_1 + 2ni\omega_2$; $m, n = 0, \pm 1, \dots$ or with no poles. By the Theorems 1.1 and 1.2 we obtain $\Phi_1(z) - \Phi_2(z) = C$, where C is an arbitrary constant.

Now consider the case, when the solution has the poles at the points $\beta_1 + 2m\omega_1 + 2ni\omega_2$, $\beta_2 + 2m\omega_1 + 2ni\omega_2, \dots, \beta_q + 2m\omega_1 + 2ni\omega_2$; $m, n = 0, \pm 1, \pm 2, \dots$; $\beta_1, \dots, \beta_q \in S_{00} - L_{00}$, $q > 1$.

The function given by (2.5) is the particular solution of the Problem 2.3 in case of $q > 1$. Now let us check the solutions of the homogeneous problem. The solution $\Phi_0(z)$ of the corresponding homogeneous problem is doubly-periodic having simple poles at the points $\beta_1 + 2m\omega_1 + 2ni\omega_2, \beta_2 + 2m\omega_1 + 2ni\omega_2, \dots, \beta_q + 2m\omega_1 + 2ni\omega_2$; $m, n = 0, \pm 1, \pm 2, \dots$. As the function $\Phi_0(z)$ satisfies the condition $\Phi_0^+ = \Phi_0^-$, by the Theorem 1.6 it is representable in the form

$$\Phi_0(z) = C \, \frac{\sigma(z-\alpha_1)\,\sigma(z-\alpha_2)\cdots\sigma(z-\alpha_q)}{\sigma(z-\beta_1)\,\sigma(z-\beta_2)\cdots\sigma(z-\beta_q)}\,, \tag{2.6}$$

where the constants $\alpha_1, \alpha_2, \dots, \alpha_q$ satisfy the condition

$$\alpha_1 + \alpha_2 + \cdots + \alpha_q = \beta_1 + \beta_2 + \cdots + \beta_q.$$

Taking into account (2.5) and (2.6) we conclude

Theorem 2.2. *The solution of the Problem* 2.3 *exists and in case of* $q > 1$ *the general solution is given by*

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{L_{00}} \varphi(t) \left[\zeta(t-z) - \zeta(\beta_1 - z)\right] dt +$$
$$+ C_1 \frac{\sigma(z-\alpha_1)\,\sigma(z-\alpha_2)\cdots\sigma(z-\alpha_q)}{\sigma(z-\beta_1)\,\sigma(z-\beta_2)\cdots\sigma(z-\beta_q)} + C_2, \tag{2.7}$$

where C_1 *and* C_2 *are an arbitrary constant and the constants* $\alpha_1, \alpha_2, \dots, \alpha_q$ *satisfy the condition*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_q = \beta_1 + \beta_2 + \cdots + \beta_q.$$

In case of $q = 1$ *the solution of the Problem* 2.3 *is given by* (2.7), *where* $C_1 = 0$.

Now let us consider the Problem of the type 2.2 for the class $P_e(0)$.

Problem 2.4. Find the function $\Phi(z)$ of the $P_e(0)$ class with the jump line L sectionally holomorphic everywhere with the possible exception of the points $\beta_1+2m\omega_1+2ni\omega_2, \beta_2+2m\omega_1+2ni\omega_2, \dots, \beta_q+2m\omega_1+2ni\omega_2$; $m,n=0,\pm1,\dots$; $\beta_1,\beta_2,\dots,\beta_q \in S_{00}-L_{00}, \beta_i \neq \beta_j$, $i \neq j$, where it may has the simple poles, also the following boundary condition is satisfied

$$\Phi^+(t_0)-\Phi^-(t_0)=\varphi(t_0), \quad t_0 \in L, \tag{2.8}$$

where $\varphi(t_0)$ is the given Holder continuous on L_{00} function of $P_e(0)$ class on L i.e.,

1. $\varphi(t_0+2\omega_1)=\varphi(t_0)\exp(\gamma_1)$,
2. $\varphi(t_0+2i\omega_2)=\varphi(t_0)\exp(\gamma_2), \quad t_0 \in L,$

where γ_1 and γ_2 are the given constants.

Consider the function

$$\Phi_*(z)=\Phi(z)\frac{\sigma(z-b_0)}{\sigma(z-a_0)}\exp(Bz), \quad z \notin L$$

where the constants a_0, b_0, ($b_0 \notin L$, $a_0 \in S_{00}$) and B satisfy the system

$$\delta_1(a_0-b_0)+B2\omega_1=-\gamma_1, \quad \delta_2(a_0-b_0)+B2i\omega_2=-\gamma_2.$$

By (1.4) the solution of this system exists.

We obtain the doubly-periodic function with the poles at the points $a_0+2m\omega_1+2ni\omega_2, \beta_1+2m\omega_1+2ni\omega_2, \dots, \beta_q+2m\omega_1+2ni\omega_2$; $m,n=0,\pm1,\dots$; $a_0 \in S_{00}-L_{00}$. Multiplying both sides of (2.8) by the function

$$\frac{\sigma(t-b_0)}{\sigma(t-a_0)}\exp(Bt),$$

we obtain

$$\Phi_*^+(t_0)-\Phi_*^-(t_0)=\varphi(t_0)\frac{\sigma(t_0-b_0)}{\sigma(t_0-a_0)}\exp(Bt_0), \quad t_0 \in L.$$

Hence the function $\Phi_*(z)$ is the solution of the Problem 2.3 and we conclude

Theorem 2.3. *The solution of the Problem* 2.3 *exists and is given by*

$$\Phi(z)=\frac{\sigma(z-a_0)}{\sigma(z-b_0)}\exp(-Bz)\bigg[\frac{1}{2\pi i}\int\limits_{L_{00}}\varphi_0(t)\left[\zeta(t-z)-\zeta(a_0-z)\right]dt+$$
$$+C_1\frac{\sigma(z-b_0^*)\,\sigma(z-\alpha_1)\cdots\sigma(z-\alpha_q)}{\sigma(z-a_0)\,\sigma(z-\beta_1)\cdots\sigma(z-\beta_q)}+C_2\bigg], \tag{2.9}$$

where

$$C_2=-\frac{1}{2\pi i}\int\limits_{L_{00}}\varphi_0(t)\left[\zeta(t-b_0)-\zeta(a_0-b_0)\right]dt,$$

$$a_0-b_0=\frac{-\gamma_1 i\omega_2+\gamma_2\omega_1}{\pi i}, \quad B=\frac{-\delta_1\gamma_2+\delta_2\gamma_1}{2\pi i},$$

C_1 is an arbitrary chosen constant, the constants $\alpha_1, \alpha_2, \dots, \alpha_q$ satisfy the condition

$$b_0^* + \alpha_1 + \alpha_2 + \dots + \alpha_q = a_0 + \beta_1 + \dots + \beta_q,$$

b_0^*, is the point congruent to the point b_0, $b_0^* \in S_{00}$, and the function $\varphi_0(t)$ is given by

$$\varphi_0(t) = \varphi(t)\frac{\sigma(t-b_0)}{\sigma(t-a_0)}\exp(Bt).$$

The solution with no poles is given by (2.9) *for $C_1 = 0$.*

3. Definition of a Sectionally Holomorphic Linearly Doubly Quasi-Periodic Function for a Given Discontinuity

Let the line L_{00} be the single closed contour and let us consider the following problem

Problem 3.1. Find a sectionally holomorphic function $\Phi(z)$ of the class $\mathcal{P}(1)$, with the jump line L, for the given discontinuity

$$\Phi^+(t_0) - \Phi^-(t_0) = \varphi(t_0), \quad t_0 \in L, \tag{3.1}$$

where $\varphi(t)$ is the given linearly doubly quasi-periodic function on L of H^* class on L_{00}, with the proper polynomials

$$\begin{aligned} P_1(z) &= \alpha_1 z + \beta_1, \\ Q_1(z) &= \alpha_2 z + \beta_2, \end{aligned} \tag{3.2}$$

$\alpha_1, \alpha_2, \beta_1, \beta_2$ are the given constants.

Let us consider the function

$$\Psi(z) = \begin{cases} A\ln\sigma(z-a) + B\zeta(z-b) + A_1 z^2 + B_1 z + C_1, & z \in S^+, \\ 0, & z \in S^-, \end{cases} \tag{3.3}$$

where a and b are an arbitrary fixed points of the area $S_{00} - (S_{001}^+ + L_{00})$, ($S_{001}^+$ and S_{00} are the areas defined in §1), $\ln\sigma(z-a)$ is any fixed branch holomorphic in S^+, the constants A, B, A_1, B_1 will be defined in the sequel.

By (1.4) and (1.5) the function $\Psi(z)$ is sectionally holomorphic linearly doubly quasi-periodic with the jump line L and with the proper polynomials

$$\begin{aligned} P_1(z) &= A\delta_1(z - a + \omega_1) + A\ln(-1) + B\delta_1 + 4A_1\omega_1 z + \\ &\quad + 4A_1\omega_1^2 + 2B_1\omega_1, \\ Q_1(z) &= A\delta_2(z - a + i\omega_2) + A\ln(-1) + B\delta_2 + 4A_1 i\omega_2 z - \\ &\quad - 4A_1\omega_2^2 + 2B_1 i\omega_2, \end{aligned} \tag{3.4}$$

satisfies the boundary conditions

$$\begin{aligned} &\Psi^+(t_0) - \Psi^-(t_0) = \\ &= A\ln\sigma(t_0-a) + B\zeta(t_0-b) + A_1 t_0^2 + B_1 t_0 + C_1, \quad t_0 \in L, \end{aligned} \tag{3.5}$$

where $\ln\sigma(t_0-a)$ is the limiting value of the branch of the function $\ln\sigma(z-a)$ holomorphic in S^+.

Consider a new unknown sectionally-holomorphic function $\Phi(z)-\Psi(z)$. By (3.1) and (3.3) this function satisfies the following boundary condition

$$\begin{gathered}\big(\Phi(t_0)-\Psi(t_0)\big)^+ - \big(\Phi(t_0)-\Psi(t_0)\big)^- = \\ = \varphi(t_0) - A\ln\sigma(t_0-a) - B\zeta(t_0-b) - A_1t_0^2 - B_1t_0 - C_1,\quad t_0\in L. \end{gathered} \tag{3.6}$$

The constants A, B, A_1, B_1, we chose in such a way that the function in the right-hand side of (3.6) will be doubly-periodic. By (3.2) and (3.4) they should satisfy the system

$$\begin{cases} \alpha_1 z+\beta_1 = A\big(\delta_1 z+\delta_1\omega_1-\delta_1 a+\ln(-1)\big)+ \\ \qquad +B\delta_1+4A_1\omega_1 z+4A_1\omega_1^2+2B_1\omega_1, \\ \alpha_2 z+\beta_2 = A\big(\delta_2 z+\delta_2 i\omega_2-\delta_2 a+\ln(-1)\big)+ \\ \qquad +B\delta_2+4A_1 i\omega_2 z-4A_1\omega_2^2+2B_1 i\omega_2, \end{cases} \tag{3.7}$$

The solution of this system exists and this fact provides that the function in the right-hand side of (3.6) is doubly-periodic.The solution of this system is given in the sequel by the formula (3.9). Hence the function $\Phi(z)-\Psi(z)$ is the linearly doubly quasi-periodic solution of the Problem 2.2

$$\Phi(z)-\Psi(z) = \frac{1}{2\pi i}\int\limits_{L_{00}} \varphi^*(t)\zeta(t-z)\,dt + A_0z^2+B_0z+C_0, \tag{3.8}$$

where A_0, B_0, C_0 are an arbitrary fixed constants,

$$\varphi^*(t_0)=\varphi(t_0)-A\ln\sigma(t_0-a)-B\zeta(t_0-b)-A_1t_0^2-B_1t_0-C_1,\quad t_0\in L.$$

Let us check the uniqueness. If $\Phi_1(z)$ and $\Phi_2(z)$ are two possible solutions, then $(\Phi_1(t)-\Phi_2(t))^+ = (\Phi_1(t)-\Phi_2(t))^-$, $t\in L$, so the function $\Phi_1(z)-\Phi_2(z)$ is of the class $P(1)$ holomorphic in every finite region of z-plane with no poles. By the Theorem 1.8 we obtain $\Phi_1(z)-\Phi_2(z)=P_2(z)$, where $P_2(z)$ is an arbitrary polynomial of the second order.

Taking into account (3.3), (3.8) and the equalities

$$\int\limits_{L_{00}} \ln\sigma(t-a)\zeta(t-z)\,dt = \begin{cases} -2\pi i\ln\sigma(z-a), & z\in S^+, \\ 0, & z\in S^-, \end{cases}$$

$$\int\limits_{L_{00}} \zeta(t-b)\zeta(t-z)\,dt = \begin{cases} -2\pi i\zeta(z-b), & z\in S^+, \\ 0, & z\in S^-, \end{cases}$$

$$\int\limits_{L_{00}} \zeta(t-b)\,dt = 0, \qquad \int\limits_{L_{00}} \ln\sigma(t-a)\,dt = 0,$$

we conclude

Theorem 3.1. *The solution of the Problem* 3.1 *exists and is representable in the form*

$$\Phi(z)=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int\limits_{L_{00}}\varphi(t)\zeta(t-z)\,dt+2A\ln\sigma(z-a)+2B\zeta(z-b)+\\ \qquad +2A_1z^2+2B_1z+2C_1+A_0z^2+B_0z+C_0, & z\in S^+,\\ \dfrac{1}{2\pi i}\displaystyle\int\limits_{L_{00}}\varphi(t)\zeta(t-z)\,dt+A_0z^2+B_0z+C_0, & z\in S^-,\end{cases}$$

where C_1, C_0, A_0, B_0 *are an arbitrary constants,* a, $b\in S_{00}-(S^+_{001}+L_{00})$.

$$\begin{cases}A &= \dfrac{\alpha_1 i\omega_2-\alpha_2\omega_1}{\pi i}, \quad A_1=\dfrac{\delta_1\alpha_2-\delta_2\alpha_1}{4\pi i},\\ B &= \dfrac{1}{\pi i}\Big\{\beta_1 i\omega_2-\beta_2\omega_1-Ai\omega_2\omega_1(\delta_1-\delta_2)-A\pi i(i\omega_2-\omega_1-a)-\\ & -4A_1 i\omega_2\omega_1(\omega_1+i\omega_2)\Big\},\\ B_1 &= \dfrac{1}{2\pi i}\Big\{\beta_2\delta_1-\beta_1\delta_2+A\delta_1\delta_2(\omega_1-i\omega_2)+A\pi i(\delta_2-\delta_1)+\\ & +4A_1(\omega_1^2\delta_2-\omega_2^2\delta_1)\Big\}.\end{cases} \tag{3.9}$$

4. On the Homogeneous Linear Conjugation Problem for the class $P_e(1)$

Problem 4.1. Find a sectionally holomorphic function $\Phi_0(z)$ of the $P_e(1)$ class with the jump line L, satisfying the condition $\Phi_0(z)\neq 0$ and the following boundary condition

$$\Phi_0^+(t_0)=G(t_0)\,\Phi_0^-(t_0),\ \ t_0\in L, \tag{4.1}$$

where $G(t_0)$ is the given function on L of the $P_e(1)$ class, $G(t_0)\neq 0$, belonging to the H class on L_{00},:

$$\begin{aligned}G(t_0+2\omega_1)&=G(t_0)\exp(\gamma_{11}t_0+\gamma_{12}),\\ G(t_0+2i\omega_2)&=G(t_0)\exp(\gamma_{21}t_0+\gamma_{22}),\ \ t_0\in L,\end{aligned} \tag{4.2}$$

where γ_{11}, γ_{12}, γ_{21}, γ_{22} are the given constants.

Let us denote by $\varkappa=\frac{1}{2\pi}[\arg G(t)]_{L_{00}}$, the symbol $[\ \ ]_{L_{00}}$ denotes the increment of the expression in the brackets as the result of a circuit around L_{00}.

Let d be an arbitrary fixed point of S^+_{001} and let us introduce the function $G_0(t)=\sigma^{-\varkappa}(t-d)G(t)$, where $\sigma(t-d)$ is the Weierstrass σ-function. The argument of $\ln G_0(t)$ will return to its initial value after any circuit of the contour L_{00}. A branch of this function may be fixed arbitrary on L_{00}. Hence $\ln G_0(t)$ is a definite one-valued function on L_{00}. By (1.5) on every L_{mn} the function $\ln G_0(t)$ is defined as follows

$$\begin{aligned}\ln G_0(t)&=\ln G_0(t-2m\omega_1-2ni\omega_2)+mP_1(t-2m\omega_1-2ni\omega_2)+\\ &\quad +nQ_1(t-2m\omega_1-2ni\omega_2),\ \ t\in L_{mn},\ \ m,n=0,\pm1,\dots,\end{aligned}$$

where

$$\begin{aligned}&P_1(t)=(\gamma_{11}-\varkappa\delta_1)t+\gamma_{12}-\varkappa(-\delta_1 d+\delta_1\omega_1+\ln(-1)),\\&Q_1(t)=(\gamma_{21}-\varkappa\delta_2)t+\gamma_{22}-\varkappa(-\delta_2 d+\delta_2 i\omega_2+\ln(-1)),\quad t_0\in L.\end{aligned}\tag{4.3}$$

Hence $\ln G_0(t)$ is the linearly doubly quasi-periodic with the proper polynomials $P_1(t)$, $Q_1(t)$.

Let us introduce the new unknown, sectionally holomorphic function

$$\Psi(z)=\begin{cases}\Phi_0(z), & z\in S^+,\\ \sigma^{\varkappa}(z-d)\,\Phi_0(z), & z\in S^-.\end{cases}$$

The condition (4.1) may be written

$$\Psi^+(t)=G_0(t)\,\Psi^-(t),\quad t\in L.$$

By taking logarithms of this equality one obtains

$$\ln\Psi^+(t)-\ln\Psi^-(t)=\ln G_0(t).\tag{4.4}$$

It will be sufficient to obtain only some particular solution of the problem (4.4). Assuming $\ln\Psi(z)$ to be one valued by (1.5) we conclude that this function is sectionally holomorphic linearly doubly quasi-periodic. Taking into account the Theorem 3.1 we obtain a particular solution of the problem (4.4)

$$\ln\Psi(z)=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_{L_{00}}\big[\ln G_0(t)-A\ln\sigma(t-a)-B\zeta(t-b)\big]\zeta(t-z)\,dt+\\ \quad+A\ln\sigma(z-a)+B\zeta(z-b)+2A_1z^2+2B_1z+P_2(z), & z\in S^+,\\ \dfrac{1}{2\pi i}\displaystyle\int_{L_{00}}\big[\ln G_0(t)-A\ln\sigma(t-a)-B\zeta(t-b)\big]\times\\ \quad\times\zeta(t-z)\,dt+P_2(z), & z\in S^-.\end{cases}$$

where $P_2(z)$ is an arbitrary polynomial of the second order and the constants A, B, A_1, B_1 are defined by the equalities (3.9), where

$$\begin{aligned}&\alpha_1=\gamma_{11}-\varkappa\delta_1,\quad \beta_1=\gamma_{12}-\varkappa\big(-\delta_1 d+\delta_1\omega_1+\ln(-1)\big),\\&\alpha_2=\gamma_{21}-\varkappa\delta_2,\quad \beta_2=\gamma_{22}-\varkappa\big(-\delta_2 d+\delta_2 i\omega_2+\ln(-1)\big).\end{aligned}\tag{4.5}$$

$\Psi(z)$ is sectionally-holomorphic everywhere different from zero. By direct verification it is seen that a particular solution of the Problem 4.4 is

$$\Psi(z)=\begin{cases}\sigma^A(z-a)\times\\ \quad\times\exp\big\{\Gamma(z)+B\zeta(z-b)+2A_1z^2+2B_1z+P_2(z)\big\}, & z\in S^+,\\ \exp\big\{\Gamma(z)+P_2(z)\big\}, & z\in S^-,\end{cases}\tag{4.6}$$

where

$$\Gamma(z)=\frac{1}{2\pi i}\int_{L_{00}}\big[\ln G_0(t)-A\ln\sigma(t-a)-B\zeta(t-b)\big]\,\zeta(t-z)\,dt.\tag{4.7}$$

From the particular solution (4.6) of the problem (4.4) the particular solution of the original Problem 4.1 follows directly

$$\Phi_0(z)=\begin{cases}\sigma^A(z-a)\times \\ \quad\times\exp\left\{\Gamma(z)+B\zeta(z-b)+2A_1z^2+2B_1z+P_2(z)\right\}, & z\in S^+, \\ \sigma^{-\varkappa}(z-d)\exp\left\{\Gamma(z)+P_2(z)\right\}, & z\in S^-,\end{cases} \tag{4.8}$$

where $\Gamma(z)$ is given by (4.7).

Let us check the uniqueness. Let $\Phi_1(z)$ and $\Phi_2(z)$ will be two possible solutions of the Problem 4.1, then by (4.1) we have

$$\left(\frac{\Phi_1(t)}{\Phi_2(t)}\right)^+=\left(\frac{\Phi_1(t)}{\Phi_2(t)}\right)^-.$$

Consequently the function $\ln\frac{\Phi_1(z)}{\Phi_2(z)}$ is holomorphic in every finite region of z-plane and of the class $\mathcal{P}_e(1)$. The Theorem 1.8 implies

$$\frac{\Phi_1(z)}{\Phi_2(z)}=\exp(Az^2+Bz+C).$$

Hence (4.8) is the general solution of the Problem 4.1.

If the function $\ln G(t)$ is doubly quasi-periodic and it is required to find solutions of the Problem 4.1 for which $\ln\Phi_0(z)$ is doubly quasi-periodic then the following condition must be fulfilled

$$\Phi_0(z+2\omega_1)=\Phi_0(z)\exp\gamma_1,$$
$$\Phi_0(z+2i\omega_2)=\Phi_0(z)\exp\gamma_2.$$

This conditions and (1.5), (4.8) implies

$$-\varkappa\delta_1+4A_0\omega_1=0,\quad -\varkappa\delta_2+4A_0i\omega_2=0,$$
$$-\varkappa\delta_1+4\omega_1(2A_1+A_0)=0,\quad -\varkappa\delta_2+4i\omega_2(2A_1+A_0)=0$$

and by the condition (1.4) we obtain $\varkappa=A=A_0=A_1=0$.

So we conclude

Theorem 4.1. *The solution of the Problem* 4.1 *exists and all solutions are representable in the form*

$$\Phi_0(z)=\begin{cases}\sigma^A(z-a)\exp\left\{\dfrac{1}{2\pi i}\displaystyle\int_{L_{00}}\varphi^*(t)\,\zeta(t-z)\,dt+\right. \\ \qquad\qquad \left.+B\zeta(z-b)+2A_1z^2+2B_1z+P_2(z)\right\}, & z\in S^+, \\ \sigma^{-\varkappa}(z-d)\exp\left\{\dfrac{1}{2\pi i}\displaystyle\int_{L_{00}}\varphi^*(t)\zeta(t-z)dt+P_2(z)\right\}, & z\in S^-,\end{cases} \tag{$*$}$$

where $\varphi^*(t)$ is given by

$$\varphi^*(t)=\ln G_0(t)-A\ln\sigma(t-a)-B\zeta(t-b).$$

a, b, d are an arbitrary fixed points, a, $b \in S_{00} - (S^{+}_{001} + L_{00})$, $d \in S^{+}_{001}$, A, A_1, B, B_1 are the certain constants given by the (3.9), (4.5)*, $P_2(z)$ is an arbitrary polynomial of the second order, $P_2(z) = A_0 z^2 + B_0 + C_0$,*

$$G_0(t) = \sigma^{-\varkappa}(t-d)G(t), \quad \varkappa = \frac{1}{2\pi}\,[\arg G(t)]_{L_{00}}.$$

If $\ln G(t)$ is doubly quasi-periodic the solution of the Problem 4.1 *for which $\ln \Phi_0(z)$ is doubly quasi-periodic exists if and only if $\varkappa = 0$ and is given by* $(*)$*, where*

$$A = A_1 = 0, \quad B = \frac{1}{\pi i}(\gamma_{12} i\omega_2 - \gamma_{22}\omega_1), \quad B_1 = \frac{1}{2\pi i}(\gamma_{22}\delta_1 - \gamma_{12}\delta_2).$$

In the case when the function $G(t)$ is doubly-periodic the solution of the Problem 4.1 *is given by* $(*)$*, where $A = -\varkappa$, $B = -\varkappa(a-d)$, the solutions for which $\ln \Phi_0(z)$ is doubly quasi-periodic exists if and only if $\varkappa = 0$ and is given by*

$$\Phi_0(z) = \exp\left\{\frac{1}{2\pi i}\int\limits_{L_{00}} \varphi^*(t)\,\zeta(t-z)\,dt + Bz + C\right\},$$

where B and C are an arbitrary chosen constants.

Note. In the case when the function $G(t)$ is doubly-periodic and $a = d \in L_{00}$ the solution of the Problem 4.1 *is given by*

$$\Phi_0(z) = \sigma^{-\varkappa}(z-d)\exp\left\{\frac{1}{2\pi i}\int\limits_{L_{00}} \ln G(t)\zeta(t-z)\,dt + P_2(z)\right\},$$

Now we consider the case when the function $G(t)$ is of the $P_e(0)$ class.

Problem 4.2. Find the function $\Psi_0(z)$ of the $P_e(0)$ class with the jump line L sectionally holomorphic everywhere with the possible exception of the finite number of points of the area $S_{00} - L_{00}$ where it can have simple poles and $\Psi_0(z)$ satisfies the boundary condition

$$\Psi_0^{+}(t_0) = G(t_0)\,\Psi_0^{-}(t_0), \quad t_0 \in L, \tag{4.9}$$

where $G(t)$ is the given function on L Holder continuous on L_{00}, $G(t) \neq 0$, and of $P_e(0)$ class, i.e.,

$$G(t_0 + 2\omega_1) = G(t_0)\exp(\gamma_{12}), \quad G(t_0 + 2i\omega_2) = G(t_0)\exp(\gamma_{22}), \quad t_0 \in L,$$

The Theorem 4.1 implies that the solutions which have not zeros or poles of this problem exists only in the case $\varkappa = 0$. So we admit that the solution of the Problem 4.2 is non-zero everywhere in z except the points $a_1 + 2m\omega_1 + 2i\omega_2, a_2 + 2m\omega_1 + 2i\omega_2, \ldots, a_\lambda + 2m\omega_1 + 2i\omega_2$; $m, n = 0, \pm 1, \pm 2, \ldots$; $a_1, a_2, \ldots, a_\lambda \in S_{00} - L_{00}$, $\lambda > 0$, where (as it will be clear in the sequel $\lambda = |\varkappa|$) it may have simple poles or zeros. By the formulas (1.5) and (4.9) the function

$$\Phi(z) = \frac{\Psi_0(z)}{\sigma(z-a_1)\,\sigma(z-a_2)\ldots\sigma(z-a_\lambda)}$$

in case of zeros, or

$$\Phi(z) = \Psi_0(z)\,\sigma(z-a_1)\,\sigma(z-a_2)\ldots\sigma(z-a_\lambda),$$

in case of poles, satisfies the conditions of the Problem 4.1. Thus, applying the Theorem 4.1 we conclude

Theorem 4.2. *The solution of the Problem* 4.2 *exists*
1. *For $\varkappa > 0$, the solutions has simple zeros, and is given by*

$$\Psi_0(z)=\begin{cases} C\sigma(z-a_1)\ldots\sigma(z-a_\varkappa)\sigma^{-\varkappa}(z-a)\times \\ \quad\times\exp\Big\{\dfrac{1}{2\pi i}\displaystyle\int\limits_{L_{00}}\varphi^*(t)\zeta(t-z)dt+\dfrac{1}{2\pi i}(\gamma_{22}\delta_1-\gamma_{12}\delta_2)z+ \\ \quad+\dfrac{1}{\pi i}(\gamma_{12}i\omega_2-\gamma_{22}\omega_1)\zeta(z-b)+B_0z+C_0\Big\}, \quad z\in S^+, \\ C\sigma(z-a_1)\ldots\sigma(z-a_\varkappa)\sigma^{-\varkappa}(z-d)\times \\ \quad\times\exp\Big\{\dfrac{1}{2\pi i}\displaystyle\int\limits_{L_{00}}\varphi^*(t)\zeta(t-z)\,dt+B_0z+C_0\Big\}, z\in S^-, \end{cases} \tag{4.10}$$

where $\varphi^*(t)$ is given by

$$\varphi^*(t) = \ln G_0(t) + \varkappa\ln\sigma(t-a) - \frac{1}{\pi i}(\gamma_{12}i\omega_2-\gamma_{22}\omega_1)\zeta(t-b).$$

a, b, d are an arbitrary fixed points, a, $b\in S_{00}-(S^+_{001}+L_{00})$, $d\in S^+_{001}$, B_0, C_0 are an arbitrary fixed constants, $a_1,\ldots,a_\varkappa$ are an arbitrary constants,

$$G_0(t) = \sigma^{-\varkappa}(t-d)G(t), \quad \varkappa = \frac{1}{2\pi}\,[\arg G(t)]_{L_{00}}.$$

2. *For $\varkappa < 0$, the solutions of the Problem* 4.2 *are not holomorphic everywhere. They have the simple poles at the points $a_1+2m\omega_1+2i\omega_2, a_2+2m\omega_1+2i\omega_2,\ldots,a_{-\varkappa}+2m\omega_1+2i\omega_2$; $m,n=0,\pm1,\pm2,\ldots$, $a_1,\ldots,a_{-\varkappa}\in S_{00}-L_{00}$, and are given by*

$$\Psi_0(z)=\begin{cases} C\sigma^{-1}(z-a_1)\ldots\sigma^{-1}(z-a_{-\varkappa})\sigma^{-\varkappa}(z-a)\times \\ \quad\times\exp\Big\{\dfrac{1}{2\pi i}\displaystyle\int\limits_{L_{00}}\varphi^*(t)\zeta(t-z)\,dt+ \\ \quad+\dfrac{1}{\pi i}(\gamma_{12}i\omega_2-\gamma_{22}\omega_1)\zeta(z-b)+ \\ \quad+\dfrac{1}{2\pi i}(\gamma_{22}\delta_1-\gamma_{12}\delta_2)z+B_0z+C_0\Big\}, \qquad z\in S^+, \\ C\sigma^{-1}(z-a_1)\ldots\sigma^{-1}(z-a_{-\varkappa})\sigma^{-\varkappa}(z-d)\times \\ \quad\times\exp\Big\{\dfrac{1}{2\pi i}\displaystyle\int\limits_{L_{00}}\varphi^*(t)\zeta(t-z)dt+B_0z+C_0\Big\}, \quad z\in S^-, \end{cases} \tag{4.11}$$

3. *For $\varkappa = 0$ the solution of the Problem 4.2 exists and is given by*

$$\Psi_0(z) = \begin{cases} \exp\Big\{\dfrac{1}{2\pi i}\displaystyle\int_{L_{00}} \varphi^*(t)\,\zeta(t-z)\,dt + \\ \quad + \dfrac{1}{\pi i}(\gamma_{12} i\omega_2 - \gamma_{22}\omega_1)\zeta(z-b) + \\ \quad + \dfrac{1}{2\pi i}(\gamma_{22}\delta_1 - \gamma_{12}\delta_2)z + B_0 z + C_0\Big\}, & z \in S^+, \\ \exp\Big\{\dfrac{1}{2\pi i}\displaystyle\int_{L_{00}} \varphi^*(t)\,\zeta(t-z)\,dt + B_0 z + C_0\Big\}, & z \in S^-, \end{cases} \tag{4.12}$$

where $\varphi^*(t)$ is given by

$$\varphi^*(t) = \ln G_0(t) - \frac{1}{\pi i}(\gamma_{12} i\omega_2 - \gamma_{22}\omega_1)\zeta(t-b),$$

B_0 and C_0 are an arbitrary chosen constants.

If the function $G(t)$ is doubly-periodic,then the solutions of the Problem 4.2 *are given by* (4.10), (4.11), (4.12) *respectively for $\gamma_{12} = \gamma_{22} = 0$, and the doubly-periodic are obtained when the constants $a_1, \dots, a_\varkappa$, B_0 satisfy the conditions:*

1. *For $\varkappa > 0$*

$$-\delta_1\bigg(a_1 + \dots + a_\varkappa + \frac{1}{2\pi i}\int_{L_{00}} \varphi^*(t)\,dt\bigg) + \varkappa\delta_1 d + 2B_0\omega_1 = 2\pi i k_1,$$

$$-\delta_2\bigg(a_1 + \dots + a_\varkappa + \frac{1}{2\pi i}\int_{L_{00}} \varphi^*(t)\,dt\bigg) + \varkappa\delta_2 d + 2B_0 i\omega_2 = 2\pi i k_2,$$

2. *For $\varkappa < 0$*

$$\delta_1\bigg(a_1 + \dots + a_{-\varkappa} - \frac{1}{2\pi i}\int_{L_{00}} \varphi^*(t)\,dt\bigg) + \varkappa\delta_1 d + 2B_0\omega_1 = 2\pi i k_1,$$

$$\delta_2\bigg(a_1 + \dots + a_{-\varkappa} - \frac{1}{2\pi i}\int_{L_{00}} \varphi^*(t)\,dt\bigg) + \varkappa\delta_2 d + 2B_0 i\omega_2 = 2\pi i k_2,$$

3. *For $\varkappa = 0$*

$$-\delta_1\frac{1}{2\pi i}\int_{L_{00}} \varphi^*(t)\,dt + 2B_0\omega_1 = 2\pi i k_1,$$

$$-\delta_2\frac{1}{2\pi i}\int_{L_{00}} \varphi^*(t)\,dt + 2B_0 i\omega_2 = 2\pi i k_2,$$

where k_1 and k_2 are the definite integers.

5. The Non-homogeneous Linear Conjugation Problem in the Case of Closed Contours

Let the doubly-periodic line L defined in §1 consists of closed contours and let the line L_{00} be the single closed contour.

Problem 5.1. Find sectionally holomorphic function $\Phi(z)$ of the $P_e(1)$ class with the jump line L satisfying the boundary condition

$$\Phi^+(t_0) = G(t_0)\,\Phi^-(t_0) + g(t_0), \quad t_0 \in L, \tag{5.1}$$

where $G(t_0)$ and $g(t_0)$ are the functions given on L, $G(t_0) \neq 0$, Holder continuous on L_{00} and of the class $P_e(1)$, i.e.,

1. $G(t+2\omega_1) = G(t)\exp(\gamma_{11}t + \gamma_{12})$,
2. $G(t+2i\omega_2) = G(t)\exp(\gamma_{21}t + \gamma_{22})$,

1. $g(t+2\omega_1) = g(t)\exp(\alpha_{11}t + \alpha_{12})$,
2. $g(t+2i\omega_2) = g(t)\exp(\alpha_{21}t + \alpha_{22})$,

where γ_{ij}, α_{ij}; $i,j = 1,2$ are the given constants.

The homogeneous problem corresponding to the Problem 5.1 is the Problem 4.1, where we assume

$$\begin{cases} \alpha_{11}z + \alpha_{12} = A(\delta_1 z + \delta_1\omega_1 - \delta_1 a + \ln(-1)) + B\delta_1 + \\ \qquad +(4A_1 + 4A_0)\omega_1 z + (4A_1 + 4A_0)\omega_1^2 + \\ \qquad +(2B_1 + 2B_0)\omega_1 - \dfrac{\delta_1}{2\pi i}\displaystyle\int\limits_{L_{00}} \varphi^*(t)\,dt, \\ \alpha_{21}z + \alpha_{22} = A(\delta_2 z + \delta_2 i\omega_2 - \delta_2 a + \ln(-1)) + B\delta_2 + \\ \qquad +(4A_1 + 4A_0)i\omega_2 z - (4A_1 + 4A_0)\omega_2^2 + \\ \qquad +(2B_1 + 2B_0)i\omega_2 - \dfrac{\delta_2}{2\pi i}\displaystyle\int\limits_{L_{00}} \varphi^*(t)\,dt, \end{cases} \tag{5.2}$$

where A, B, A_1, B_1 are the definite constants given by (3.9) and (4.5). From the system (5.2) we obtain

$$\begin{cases} A_0 = \dfrac{\delta_1(\alpha_{21} - \gamma_{2.1}) - \delta_2(\alpha_{11} - \gamma_{11})}{4\pi i}, \\ B_0 = \dfrac{1}{2\pi i}\Big\{(\alpha_{22} - \gamma_{22})\delta_1 - (\alpha_{12} - \gamma_{12})\delta_2 - \\ \qquad -\varkappa\pi i(\delta_2 - \delta_1) + 4A_0(\omega_1^2\delta_2 - \omega_2^2\delta_1)\Big\}. \end{cases} \tag{5.3}$$

The formula (5.2) provides that the function $\frac{g(t)}{\Phi_0^+(t)}$ will be doubly-periodic in its domain of definition. From (4.1) and (5.1) we get

$$\frac{\Phi^+(t)}{\Phi_0^+(t)} - \frac{\Phi^-(t)}{\Phi_0^-(t)} = \frac{g(t)}{\Phi_0^+(t)}, \quad t \in L.$$

We admit that the function $\frac{\Phi(z)}{\Phi_0(z)}$ is doubly-periodic sectionally holomorphic with no poles.So applying the Theorem 2.1 we obtain the following

Theorem 5.1. *The solution of the Problem* 5.1 *exists if and only if*

$$\int\limits_{L_{00}} \frac{g(t)}{\Phi_0^+(t)}\, dt = 0,$$

and is given by

$$\Phi(z) = \frac{\Phi_0(z)}{2\pi i} \int\limits_{L_{00}} \frac{g(t)}{\Phi_0^+(t)}\, \zeta(t-z)\, dt + C\Phi_0(z),$$

$\Phi_0(z)$ *is given by* (4.8)*, where* A_0, B_0 *are given by* (5.3) *and* C *is an arbitrary chosen constant.*

The case when $G(t)$ is of the $P_e(0)$ class we will consider separately.

Problem 5.2. Find sectionally holomorphic function $\Phi(z)$ of $P_e(0)$ class with the jump line L satisfying the boundary condition

$$\Phi^+(t_0) = G(t_0)\, \Phi^-(t_0) + g(t_0), \quad t_0 \in L, \tag{5.4}$$

where $G(t_0)$ and $g(t_0)$ are Holder continuous on L_{00} functions of class $P_e(0)$ given on L, $G(t_0) \neq 0$,

1. $G(t+2\omega_1) = G(t)\exp(\gamma_{12})$, 2. $G(t+2i\omega_2) = G(t)\exp(\gamma_{22})$,
1. $g(t+2\omega_1) = g(t)\exp(\alpha_{12})$, 2. $g(t+2i\omega_2) = g(t)\exp(\alpha_{22})$,

where $\gamma_{12}, \gamma_{22}, \alpha_{12}, \alpha_{22}$ are the given constants.

Let us consider the homogeneous problem corresponding to the Problem 5.2. By the Theorem 4.3 the solutions of the homogeneous problem in the case $\varkappa < 0$ are only zeros, in the case $\varkappa > 0$ are given by (4.10) and in the case $\varkappa = 0$ the solution is given by (4.12).

For the constructing of the solution of non-homogeneous problem we will use the solutions of the Problem 4.2 given by the formulas (4.10), (4.11), (4.12).

From (4.9) and (5.4) we get

$$\frac{\Phi^+(t)}{\Psi_0^+(t)} - \frac{\Phi^-(t)}{\Psi_0^-(t)} = \frac{g(t)}{\Psi_0^+(t)}, \quad t \in L.$$

where $\Psi_0(z)$ is given by (4.10), (4.11) or (4.12).

In case of $\varkappa > 0$ the function $\frac{\Phi(z)}{\Psi_0(z)}$ is doubly-periodic sectionally holomorphic except the points $a_1 + 2m\omega_1 + 2ni\omega_2, a_2 + 2m\omega_1 + 2ni\omega_2, \dots, a_\varkappa + \omega_1 + 2ni\omega_2$; $m, n = \pm 1, \pm 2, \dots$, where it has the simple poles.

In case of $\varkappa \leq 0$ the function $\frac{\Phi(z)}{\Psi_0(z)}$ is doubly-periodic sectionally holomorphic.

Hence applying the Theorems 2.1 and 4.3 we conclude

Theorem 5.2.

1. *In the case* $\varkappa > 0$ *the solution of the Problem* 5.2 *exists and is given by*

$$\Phi(z) = \frac{\Psi_0(z)}{2\pi i} \int\limits_{L_{00}} \frac{g(t)}{\Psi_0^+(t)} \left[\zeta(t-z) - \zeta(a_1 - z)\right] dt$$
$$+ C_1 \frac{\sigma(z-\beta_1)\sigma(z-\beta_2)\cdots\sigma(z-\beta_\varkappa)}{\sigma(z-a_1)\sigma(z-a_2)\cdots\sigma(z-a_\varkappa)} \Psi_0(z) + C_2\Psi_0(z), \qquad (5.5)$$

$\Psi_0(z)$ *is given by* (4.10), *where* A_0, B_0 *are given by (5.3) for* $\gamma_{11} = \gamma_{12} = \alpha_{11} = \alpha_{12} = 0$, C_1, C_2 *are an arbitrary constants and the constants* $\beta_1, \dots, \beta_\varkappa$ *satisfying the condition* $\beta_1 + \cdots + \beta_\varkappa = a_1 + \cdots + a_\varkappa$.

In case of $\varkappa = 1$ *the solution of the problem is given by* (4.3) *for* $C_1 = 0$.

2. *In case of* $\varkappa < 0$ *the unique solution of the Problem* 5.2 *exists if and only if*

$$\int\limits_{L_{00}} \frac{g(t)}{\Psi_0^+(t)}\, dt = 0,$$
$$\frac{1}{2\pi i} \int\limits_{L_{00}} \frac{g(t)}{\Psi_0^+(t)} \zeta(t-a_1)\, dt - \frac{1}{2\pi i} \int\limits_{L_{00}} \frac{g(t)}{\Psi_0^+(t)} \zeta(t-a_k)\, dt = 0,$$
$$k = 2, \dots, -\varkappa,$$

and is given by

$$\Phi(z) = \frac{\Psi_0(z)}{2\pi i} \int\limits_{L_{00}} \frac{g(t)}{\Psi_0^+(t)} \zeta(t-z)\, dt -$$
$$- \frac{\Psi_0(z)}{2\pi i} \int\limits_{L_{00}} \frac{g(t)}{\Psi_0^+(t)} \zeta(t-a_1)\, dt, \qquad (5.6)$$

where $\Psi_0(z)$ *is given by* (4.11), *where* A_0, B_0 *are given by* (5.3) *for* $\gamma_{11} = \gamma_{12} = \alpha_{11} = \alpha_{12} = 0$. *For* $\varkappa = 0$ *the unique solution exists if and only if*

$$\int\limits_{L_{00}} \frac{g(t)}{\Psi_0(t)}\, dt = 0$$

and is given by

$$\Phi(z) = \frac{\Psi_0(z)}{2\pi i} \int\limits_{L_{00}} \frac{g(t)}{\Psi_0^+(t)} \zeta(t-z)\, dt + C\Psi_0(z),$$

where $\Psi_0(z)$ *is given by* (4.12), *where* A_0, B_0 *are given by* (5.3) *for* $\gamma_{11} = \gamma_{12} = \alpha_{11} = \alpha_{12} = 0$, *and* C *is an arbitrary constant.*

If the function $G(t)$ *is doubly-periodic the solutions of the Problem* 5.2 *are given by* (5.5), (5.6), (5.7) *respectively for* $\gamma_{ij} = 2\pi i k_{ij}$; $i, j = 1, 2$, *where* k_{ij} *are the definite integers.*

Note. In the case when the contour L_{00} consists of several non-intersected closed contours L_{00}^{j}, $j = 1, 2, \dots, k$, the solution of the corresponding homogeneous Problem 5.1 is representable by

$$\Phi_0(z) = \Phi_1(z)\Phi_2(z)\dots\Phi_k(z),$$

where $\Phi_1(z), \Phi_2(z), \dots, \Phi_k(z)$ are the Canonical functions of the homogeneous problems of 4.2 type for L_{00}^{j}, $j = 1, 2, \dots, k$, respectively, we will chose the different zeros or poles for every $\Phi_j(z)$, $j = 1, 2, \dots, k$. So the Theorems 5.1 and 5.2 are also true for this case.

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Author's address:
I. Vekua Institute of Applied Mathematics
Tbilisi State University
2, University St., Tbilisi 0143
Georgia