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ON THE NUMBER OF REPRESENTATIONS OF POSITIVE INTEGERS BY THE QUADRATIC FORMS $x_1^2+x_2^2+4(x_3^2+\cdots+x_9^2)$ AND $x_1^2+4(x_2^2+\cdots+x_9^2)$

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Abstract. Explicit exact (non-asymptotic) formulas are derived for the number of representations of positive integers by the quadratic forms $x_1^2+x_2^2+4(x_3^2+\cdots+x_9^2)$ and $x_1^2+4(x_2^2+\cdots+x_9^2)$.

1. Preliminaries

1.1. In this paper N, a, d, n, q, r, s denote positive integers; u, v are odd positive integers; ω is a square–free integer; p is a prime number; k, ℓ are non-negative integers; c, g, h, j, m, x, α, β, γ, δ are integers; z, τ are complex variables ($\operatorname{Im}\tau>0$); $e(z)=\exp(2\pi iz)$; $Q=e(\tau)$; $(\frac{h}{u})$ is a generalized Jacobi symbol; $\eta(\gamma)=1$ if $\gamma\geq 0$ and $\eta(\gamma)=-1$ if $\gamma<0$. Further, $\sum\limits_{h \bmod q}$ and $\sum\limits_{h \bmod q}{}'$ denote, respectively, the sums in which h runs a complete and a reduced residue system modulo q; $r(n;f)$ denote the number of representations of a positive integer n by the positive quadratic form $f=a_1x_1^2+a_2x_2^2+\cdots+a_9x_9^2$ with the determinant Δ and least common multiple a of its coefficients; $\rho(n;f)$ is a sum of the singular series corresponding to $r(n;f)$.

It is well-known that

$$\vartheta_{gh}(z\mid\tau;c,N)=$$
$$=\sum_{m\equiv c \pmod N}(-1)^{h(m-c)/N}e\Big(\frac{1}{2N}\Big(m+\frac{g}{2}\Big)^2\tau\Big)e\Big(\Big(m+\frac{g}{2}\Big)z\Big)\quad(1.1)$$

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(the theta-function with characteristics g, h), hence

$$\frac{\partial}{\partial z}\,\vartheta_{gh}(z\mid\tau;c,N)=\pi i\sum_{m\equiv c\ (\mathrm{mod}\ N)}(-1)^{h(m-c)/N}(2m+g)\times$$
$$\times\, e\Big(\frac{1}{2N}\Big(m+\frac{g}{2}\Big)^2\tau\Big)e\Big(\Big(m+\frac{g}{2}\Big)z\Big). \qquad (1.2)$$

Suppose

$$\begin{aligned}\vartheta_{gh}(\tau;c,N)&=\vartheta_{gh}(0\mid\tau;c,N),\\ \vartheta'_{gh}(\tau;c,N)&=\frac{\partial}{\partial z}\,\vartheta_{gh}(z\mid\tau;c,N)\Big|_{z=0}.\end{aligned} \qquad (1.3)$$

Obviously,

$$\begin{aligned}\vartheta_{g+2j,h}(\tau;c,N)&=\vartheta_{gh}(\tau;c+j,N),\\ \vartheta'_{g+2j,h}(\tau;c,N)&=\vartheta'_{gh}(\tau;c+j,N),\end{aligned} \qquad (1.4)$$

$$\begin{aligned}\vartheta_{gh}(\tau;c+Nj,N)&=(-1)^{hj}\vartheta_{gh}(\tau;c,N),\\ \vartheta'_{gh}(\tau;c+Nj,N)&=(-1)^{hj}\vartheta'_{gh}(\tau;c,N).\end{aligned} \qquad (1.5)$$

$$\vartheta_{gh}(\tau;0,N)=\sum_{m=-\infty}^{\infty}(-1)^{hm}Q^{(2Nm+g)^2/8N}, \qquad (1.6)$$

$$\vartheta'_{gh}(\tau;0,N)=\pi i\sum_{m=-\infty}^{\infty}(-1)^{hm}(2Nm+g)Q^{(2Nm+g)^2/8N}. \qquad (1.7)$$

$$\vartheta_{-g,h}(\tau;0,N)=\vartheta_{gh}(\tau;0,N),\quad \vartheta'_{-g,h}(\tau;0,N)=-\vartheta'_{gh}(\tau;0,N). \qquad (1.8)$$

It is clear that

$$\prod_{k=1}^{9}\vartheta_{00}(\tau;0,2a_k)=1+\sum_{n=1}^{\infty}r(n;f)Q^n. \qquad (1.9)$$

Further, we put

$$\Theta(\tau;f)=1+\sum_{n=1}^{\infty}\rho(n;f)Q^n. \qquad (1.10)$$

1.2. For the sake of convenience, we quote here some known results in the form of the following lemmas.

Lemma 1 ([1], p. 811, the end of p. 954). *The entire modular form $F(\tau)$ of weight r on the convergence subgroup $\Gamma_0(4N)$ is identically zero, if in its expansion of the series*

$$F(\tau)=C_0+\sum_{n=1}^{\infty}C_nQ^n,$$

$C_0 = C_n = 0$ *for all* $n \leq \frac{r}{12}\, 4N \prod\limits_{p|4N} (1 + \frac{1}{p})$.

Lemma 2 ([4], p. 64). *For a given* $N \geq a$, *the function*

$$\psi(\tau; f) = \prod_{k=1}^{9} \vartheta_{00}(\tau; 0, 2a_k) - \Theta(\tau; f) - \lambda \prod_{k=1}^{3} \vartheta'_{g_k h_k}(\tau; 0, 2N_k),$$

where λ *is an arbitrary constant* V, *is an entire modular form of weight* $9/2$, *and the multiplier system*

$$v(M) = i^{\eta(\gamma)(\operatorname{sgn}\delta - 1)/2} \cdot i^{(|\delta|-1)^2/4} \Big(\frac{\beta\Delta \operatorname{sgn}\delta}{|\delta|}\Big)$$

on $\Gamma_0(4N)$ ($M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *is a substitution matrix from* $\Gamma_0(4N)$), *if the following conditions hold:*

1) $2 \mid g_k$, $N_k \mid N$ $(k = 1, 2, 3)$, $a \mid N$,

2) $4 \mid N \sum\limits_{k=1}^{3} \frac{h_k^2}{N_k}$, $4 \mid \sum\limits_{k=1}^{3} \frac{g_k^2}{4N_k}$,

3) *for all* α *and* δ *with* $\alpha\delta \equiv 1 \pmod{4N}$

$$\operatorname{sgn}\delta \Big(\frac{N_1 N_2 N_3}{|\delta|}\Big) \prod_{k=1}^{3} \vartheta'_{\alpha g_k, h_k}(\tau; 0, 2N_k) = \Big(\frac{-\Delta}{|\delta|}\Big) \prod_{k=1}^{3} \vartheta'_{g_k h_k}(\tau; 0, 2N_k).$$

Lemma 3 (see, e.g., [5], formula (1.13). *If* $2^k \| n$, $2^{k'} \| \Delta$, $\Delta n = 2^{k+k'} uv = r^2\omega$, $u = r_1^2 \omega_1$, $v = r_2^2 \omega_2$, *then*

$$\rho(n; f) = \frac{n^{7/2} \cdot 48 \cdot 30}{\pi^4 \Delta^{1/2} r_1^7}\, \chi_2 \prod_{\substack{p|\Delta \\ p>2}} \chi_p \prod_{p|2\Delta} (1 - p^{-8})^{-1} L(4, \omega) \times$$
$$\times \prod_{\substack{p|r_2 \\ p>2}} \Big(1 - \Big(\frac{\omega}{p}\Big) p^{-4}\Big) \sum_{d|r_1} d^7 \prod_{p|d} \Big(1 - \Big(\frac{\omega}{p}\Big) p^{-4}\Big).$$

Lemma 4 ([4], [2], p. 298). *The sum of Dirichlet* $L(4, \omega)$ *series is equal to*

$$L(4, 1) = \frac{\pi^4}{2^5 \cdot 3}, \quad L(4, 2) = \frac{11\pi^4}{2^8 \cdot 3\sqrt{2}},$$
$$L(4, \omega) = -2\pi^4 \omega^{-1/2} \sum_{0<h\leq\omega/2} \Big(\frac{h}{\omega}\Big) \Big(\frac{h^2}{2^2\omega^2} - \frac{h^3}{3\omega^3}\Big)$$
$$\text{if} \quad \omega > 1, \quad \omega \equiv 1 \pmod 4,$$
$$L(4, \omega) = 2\pi^4 \omega^{-1/2} \Big\{ \sum_{0<h\leq\omega/4} \Big(\frac{h}{\omega}\Big) \Big(\frac{h}{2^4\omega} - \frac{h^3}{3\omega^3}\Big)$$

$$- \sum_{\omega/4<h\le\omega/2} \Big(\frac{h}{\omega}\Big) \Big(\frac{1}{2^5\cdot 3} - \frac{3h}{2^4\omega} + \frac{h^2}{2\omega^2} - \frac{h^3}{3\omega^3}\Big)\Big\}$$

$$\text{if} \quad \omega \equiv 3 \pmod 4,$$

$$L(4,\omega) = 2\pi^4\omega^{-1/2}\Big\{ \sum_{0<h\le\omega/16} \Big(\frac{h}{\omega/2}\Big)\Big(\frac{11}{2^8\cdot 3} - \frac{h^2}{\omega^2}\Big) +$$

$$+ \sum_{\omega/16<h\le 3\omega/16} \Big(\frac{h}{\omega/2}\Big)\Big(\frac{5}{2^7\cdot 3} + \frac{h}{2^4\omega} - \frac{2h^2}{\omega^2} + \frac{2^4}{3}\,\frac{h^3}{\omega^3}\Big)$$

$$+ \sum_{3\omega/16<h\le\omega/4} \Big(\frac{h}{\omega/2}\Big)\Big(\frac{37}{2^8\cdot 3} - \frac{h}{2\omega} + \frac{h^2}{\omega^2}\Big)\Big\}$$

$$\text{if} \quad \omega > 2, \ \ \omega \equiv 2 \pmod 8,$$

$$L(4,\omega) = 2\pi^4\omega^{-1/2}\Big\{ \sum_{0<h\le\omega/16} \Big(\frac{h}{\omega/2}\Big)\Big(\frac{3h}{2^4\omega} - \frac{2^4h^3}{3\omega^3}\Big)$$

$$- \sum_{\omega/16<h\le 3\omega/16} \Big(\frac{h}{\omega/2}\Big)\Big(\frac{1}{2^8\cdot 3} - \frac{h}{2^2\omega} + \frac{h^2}{\omega^2}\Big)$$

$$- \sum_{3\omega/16<h\le\omega/4} \Big(\frac{h}{\omega/2}\Big)\Big(\frac{7}{2^6\cdot 3} - \frac{13h}{2^4\omega} + \frac{2^2h^2}{\omega^2} - \frac{2^4h^3}{3\omega^3}\Big)\Big\}$$

$$\text{if} \quad \omega \equiv 6 \pmod 8.$$

2. Formula for $r(n;f)$ if $f = x_1^2 + x_2^2 + 4(x_3^2 + \cdots + x_9^2)$

Lemma 5. *The function*

$$\begin{aligned}\psi(\tau;f) &= \vartheta_{00}^2(\tau;0,2)\vartheta_{00}^7(\tau;0,8) - \Theta(\tau;f)\\ &- \frac{64}{17}\,\frac{1}{512(\pi i)^3}\,\vartheta_{80}'^3(\tau;0,24)\\ &- \frac{46}{17}\,\frac{1}{1024(\pi i)^3}\,\vartheta_{80}'^2(\tau;0,24)\vartheta_{16,0}'(\tau;0,24)\\ &- \frac{32}{17}\,\frac{1}{4096(\pi i)^3}\,\vartheta_{16,0}'^3(\tau;0,24)\end{aligned} \tag{2.1}$$

is an entire modular form of weight $9/2$*, and the multiplier system*

$$v(M) = i^{\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(|\delta|-1)^2/4}\Big(\frac{4^7\beta\operatorname{sgn}\delta}{|\delta|}\Big)$$

on $\Gamma_0(48)$ ($M = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ *is a matrix of the substitution from* $\Gamma_0(48)$).

Proof. In Lemma 2 we put $a=4$, $\Delta = 4^7$; $g_1=g_2=g_3=8$; $g_1=g_2=8$, $g_3=16$; $g_1=g_2=g_3=16$; $h_1=h_2=h_3=0$; $N_1=N_2=N_3=N=12$.

It is easy to verify that the last three summands in (2.1) satisfy conditions 1) and 2) of this lemma. We have

$$\Big(\frac{N_1N_2N_3}{|\delta|}\Big)=\Big(\frac{12^3}{|\delta|}\Big)=\Big(\frac{3}{|\delta|}\Big),\quad \Big(\frac{-\Delta}{|\delta|}\Big)=\Big(\frac{-1}{|\delta|}\Big). \tag{2.2}$$

If $\alpha\delta\equiv 1 \pmod{48}$, then $\alpha\delta\equiv 1 \pmod{24}$, whence, in particular,

$$\alpha\equiv\pm1 \pmod{24},\ \text{ and respectively }\ \delta\equiv\pm1 \pmod{24}. \tag{2.3}$$

and vice versa.

Reasoning just as in ([5], p. 117) we get

$$\operatorname{sgn}\delta\Big(\frac{N_1N_2N_3}{|\delta|}\Big)=\begin{cases}\Big(\frac{-1}{|\delta|}\Big) & \text{if}\quad \delta\equiv 1(\pmod 4),\\ -\Big(\frac{-1}{|\delta|}\Big) & \text{if}\quad \delta\equiv -1(\pmod 4).\end{cases} \tag{2.4}$$

Since in the quadratic form $\Delta=4^7$, we have $(\frac{-1}{|\delta|})=(\frac{-4^7}{|\delta|})$, and formula (2.4) can, according to (2.3), be rewritten as

$$\operatorname{sgn}\delta\Big(\frac{N_1N_2N_3}{|\delta|}\Big)=$$
$$=\begin{cases}\Big(\frac{-4^7}{|\delta|}\Big) & \text{for }\ \delta\equiv 1 \pmod 4,\ \text{ i.e., also for }\ \alpha\equiv 1 \pmod 4\\ -\Big(\frac{-4^7}{|\delta|}\Big) & \text{for }\ \delta\equiv -1 \pmod 4,\ \text{ i.e., also for }\ \alpha\equiv -1 \pmod 4.\end{cases} \tag{2.5}$$

In [5] (formulas (2.5)–(2.7) and (2.8)) it is shown that

$$\vartheta'^3_{8\alpha,0}(\tau;0,24)=\begin{cases}\vartheta'^3_{8,0}(\tau;0,24) & \text{for}\quad \alpha\equiv 1 \pmod{24},\\ -\vartheta'^3_{8,0}(\tau;0,24) & \text{for}\quad \alpha\equiv -1 \pmod{24},\end{cases} \tag{2.6}$$

$$\vartheta'^2_{8\alpha,0}(\tau;0,24)\vartheta'_{16\alpha,0}(\tau;0,24)=$$
$$=\begin{cases}\vartheta'^2_{80}(\tau;0,24)\vartheta'_{16,0}(\tau;0,24) & \text{for}\quad \alpha\equiv 1 \pmod{24},\\ -\vartheta'^2_{80}(\tau;0,24)\vartheta'_{16,0}(\tau;0,24) & \text{for}\quad \alpha\equiv -1 \pmod{24},\end{cases} \tag{2.7}$$

$$\vartheta'^3_{16\alpha,0}(\tau;0,24)=\begin{cases}\vartheta'^3_{16,0}(\tau;0,24) & \text{for}\quad \alpha\equiv 1 \pmod{24},\\ -\vartheta'^3_{16,0}(\tau;0,24) & \text{for}\quad \alpha\equiv -1 \pmod{24}.\end{cases} \tag{2.8}$$

Hence, by (2.5)–(2.8), for all α and δ with $\alpha\delta\equiv 1 \pmod{48}$ we have

$$\operatorname{sgn}\delta\Big(\frac{N_1N_2N_3}{|\delta|}\Big)\vartheta'^{3}_{8\alpha,0}(\tau;0,24)=\Big(\frac{-4^7}{|\delta|}\Big)\vartheta'^{3}_{80}(\tau;0,24),$$

$$\operatorname{sgn}\delta\Big(\frac{N_1N_2N_3}{|\delta|}\Big)\vartheta'^{2}_{8\alpha,0}(\tau;0,24)\vartheta'_{16\alpha,0}(\tau;0,24)=$$
$$=\Big(\frac{-4^7}{|\delta|}\Big)\vartheta'^{2}_{80}(\tau;0,24)\vartheta'_{16,0}(\tau;0,24),$$

$$\operatorname{sgn}\delta\Big(\frac{N_1N_2N_3}{|\delta|}\Big)\vartheta'^{3}_{16\alpha,0}(\tau;0,24)=\Big(\frac{-4^7}{|\delta|}\Big)\vartheta'^{3}_{16,0}(\tau;0,24).$$

Thus condition 3) of Lemma 2 is also satisfied. □

Theorem 1. *The following identity takes place:*

$$\vartheta^2_{00}(\tau;0,2)\vartheta^7_{00}(\tau;0,8)=\Theta(\tau;f)+\frac{64}{17}\,\frac{1}{512(\pi i)^3}\,\vartheta'^{3}_{80}(\tau;0,24)$$
$$+\frac{46}{17}\,\frac{1}{1024(\pi i)^3}\,\vartheta'^{2}_{8,0}(\tau;0,24)\vartheta'_{16,0}(\tau;0,24)$$
$$+\frac{32}{17}\,\frac{1}{4096(\pi i)^3}\,\vartheta'^{3}_{16,0}(\tau;0,24).$$

Proof. According to Lemma 1, the function $\psi(\tau;f)$ will be identically zero, if all coefficients of Q^n $(n\le 36)$ in its expansion in powers of Q are zero.

If in Lemma 3 we put $\Delta=4^7$, i.e., $v=1$, $r_2=1$ and $n=2^k u$, then

$$\rho(n;f)=\frac{2^{7k/2}u^{7/2}}{\pi^4\cdot 128 r_1^7}\cdot 48\cdot 30\chi_2\,\frac{256}{255}\,L(4,\omega)\sum_{d|r_1}d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)=$$
$$=\frac{2^{7k/2}u^{7/2}\cdot 192}{17\pi^4\cdot r_1^7}\cdot\chi_2 L(4,\omega)\sum_{d|r_1}d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big). \tag{2.9}$$

The method of finding the values of χ_2 for an arbitrary quadratic form has been developed in [6]. For the quadratic form $x_1^2+x_2^2+4(x_3^2+\dots+x_9^2)$ we get

$$\begin{aligned}\chi_2&=1+(-1)^{(u-1)/2}\quad\text{if}\quad k=0,\\ &=1\quad\text{if}\quad k=0,\\ &=\frac{135+((-1)^{(u^2-1)/8}\cdot 127-8)2^{-7k|2}}{127}\quad\text{if}\ k\ge 2,\ \ 2\mid k,\ \ u\equiv 1 \pmod 4,\\ &=\frac{135-2040\cdot 2^{-7k/2}}{127}\quad\text{if}\ k\ge 2,\ \ 2\mid k,\ \ u\equiv 3 \pmod 4,\\ &=\frac{135-255\cdot 2^{-7k/2+13/2}}{127}\\ &\qquad\qquad\text{if}\quad k\ge 3,\ \ 2\mid k.\end{aligned}\tag{2.10}$$

In particular,

$$\chi_2 = \frac{136 + (-1)^{(u^2-1)/8}}{128} \quad \text{if} \quad k = 2, \ \ u \equiv 1 \pmod 4),$$
$$= \frac{15}{16} \quad \text{if} \quad k = 2, \ \ u \equiv 3 \pmod 4), \tag{2.11}$$
$$= \frac{15}{16} \ \ \text{if} \ \ k = 3.$$

Our aim now is to find the values of $\rho(n; f)$ for all $1 \le n \le 36$. In the table below, the values of χ_2 are obtained by virtue of (2.10)–(2.11), the values of $L(4, \omega)$ are calculated by means of Lemma 4 and placed after Lemma 5 from [5], the values of $\rho(n; f)$ are calculated by (2.9).

n	u	r_1	Δn	r	ω	χ_2	$L(4,\omega)$	$\rho(n;f)$
1	1	1	$4^7 \cdot 1$	128	1	2	$\frac{\pi^4}{2^5 \cdot 3}$	$\frac{4}{17}$
3	3	1	$4^7 \cdot 3$	128	3	0	$\frac{23\pi^4}{2^4 \cdot 3^4\sqrt{3}}$	0
5	5	1	$4^7 \cdot 5$	128	5	2	$\frac{17\pi^4}{2 \cdot 3 \cdot 5^3\sqrt{5}}$	64
7	7	1	$4^7 \cdot 7$	128	7	0	$\frac{113\pi^4}{2^2 \cdot 3 \cdot 7^3\sqrt{7}}$	0
9	9	3	$4^7 \cdot 9$	$128 \cdot 3$	1	2	$\frac{\pi^4}{2^5 \cdot 3}$	$\frac{4 \cdot 2161}{17}$
11	11	1	$4^7 \cdot 11$	128	11	0	$\frac{2153\pi^4}{2^4 \cdot 3 \cdot 11^3\sqrt{11}}$	0
13	13	1	$4^7 \cdot 13$	128	13	2	$\frac{493\pi^4}{2 \cdot 3 \cdot 13^3\sqrt{13}}$	1856
15	15	1	$4^7 \cdot 15$	128	15	0	$\frac{179\pi^4}{20 \cdot 15^2\sqrt{15}}$	0
17	17	1	$4^7 \cdot 17$	128	17	2	$\frac{205\pi^4}{17^3\sqrt{17}}$	$\frac{128 \cdot 615}{17}$
19	19	1	$4^7 \cdot 19$	128	19	0	$\frac{14933\pi^4}{48 \cdot 19^3\sqrt{19}}$	0
21	21	1	$4^7 \cdot 21$	128	21	2	$\frac{187\pi^4}{9 \cdot 21^2\sqrt{21}}$	9856

n	u	r_1	Δn	r	ω	χ_2	$L(4,\omega)$	$\rho(n;f)$
23	23	1	$4^7 \cdot 23$	128	23	0	$\frac{7093\pi^4}{2^2 \cdot 3 \cdot 23^3\sqrt{23}}$	0
25	25	5	$4^7 \cdot 25$	$128 \cdot 5$	1	2	$\frac{\pi^4}{2^5 \cdot 3}$	$\frac{4 \cdot 78001}{17}$
27	27	3	$4^7 \cdot 27$	$128 \cdot 3$	3	0	$\frac{23\pi^4}{2^4 \cdot 3^4\sqrt{3}}$	0
29	29	1	$4^7 \cdot 29$	128	29	2	$\frac{2669\pi^4}{2 \cdot 29^3\sqrt{29}}$	30144
31	31	1	$4^7 \cdot 31$	128	31	0	$\frac{10357\pi^4}{6 \cdot 31^3\sqrt{31}}$	0
33	33	1	$4^7 \cdot 33$	128	33	2	$\frac{2115\pi^4}{33^3\sqrt{33}}$	$\frac{128 \cdot 6345}{17}$
35	35	1	$4^7 \cdot 35$	128	35	0	$\frac{61733\pi^4}{8 \cdot 3 \cdot 35^3\sqrt{35}}$	0
2	1	1	$4^7 \cdot 2$	128	2	1	$\frac{11\pi^4}{2^8 \cdot 3\sqrt{2}}$	$\frac{2 \cdot 11}{17}$
6	3	1	$4^7 \cdot 6$	128	6	1	$\frac{29\pi^4}{2^7 \cdot 3^2\sqrt{6}}$	$\frac{4 \cdot 261}{17}$
10	5	1	$4^7 \cdot 10$	128	10	1	$\frac{1577\pi^4}{2^7 \cdot 3 \cdot 5^3\sqrt{10}}$	$\frac{4 \cdot 1577}{17}$
14	7	1	$4^7 \cdot 14$	128	14	1	$\frac{2503\pi^4}{2^6 \cdot 3 \cdot 7^3\sqrt{14}}$	$\frac{8 \cdot 2503}{17}$
18	9	3	$4^7 \cdot 18$	$128 \cdot 3$	2	1	$\frac{11\pi^4}{2^8 \cdot 3\sqrt{2}}$	$\frac{2 \cdot 24365}{17}$
22	11	1	$4^7 \cdot 22$	128	22	1	$\frac{24889\pi^4}{2^7 \cdot 3 \cdot 11^3\sqrt{22}}$	$\frac{4 \cdot 24889}{17}$
26	13	1	$4^7 \cdot 26$	128	26	1	$\frac{43679\pi^4}{2^7 \cdot 3 \cdot 13^3\sqrt{26}}$	$\frac{4 \cdot 43679}{17}$
30	15	1	$4^7 \cdot 30$	128	30	1	$\frac{36451\pi^4}{2^6 \cdot 3^4 \cdot 5^3\sqrt{30}}$	$\frac{8 \cdot 36451}{17}$
34	17	1	$4^7 \cdot 34$	128	34	1	$\frac{57241\pi^4}{2^6 \cdot 3 \cdot 17^3\sqrt{34}}$	$\frac{8 \cdot 57241}{17}$

n	u	r_1	Δn	r	ω	χ_2	$L(4,\omega)$	$\rho(n;f)$
4	1	1	$4^8\cdot 1$	256	1	$\frac{137}{128}$	$\frac{\pi^4}{2^5\cdot 3}$	$\frac{2\cdot 137}{17}$
12	3	1	$4^8\cdot 3$	256	3	$\frac{15}{16}$	$\frac{23\pi^4}{2^4\cdot 3^4\sqrt{2}}$	$\frac{32\cdot 345}{17}$
20	5	1	$4^8\cdot 5$	256	5	$\frac{135}{128}$	$\frac{17\pi^4}{2\cdot 3\cdot 5^3\sqrt{5}}$	4320
28	7	1	$4^8\cdot 7$	256	7	$\frac{15}{16}$	$\frac{113\pi^4}{3\cdot 4\cdot 7^3\sqrt{7}}$	$\frac{128\cdot 1695}{17}$
36	9	3	$4^8\cdot 9$	$256\cdot 3$	1	$\frac{137}{126}$	$\frac{\pi^4}{2^5\cdot 3}$	$\frac{2\cdot 296057}{17}$
8	1	1	$4^8\cdot 2$	256	2	$\frac{15}{16}$	$\frac{11\pi^4}{2^8\cdot 3\sqrt{2}}$	$\frac{16\cdot 165}{17}$
24	3	1	$4^8\cdot 6$	256	6	$\frac{15}{16}$	$\frac{29\pi^4}{2^7\cdot 3^2\cdot\sqrt{6}}$	$\frac{32\cdot 3915}{17}$
16	1	1	$4^9\cdot 1$	512	1	$\frac{2211959}{2080768}$	$\frac{\pi^4}{2^5\cdot 3}$	$\frac{2\cdot 17417}{17}$
32	1	1	$4^9\cdot 2$	512	2	$\frac{276225}{260096}$	$\frac{11\pi^4}{2^8\cdot 3\sqrt{2}}$	$\frac{16\cdot 23925}{17}$

Thus, according to (1.10), we get

$$\begin{aligned}\Theta(\tau;f) = 1 + \frac{4}{17}Q + \frac{2\cdot 11}{17}Q^2 + \frac{2\cdot 137}{17}Q^4 + 64Q^5 + \frac{4\cdot 261}{17}Q^6 + \\ + \frac{16\cdot 165}{17}Q^8 + \frac{4\cdot 2161}{17}Q^9 + \frac{4\cdot 1577}{17}Q^{10} + \frac{32\cdot 345}{17}Q^{12} + 1856\,Q^{13} + \\ \frac{8\cdot 2503}{17}Q^{14} + \frac{2\cdot 17417}{17}Q^{16} + \frac{128\cdot 615}{17}Q^{17} + \frac{2\cdot 24365}{17}Q^{18} + 4320\,Q^{20} + \\ +9856Q^{21} + \frac{4\cdot 24889}{17}Q^{22} + \frac{32\cdot 3915}{17}Q^{24} + \frac{4\cdot 78001}{17}Q^{25} + \\ \frac{4\cdot 43679}{17}Q^{26} + \frac{128\cdot 1695}{17}Q^{28} + +30144Q^{29} + \frac{8\cdot 36451}{17}Q^{30} + \\ \frac{16\cdot 23925}{17}Q^{32} + \frac{128\cdot 6345}{17}Q^{33} + \frac{8\cdot 57241}{17}Q^{34} + \\ + \frac{2\cdot 296057}{17}Q^{36} + \cdots . \qquad (2.12)\end{aligned}$$

By (1.6) we have

$$\vartheta_{00}(\tau;0,2)=\sum_{m=-\infty}^{\infty}Q^{m^2}=$$
$$=1+2Q+2Q^4+2Q^9+2Q^{16}+2Q^{25}+2Q^{36}+\cdots \tag{2.13}$$

and

$$\vartheta_{00}(\tau;0,8)=\sum_{m=-\infty}^{\infty}Q^{4m^2}=1+2Q^4+2Q^{16}+2Q^{36}+\cdots. \tag{2.14}$$

From (2.13) we get

$$\vartheta_{00}^2(\tau;0,2)=1+4Q+4Q^2+4Q^4+8Q^5+4Q^8+4Q^9+8Q^{10}+8Q^{13}+$$
$$+4Q^{16}+8Q^{17}+4Q^{18}+8Q^{20}+12Q^{25}+8Q^{26}+$$
$$+8Q^{29}+4Q^{32}+8Q^{34}+4Q^{36}+\cdots. \tag{2.15}$$

and from (2.14) we obtain

$$\vartheta_{00}^2(\tau;0,8)=1+4Q^4+4Q^8+4Q^{16}+8Q^{20}+4Q^{32}+4Q^{36}+\cdots. \tag{2.16}$$

Hence, by (2.16) and (2.14) we arrive at

$$\vartheta_{00}^3(\tau;0,8)=1+6Q^4+12Q^8+8Q^{12}+6Q^{16}+24Q^{20}+24Q^{24}+$$
$$+12Q^{32}+30Q^{36}\cdots. \tag{2.17}$$

and by (2.17) and (2.14) we find that

$$\vartheta_{00}^4(\tau;0,8)=1+8Q^4+24Q^8+32Q^{12}+24Q^{16}+48Q^{20}+$$
$$96Q^{24}+64Q^{28}+24Q^{32}+104Q^{36}+\cdots. \tag{2.18}$$

By tedious enough calculations it is not difficult to verify that (2.18) and (2.17) result in

$$\vartheta_{00}^7(\tau;0,8)=1+14Q^4+84Q^8+280Q^{12}+574Q^{16}+840Q^{20}+1288Q^{24}$$
$$+2368Q^{28}+3444Q^{32}+3542Q^{36}+\cdots. \tag{2.19}$$

while (2.15) and (2.19) give

$$\vartheta_{00}^2(\tau;0,2)\,\vartheta_{00}^7(\tau;0,8)\;=1+4Q+4Q^2+18Q^4+64Q^5+56Q^6+$$
$$+144Q^8+452Q^9+344Q^{10}+672Q^{12}+1856Q^{13}+1232Q^{14}+2034Q^{16}+$$
$$+4992Q^{17}+2972Q^{18}+4320Q^{20}+9856Q^{21}+5656Q^{22}+7392Q^{24}+$$
$$+17092Q^{25}+10088Q^{26}+12672Q^{28}+30144Q^{29}+17424Q^{30}+$$
$$+22608Q^{32}+50304Q^{33}+27056Q^{34}+34802Q^{36}+\cdots. \tag{2.20}$$

In [5] (formulas (2.21)–(2.25) and (2.27)) it is shown that

$$\frac{1}{512(\pi i)^3}\,\vartheta'^3_{80}(\tau;0,24)=$$
$$=\frac{1}{512(\pi i)^3}\Big(\sum_{m=-\infty}^{\infty}8\pi i(6m+1)Q^{(6m+1)^2/3}\Big)^3= \tag{2.21}$$
$$=Q-15Q^9+96Q^{17}-335Q^{25}+672Q^{33}+\cdots. \tag{2.21$_1$}$$
$$\frac{1}{1024(\pi i)^3}\,\vartheta'^2_{80}(\tau;0,24)\,\vartheta'_{16,0}(\tau;0,24)=$$
$$=\frac{1}{1024(\pi i)^3}\Big(8\pi i\sum_{m=-\infty}^{\infty}(6m+1)Q^{(6m+1)^2/3}\Big)^2\times$$
$$\times\Big(16\pi i\sum_{m=-\infty}^{\infty}(3m+1)Q^{4(3m+1)^2/3}\Big)= \tag{2.22}$$
$$=Q^2-2Q^6-10Q^{10}+20Q^{14}+39Q^{18}-74Q^{22}-70Q^{26}+$$
$$+100Q^{30}+44Q^{34}+\cdots. \tag{2.22$_1$}$$
$$\frac{1}{4096(\pi i)^3}\,\vartheta'^3_{16,0}(\tau;0,24)=$$
$$=\frac{1}{4096(\pi i)^3}\Big(16\pi i\sum_{m=-\infty}^{\infty}(3m+1)Q^{4(3m+1)^2/3}\Big)^3= \tag{2.23}$$
$$=Q^4-6Q^8+12Q^{12}-8Q^{16}+12Q^{24}-48Q^{28}+$$
$$+48Q^{32}-15Q^{36}+\cdots. \tag{2.23$_1$}$$

By (2.1), (2.20), (2.12) and (2.21)–(2.23$_1$) it is easily to verify that all coefficients of Q^n $(n\le 36)$ in the expansion of $\psi(\tau;f)$ in powers of Q are zero. Thus, according to Lemma 1, Theorem 1 is proved. □

Theorem 2. *Let $n=2^k u$, $4^7 n=r^2\omega$, $r_1^2\mid u$. Then*

$$r(n;f)=\frac{132n^{7/2}\cdot 2}{17\pi^4 r_1^7}\,L(4,\omega)\sum_{d|r_1}d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)+$$
$$+\frac{64}{17}\sum_{\substack{x_1^2+x_2^2+x_3^2=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 6}}x_1x_2x_3\quad \textit{if}\quad n\equiv 1 \pmod 4,$$
$$=0\quad \textit{if}\quad n\equiv 3 \pmod 4,$$
$$=\frac{132\cdot 2^{7/2}u^{7/2}}{17\pi^4 r_1^7}\,L(4,\omega)\sum_{d|r_1}d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)+$$

$$+\frac{46}{17}\sum_{\substack{x_1^2+x_2^2+4x_3^2=3n\\ x_1\equiv x_2\equiv 1 \pmod 6\\ x_3\equiv 1 \pmod 3}} x_1x_2x_3 \quad \textit{if} \quad n\equiv 3 \pmod 4,$$

$$=\frac{132\cdot 2^{7/2}u^{7/2}}{17\pi^4 r_1^7}\,\chi_2 L(4,\omega)\sum_{d|r_1} d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)+$$

$$+\frac{32}{7}\sum_{\substack{4(x_1^2+x_2^2+x_3^2)=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 3}} x_1x_2x_3 \quad \textit{if} \quad n\equiv 3 \pmod 4.$$

The values of χ_2 *will be calculated by formulas* (2.10) *and* (2.11)*, and the values of* $L(4,\omega)$ *by Lemma* 4.

Remark. If $p^2 \nmid n$ $(p>2)$, i.e., $r_1=1$, then in all the above formulas

$$\sum_{d|r_1} d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)=1.$$

Proof. Equating the coefficients of Q^n in both sides of the identity from Theorem 1, by virtue of (1.9) and (1.10) we get

$$r(n;f)=\rho(n;f)+\frac{64}{17}\,w_1(n)+\frac{46}{17}\,w_2(n)+\frac{32}{17}\,w_3(n),$$

where $w_1(n)$, $w_2(n)$ and $w_3(n)$ denote respectively the coefficients of Q^n in the expansions of the functions

$$\frac{1}{512(\pi i)^3}\,\vartheta'^3_{80}(\tau;0,24),\quad \frac{1}{1024(\pi i)^3}\,\vartheta^{12}_{80}(\tau;0,24)\vartheta'_{16,0}(\tau;0,24),$$
$$\frac{1}{4096(\pi i)^3}\,\vartheta^{13}_{16,0}(\tau;0,24)$$

in powers of Q.

The values of $\rho(n;f)$ are given in (2.9).

Further,

(a) from (2.21) follows

$$\frac{1}{512(\pi i)^3}\,\vartheta_{80}'^{3}(\tau;0,24)=$$

$$=\sum_{m_1,m_2,m_3=-\infty}^{\infty}(6m_1+1)(6m_2+1)(6m_3+1)Q^{((6m_1+1)^2+(6m_2+1)^2+(6m_3+1)^2)/3}=$$

$$=\sum_{n=1}^{\infty}\Big(\sum_{\substack{x_1^2+x_2^2+x_3^2=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 6}} x_1x_2x_3\Big)Q^n,$$

$$\text{i.e.,}\;\; w_1(n)=\sum_{\substack{x_1^2+x_2^2+x_3^2=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 6}} x_1x_2x_3; \tag{2.24}$$

it is obvious that $w_1(n)=0$ for $n\equiv 0 \pmod 2$ and $n=3 \pmod 4$;

(b) from (2.22) follows

$$\frac{1}{1024(\pi i)^3}\,\vartheta_{80,0}'^{2}(\tau;0,24)\vartheta_{16,0}'(\tau;0,24)=$$

$$=\sum_{m_1,m_2,m_3=-\infty}^{\infty}(6m_1+1)(6m_2+1)(3m_3+1)Q^{((6m_1+1)^2+(6m_2+1)^2+4(3m_3+1)^2)/3}=$$

$$=\sum_{n=1}^{\infty}\Big(\sum_{\substack{x_1^2+x_2^2+4x_3^2=3n\\ x_1\equiv x_2\equiv 1 \pmod 6\\ x_3\equiv 1 \pmod 3}} x_1x_2x_3\Big)Q^n \;\;\text{i.e.,}\;\; w_2(n)=\sum_{\substack{x_1^2+x_2^2+4x_3^2=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 6\\ x_3\equiv 1 \pmod 3}} x_1x_2x_3;$$

it is obvious that $w_2(n)=0$ for $n\equiv 1 \pmod 4$ and $n\equiv 0 \pmod 2$;

(c) from (2.23) follows

$$\frac{1}{4096(\pi i)^3}\,\vartheta_{16,0}'^{3}(\tau;0,24)(\tau;0,24)=$$

$$=\sum_{m_1,m_2,m_3=-\infty}^{\infty}(3m_1+1)(3m_2+1)(3m_3+1)Q^{(4(3m_1+1)^2+4(3m_2+1)^2+4(3m_3+1)^2)/3}=$$

$$=\sum_{n=1}^{\infty}\Big(\sum_{\substack{4(x_1^2+x_2^2+x_3^2)=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 3}} x_1x_2x_3\Big)Q^n,\quad\text{i.e.,}$$

$$w_3(n)=\sum_{\substack{4(x_1^2+x_2^2+x_3^2)=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 3}} x_1x_2x_3; \tag{2.25}$$

it is obvious that $w_3(n)=0$ for $n\equiv 1 \pmod 2$ and $n\equiv 2 \pmod 4$. □

3. Formula for $r(n;f)$ if $f = x_1^2 + 4(x_2^2 + \cdots + x_9^2)$

Lemma 6. *The function*

$$\psi(\tau;f) = \vartheta_{00}(\tau;0,2)\vartheta_{00}^{\delta}(\tau;0,8) - Q(\tau;f) =$$
$$-\frac{32}{17}\,\frac{1}{512(\pi i)^3}\vartheta_{80}^{'3}(\tau;0,24) - \frac{32}{17}\,\frac{1}{4096(\pi i)^3}\vartheta_{16,0}^{'3}(\tau;0,24) \qquad (3.1)$$

is an entire modular form of weight $9/2$, *and the multiplier system*

$$v(M) = i^{\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(|\delta|-1)^2/4}\Big(\frac{4^8\beta\Delta\operatorname{sgn}\delta}{|\delta|}\Big)$$

on $\Gamma_0(48)$ ($M = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ *is a substitution matrix from* $\Gamma_0(4N)$).

Proof. In Lemma 2 we put $a = 4$, $\Delta = 4^8$; $g_1 = g_2 = g_3 = 8$; $g_1 = g_2 = g_3 = 16$; $h_1 = h_2 = h_3 = 0$; $N_1 = N_2 = N_3 = N = 12$. It is easy to verify that the last two summands in (3.1) satisfy conditions (1) and (2) of this lemma.

Further, reasoning as in Lemma 5(pp. 5. 6.), but taking $\Delta = 4^8$ instead of $\Delta = 4^7$, for all α and δ with $\alpha\delta \equiv 1 \pmod{48}$ we get

$$\operatorname{sgn}\delta\Big(\frac{N_1N_2N_3}{|\delta|}\Big)\vartheta_{8\alpha,0}^{'3}(\tau;0,24) = \Big(\frac{-4^8}{|\delta|}\Big)\vartheta_{80}^{'3}(\tau;0,24),$$
$$\operatorname{sgn}\delta\Big(\frac{N_1N_2N_3}{|\delta|}\Big)\vartheta_{16\alpha,0}^{'3}(\tau;0,24) = \Big(\frac{-4^8}{|\delta|}\Big)\vartheta_{16,0}^{'3}(\tau;0,24).$$

Thus condition (3) of Lemma 2 is also satisfied. □

Theorem 3. *The following identity takes place:*

$$\vartheta_{00}(\tau;0,2)\vartheta_{00}^{8}(\tau;0,\delta) = \Theta(\tau;f) =$$
$$= \frac{32}{17}\,\frac{1}{512(\pi i)^3}\vartheta_{08}^{'3}(\tau;0,24) + \frac{32}{17}\,\frac{1}{4096(\pi i)^3}\vartheta_{16,0}^{'3}(\tau;0,24).$$

Proof. According to Lemma 1, the function $\psi(\tau;f)$ will be identically zero, if all coefficients of Q^n ($\eta \le 36$) in the expansion in powers of Q are zero.

If in Lemma 3 we put $\Delta = 4^8$, i.e., $v = 1$, $r_2 = 1$ and $\eta = 2^k u$, then

$$\rho(n;f) = \frac{2^{7k/2}u^{7/2}}{\pi^4\cdot 256 r_1^7}\cdot 48\cdot 30\chi_2\,\frac{256}{255}\,L(4,\omega)\sum_{d|r_1} d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big) =$$
$$= \frac{2^{7k/2}u^{7/2}\cdot 96}{17\pi^4\cdot r_1^7}\cdot\chi_2 L(4,\omega)\sum_{d|r_1} d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big). \qquad (3.2)$$

The method of finding the values of χ_2 for an arbitrary quadratic form has been developed in [6]. In the case of the quadratic form $f = x_1^2 + 4(x_2^2 + \cdots + x_9^2)$ we get

$$\begin{aligned}
\chi_2 &= 1-(-1)^{(u+1)/2} \quad \text{if} \quad k=0,\\
&= 0 \quad \text{if} \quad k=1,\\
&= \frac{270-2^{-7k/2+11}}{127} + (8+(-1)^{(u^2-1)/\delta})2^{-7k|2+1} \text{ if } 2 \mid k,\ u \equiv 1 \pmod 4,\\
&= \frac{270-2^{-7k/2+11}}{127} - 2^{-7k/2+4} \quad \text{if} \quad 2 \mid k, \quad u \equiv 3 \pmod 4, \qquad (3.3)\\
&= \frac{270-2^{-7k/2+29/2}}{127} - 2^{-7k/2+15/2} \quad \text{if } 2 \nmid k, \quad k \geq 3.
\end{aligned}$$

In particular,

$$\begin{aligned}
\chi_2 &= 0, \quad \text{if} \quad k=0, \quad u \equiv 3 \pmod 4,\\
&= 2 \quad \text{if} \quad k=0, \quad u \equiv 1 \pmod 4,\\
&= \frac{136+(-1)^{(u^2-1)/8}}{64} \quad \text{if} \quad k=2, \quad u \equiv 1 \pmod 4, \qquad (3.4)\\
&= \frac{15}{8} \quad \text{if} \quad k=2 \quad u \equiv 3 \pmod 4.
\end{aligned}$$

Our aim now is to find the values of $\rho(\tau; f)$ for all $1 \leq n \leq 36$. In the table below, the values of χ_2 are obtained by virtue of (3.3)–(3.4), the values of $L(4,\omega)$ are calculated by means of Lemma 4 and placed after Lemma 5 from [5]; the values of $\rho(n; f)$ are calculated by (3.2). $\square$

n	u	r_1	Δn	r	ω	χ_2	$L(4,\omega)$	$\rho(n;f)$
1	1	1	$4^8 \cdot 1$	256	1	2	$\frac{\pi^4}{2^5 \cdot 3}$	$\frac{2}{17}$
3	3	1	$4^8 \cdot 3$	256	3	0	$\frac{23\pi^4}{2^4 \cdot 3^4\sqrt{3}}$	0
5	5	1	$4^8 \cdot 5$	256	5	2	$\frac{17\pi^4}{2 \cdot 3 \cdot 5^3\sqrt{5}}$	32
7	7	1	$4^8 \cdot 7$	256	7	0	$\frac{113\pi^4}{2^2 \cdot 3 \cdot 7^3\sqrt{7}}$	0
9	9	3	$4^8 \cdot 9$	$256 \cdot 3$	1	2	$\frac{\pi^4}{2^5 \cdot 3}$	$\frac{2 \cdot 2161}{17}$
11	11	1	$4^8 \cdot 11$	256	11	0	$\frac{2153\pi^4}{2^4 \cdot 3 \cdot 11^3\sqrt{11}}$	0

n	u	r_1	Δn	r	ω	χ_2	$L(4,\omega)$	$\rho(n;f)$
13	13	1	$4^8\cdot 13$	256	13	2	$\frac{493\pi^4}{2\cdot 3\cdot 13^3\sqrt{13}}$	928
15	15	1	$4^8\cdot 15$	256	15	0	$\frac{179\pi^4}{20\cdot 15^2\sqrt{15}}$	0
17	17	1	$4^8\cdot 17$	256	17	2	$\frac{205\pi^4}{17^3\sqrt{17}}$	$\frac{192\cdot 205}{17}$
19	19	1	$4^8\cdot 19$	256	19	0	$\frac{14933\pi^4}{48\cdot 19^3\sqrt{19}}$	0
21	21	1	$4^8\cdot 21$	256	21	2	$\frac{187\pi^4}{9\cdot 21^2\sqrt{21}}$	4928
23	23	1	$4^8\cdot 23$	256	23	0	$\frac{7093\pi^4}{2^2\cdot 3\cdot 23^3\sqrt{23}}$	0
25	25	5	$4^8\cdot 25$	$256\cdot 5$	1	2	$\frac{\pi^4}{2^5\cdot 3}$	$\frac{2\cdot 78001}{17}$
27	27	3	$4^8\cdot 27$	$256\cdot 3$	3	0	$\frac{23\pi^4}{2^4\cdot 3^4\sqrt{3}}$	0
29	29	1	$4^8\cdot 29$	256	29	2	$\frac{2669\pi^4}{2\cdot 29^3\sqrt{29}}$	15072
31	31	1	$4^8\cdot 31$	256	31	0	$\frac{10357\pi^4}{6\cdot 31^3\sqrt{31}}$	0
33	33	1	$4^8\cdot 33$	256	33	2	$\frac{2115\pi^4}{33^3\sqrt{33}}$	$\frac{192\cdot 2115}{17}$
35	35	1	$4^8\cdot 35$	256	35	0	$\frac{61733\pi^4}{8\cdot 3\cdot 35^3\sqrt{35}}$	0
2	1	1	$4^8\cdot 2$	256	2	0	$\frac{11\pi^4}{2^8\cdot 3\sqrt{2}}$	0
6	3	1	$4^8\cdot 6$	256	6	0	$\frac{29\pi^4}{2^7\cdot 3^2\sqrt{6}}$	0
10	5	1	$4^8\cdot 10$	256	10	0	$\frac{1577\pi^4}{2^7\cdot 3\cdot 5^3\sqrt{10}}$	0
14	7	1	$4^8\cdot 14$	256	14	0	$\frac{2503\pi^4}{2^6\cdot 3\cdot 7^3\sqrt{14}}$	0

n	u	r_1	Δn	r	ω	χ_2	$L(4,\omega)$	$\rho(n;f)$
18	9	3	$4^8 \cdot 18$	$256 \cdot 3$	2	0	$\frac{11\pi^4}{2^8 \cdot 3\sqrt{2}}$	0
22	11	1	$4^8 \cdot 22$	256	22	0	$\frac{24889\pi^4}{2^7 \cdot 3 \cdot 11^3\sqrt{22}}$	0
26	13	1	$4^8 \cdot 26$	256	26	0	$\frac{43679\pi^4}{2^7 \cdot 3 \cdot 13^3\sqrt{26}}$	0
30	15	1	$4^8 \cdot 30$	256	30	0	$\frac{36451\pi^4}{2^6 \cdot 3^4 \cdot 5^3\sqrt{30}}$	0
34	17	1	$4^8 \cdot 34$	256	34	0	$\frac{57241\pi^4}{2^6 \cdot 3 \cdot 17^3\sqrt{34}}$	0
4	1	1	$4^9 \cdot 1$	512	1	$\frac{137}{64}$	$\frac{\pi^4}{2^5 \cdot 3}$	$\frac{2 \cdot 137}{17}$
12	3	1	$4^9 \cdot 3$	512	3	$\frac{15}{8}$	$\frac{23\pi^4}{2^4 \cdot 3^4\sqrt{3}}$	$\frac{32 \cdot 345}{17}$
20	5	1	$4^9 \cdot 5$	512	5	$\frac{135}{64}$	$\frac{17\pi^4}{2 \cdot 3 \cdot 5^3\sqrt{5}}$	4320
28	7	1	$4^9 \cdot 7$	512	7	$\frac{15}{8}$	$\frac{113\pi^4}{3 \cdot 4 \cdot 7^3\sqrt{7}}$	$\frac{128 \cdot 1695}{17}$
36	9	3	$4^9 \cdot 9$	$512 \cdot 3$	1	$\frac{137}{64}$	$\frac{\pi^4}{2^5 \cdot 3}$	$\frac{2 \cdot 296057}{17}$
8	1	1	$4^9 \cdot 2$	512	2	$\frac{1905}{1016}$	$\frac{11\pi^4}{2^8 \cdot 3\sqrt{2}}$	$\frac{16 \cdot 165}{17}$
24	3	1	$4^9 \cdot 6$	512	6	$\frac{1905}{1016}$	$\frac{29\pi^4}{2^7 \cdot 3^2 \cdot \sqrt{6}}$	$\frac{32 \cdot 3915}{17}$
16	1	1	$4^{10} \cdot 1$	1024	1	$\frac{17417}{8192}$	$\frac{\pi^4}{2^5 \cdot 3}$	$\frac{2 \cdot 17417}{17}$
32	1	1	$4^{10} \cdot 2$	1024	2	$\frac{2175}{1024}$	$\frac{11\pi^4}{2^8 \cdot 3\sqrt{2}}$	$\frac{16 \cdot 23925}{17}$

Thus according to (1.10), we get

$$\Theta(\tau; f) = 1 + \frac{2}{17}\,Q + \frac{2 \cdot 137}{17}\,Q^4 + 32Q^5 + \frac{16 \cdot 165}{17}\,Q^8 + \frac{2 \cdot 2161}{17}\,Q^9 +$$

$$+\frac{32\cdot 345}{17}\,Q^{12}+928Q^{13}+\frac{2\cdot 17417}{17}\,Q^{16}+\frac{192\cdot 205}{17}\,Q^{17}+4320\,Q^{20}+$$
$$+4928Q^{21}+\frac{32\cdot 3915}{17}\,Q^{24}+\frac{2\cdot 78001}{17}\,Q^{25}+\frac{128\cdot 1695}{17}\,Q^{28}+15072\,Q^{29}+$$
$$+\frac{16\cdot 23925}{17}\,Q^{32}+\frac{192\cdot 2115}{17}\,Q^{33}+\frac{2\cdot 296057}{17}\,Q^{36}+\cdots. \tag{3.5}$$

By (1.6) we have

$$\vartheta_{00}(\tau;0,2)=$$
$$=\sum_{m=-\infty}^{\infty}Q^{m^2}=1+2Q+2Q^4+2Q^9+2Q^{16}+2Q^{25}+2Q^{36}+\cdots. \tag{3.6}$$

and

$$\vartheta_{00}(\tau;0,8)=\sum_{m=-\infty}^{\infty}Q^{4m^2}=1+2Q^4+2Q^{16}+2Q^{36}+\cdots. \tag{3.7}$$

By tedious enough calculations it is not difficult to verify that

$$\vartheta_{00}^8(\tau;0,8)=1+16Q^4+112Q^8+448Q^{12}+880Q^{16}+2016Q^{20}+3136Q^{24}+$$
$$+5504Q^{29}+9328Q^{32}+12112Q^{36}+\cdots. \tag{3.8}$$

From (3.6) and (3.8) we get

$$\vartheta_{00}(\tau;0,2)\,\vartheta_{00}^8(\tau;0,8)\;=1+2Q+18Q^4+32Q^5+144Q^8+$$
$$+226Q^5+672Q^{12}+928Q^{13}+2034Q^{16}+2496Q^{17}+4320Q^{20}+4928Q^{21}+$$
$$7392Q^{24}+8546Q^{25}+12672Q^{28}+15072Q^{29}+22608Q^{32}+$$
$$+25152Q^{33}+34802Q^{36}+\cdots. \tag{3.9}$$

Owing to (3.1), (3.9), (3.5) and (2.21_1), (2.23_1), one can verify that all coefficients of Q^n $(n\le 36)$ in the expansion of $\psi(\tau;f)$ in powers of Q are zero. Thus, according to Lemma 1, the theorem is proved. □

Theorem 4. *Let* $n=2ku$, $4^8n=r^2\omega$ *and* $r_1^2\mid u$. *Then*

$$r(n;f)=\frac{192n^{7/2}}{17\pi^4r_1^7}\,L(4,\omega)\sum_{d|r_1}d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)+$$
$$+\frac{32}{17}\sum_{\substack{x_1^2+x_2^2+x_3^2=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 6}}x_1x_2x_3\quad \textit{if}\quad n\equiv 1 \pmod 4,$$
$$=0\quad \textit{if}\quad n\equiv 3 \pmod 4,\quad \textit{and if}\;\; n\equiv 2 \pmod 4,$$
$$=\frac{96\cdot 2^{7k/2}u^{7/2}}{17\pi^4r_1^7}\,\chi_2L(4,\omega)\sum_{d|r_1}d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)+$$

$$+\frac{32}{7}\sum_{\substack{4(x_1^2+x_2^2+x_3^2=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 3}} x_1x_2x_3 \quad \textit{if} \quad n\equiv 0 \pmod 4).$$

The values of χ_2 *will be calculated by formulas* (3.3) *and* (3.4), *and the values of* $L(4,\omega)$ *by Lemma* 4.

Remark 1. If $p^2\nmid n$ $(p>2)$, i.e., $r_1=1$, then in all the above formulas

$$\sum_{d|r_1} d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)=1.$$

Proof. Equating the coefficients of Q^n in both sides of the identity from Theorem 3, we get

$$r(n;f)=\rho(n;f)+\frac{32}{17}\,w_1(n)+\frac{32}{17}\,w_2(n),$$

where $w_1(n)$ and $w_2(n)$ denote respectively the coefficients of Q^n in the expansions of the functions

$$\frac{1}{512(\pi i)^3}\,\vartheta'^3_{80}(\tau;0,24) \;\text{ and }\; \frac{1}{4096(\pi i)^3}\,\vartheta'^3_{16,0}(\tau;0,24)$$

in powers of Q. The values of $\rho(n;f)$ are given in (3.2).

Further,

(a) from (2.24) it follows that

$$\frac{1}{512(\pi i)^3}\vartheta'^3_{80}(\tau;0,24)=\sum_{n=1}^{\infty}\Big(\sum_{\substack{x_1^2+x_2^2+x_3^2=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 6}} x_1x_2x_3\Big)Q^n,$$

i.e.,

$$w_1(n)=\sum_{\substack{x_1^2+x_2^2+x_3^2=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 6}} x_1x_2x_3,$$

as well as $w_1(n)=0$ for $n\equiv 0 \pmod 2$ and $n\equiv 3 \pmod 4$;

(b) from (2.25) it follows that

$$\frac{1}{4096(\pi i)^3}\vartheta'^3_{16,0}(\tau;0,24)=\sum_{n=1}^{\infty}\Big(\sum_{\substack{4(x_1^2+x_2^2+x_3^2)=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 3}} x_1x_2x_3\Big)Q^n,$$

i.e.,

$$w_2(n)=\sum_{\substack{4(x_1^2+x_2^2+x_3^2)=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 3}} x_1x_2x_3,$$

as well as $w_2(n)=0$ for $n\equiv 1 \pmod 2$ and $n\equiv 2 \pmod 4$. $\square$

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