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MULTIPLIER AND HILBERT C^*-CATEGORIES

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Abstract. The properties of C^*-categoroids, a generalization of C^*-categories and C^*-algebras, are studied. Namely, an embedding into some C^*-category of Hilbert modules as well as the multiplier C^*-category of a C^*-categoroid are constructed. The C^*-category of countably-generated Hilbert modules is characterized.

Introduction

The category $\mathcal{H}(B)$ of countably generated right modules over a (real or complex) σ-unital C^*-algebra B plays a crucial role in the construction of Kasparov's KK-theory. The purpose of this paper is to study formal categorical properties or, more precisely, to construct a category equivalent to the category $\mathcal{H}(B)$ using only the algebra B and categorical operations.

Our approach is based on the categorical formalization of the following observations:

- The category $\mathcal{H}(B)$ has an enriched structure, namely, it is a C^*-category;
- There is a "non-unital subcategory" of compact B-homomorphisms;
- The space of B-homomorphisms is somehow "the space of centralizers" of compact homomorphisms;
- There exists a Hilbert sum of a countable set of Hilbert modules;
- When B is a σ-unital C^*-algebra, the category $\mathcal{H}(B)$ is somehow generated by B.

The results of this paper were announced in [4].

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The paper is organized as follows. First we consider a generalized version of C^*-algebras called C^*-categoroids (or C^*-pseudocategories [4]) C^*-categoroids possess the main properties of C^*-algebras, in particular, the imbedding of C^*-categoroids into the category of Hilbert spaces and bounded linear maps. The latter result generalizes the same property of C^*-algebras and C^*-categories [8], [2].

Next we define and construct the multiplier C^*-category of a given C^*-categoroid. Note in doing so we use the definition of a stable C^*-algebra of a C^*-categoroid and the properties of its multiplier C^*-algebra. The properties of the multiplier C^*-category are studied and the corresponding results for C^*-algebras are generalize (cf. [1], [8]). The analogue of Busby's approach for extensions of C^*-algebras is not considered in this paper.

The main definitions of this paper concern the Hilbert sum in the C^*-category and a Hilbert C^*-category itself. Note that the Hilbert sum is defined in the C^*-category with formal compact morphisms which is a pair (A, B), where A is a C^*-category and B is a C^*-ideal in A such that the canonical $*$-functor $A \to M(B)$ is an isomorphism. By this definition the Hilbert C^*-category is:

- a C^*-category with formal compact morphisms where there exists a Hilbert sum of any countable set of objects (in particular, it is an additive category);
- a pseudo-abelian C^*-category;
- a universal Hilbert C^*-category which contains a given C^*-categoroid as an ideal of compact morphisms. If the C^*-categoroid is given by a σ-unital C^*-algebra B, then the universal Hilbert C^*-category of B is equivalent to $\mathcal{H}(B)$.

1. Definition of a C^*-Categoroid

In this section we give some elementary properties of C^*-categories and C^* -categoroids, a natural categorical generalization of unital C^* -algebras and C^*-algebras [2], [4].

Recall that *the diagram scheme* D consists a class of objects ObD and a set $\hom(a, b)$ for any $a, b \in ObD$. By a *k-scheme* we mean a diagram scheme D such that $\hom(a, b)$ has the structure of a k-linear space, where $k = R$ or C. D is called an *involutive* k-scheme if:

- an anti-linear map $* : \hom(a, b) \to \hom(b, a)$ is given for each $a, b \in ObD$.
- the bilinear composition law

$$\hom(a, b) \times \hom(b, a) \to \hom(a, a)$$
$$\hom(a, b) \times \hom(b, b) \to \hom(a, b)$$
$$\hom(a, a) \times \hom(a, b) \to \hom(a, b)$$

  is associative for any $a, b \in ObD$.

- $(f^*)^* = f$, and $(fg)^* = g^*f^*$ if the composition fg exists.

Definition 1. By a C^*-*scheme* is meant an involutive k-scheme D such that:

1) $\hom(a,b)$ is a Banach space; 2) involution is isometry; 3) $\|f\|^2 \leq \|f^*f + g^*g\|^2$ for any $f \in \hom(a,b)$ and $g \in \hom(a,b')$ [6], [5]; 4) the morphism f^*f is a positive element in the C^*-algebra $\hom(a,a)$ for any $f \in \hom(a,b)$ and $a,b \in ObD$.

A diagram scheme $\mathcal{D}$ is called *a categoroid* if it satisfies all the axioms of category except the existence of identities of objects. Let a and b be objects from $\mathcal{D}$. Then $\hom(a,b)$ denotes a set of morphisms from a to b. The definition of morphisms between categoroids is analogous to that of a functor, and is called *a functoroid*. If $\mathcal{F} : D \to D'$ is a functoroid of categoroids and f, g are composable morphisms in $\mathcal{D}$, then $\mathcal{F}(fg) = \mathcal{F}(f)\mathcal{F}(g)$.

Definition 2. A categoroid A is called a C^*-categoroid if it has the structure of a C^*-scheme such that
1) the composition of morphisms is bilinear and $\|fg\| \leq \|f\| \cdot \|g\|$ for any composable morphisms f and g;
2) If A is both a category and a C^*-categoroid, then it is called a C^*-category.

Remark. In what follows the term "C^*-categoroid" will mean a C^*-categoroid with a small underlying categoroid, while the term "large C^*-categoroid" will mean a C^*-categoroid with a usual underlying categoroid.

Example 3. 1) The category with all Hilbert spaces as objects and all bounded linear maps as morphisms is a large C^*-category denoted by $\mathcal{H}$.

2) Let A be a C^*-algebra. The category $\mathcal{H}(A)$ with all right A-modules as objects and all bounded A-linear maps with adjoints as morphisms is a large C^*-category.

3) Let $Rep_H(A)$ be a category whose objects are non-degenerate representations of a C^*-algebra A on the Hilbert space H and whose morphisms are intertwining operators between these representations. Then $Rep_H(A)$ is a C^*-category.

4) C^*-algebra is a C^*-categoroid with one object $\diamond$ and elements of the C^*-algebra as morphisms.

5) The category of Hilbert spaces as objects and all compact linear maps as morphisms has the structure of a large C^*-categoroid.

Definition 4. Let A and B be C^*-categoroids. A $*$-functoroid $\mathcal{F} : A \to B$ is given if $\mathcal{F}$ maps the objects and morphisms of A into the objects and morphisms of B, respectively, so that:

a) $\mathcal{F}(fg) = \mathcal{F}(f)\mathcal{F}(g)$;
b) $\mathcal{F}(f+g) = \mathcal{F}(f) + \mathcal{F}(g)$;
c) $\mathcal{F}(\lambda f) = \lambda\mathcal{F}(f)$;

d) $\mathcal{F}(f^*) = \mathcal{F}(f)^*$ when the left side is defined.

If A and B are categories and $\mathcal{F}(1_a) = 1_{\mathcal{F}(1_a)}$ for any $a \in ObA$, then $\mathcal{F}$ is called a $*$-functor. We say that a $*$-functoroid between C^*-categoroids is faithful if canonical maps between objects and between morphisms are injections.

Lemma 5. *Any $*$-functoroid is norm decreasing. Moreover, the faithful $*$-functoroid is norm preserving.*

Proof. Let $\mathcal{F} : A \to B$ be a $*$-functoroid and $f : a \to a'$ be a morphism in A. By definition, there exists an element of $g \in \hom(a, a)$ such that $f^* f = g^* g$. Then one gets

$$\|\mathcal{F}(f)\|^2 = \|\mathcal{F}(f^*)\mathcal{F}(f)\| = \|\mathcal{F}(g)\|^2.$$

Since g is an element of the C^*-algebra $\hom(a, a)$, we have $\|\mathcal{F}(g)\| \leq \|g\| = \|f\|$. Therefore $\|\mathcal{F}(f)\| \leq \|f\|$. When $\mathcal{F}$ is faithful, we obtain $\|\mathcal{F}(g)\| = \|g\| = \|f\|$. Therefore $\|\mathcal{F}(f)\| = \|f\|$. $\square$

2. An Ideal and the Smallest Categorization

Let A be a C^*-categoroid and $I \subset MorA$. Let $(a, b)_I = \hom(a, b) \cap I$. Then I is called a left ideal if $(a, b)_I$ is a linear subspace of $\hom(a, b)$ and $f \in (a, b)_I, g \in \hom(b, c)$ implies $gf \in (a, c)_I$. A right ideal is defined similarly. I is a two-sided ideal if it is both a left and a right ideal. An ideal I is closed if $(a, b)_I$ is closed in $\hom(a, b)$ for each pair of objects. I determines an equivalence relation on the morphisms of A : $f \sim g$, if $f - g \in I$. If $I = I^*$ is an ideal of A, the set of equivalence classes A/I can be made into a $*$-categoroid in a unique way by requiring that the canonical map $f \mapsto \bar{f}$ of $A \to A/I$ be a $*$-functor. A/I can be made into a normed $*$-categoroid by defining

$$\|\bar{f}\| = \sup_{g \in \bar{f}} \|g\|.$$

Arguing as for C^*-algebras, one can show that the following lemma is valid.

Lemma 6. *Let A be a C^*-categoroid and I a closed two-sided ideal of A. Then $I = I^*$ and A/I is a C^*-categoroid.*

An ideal I in A is called *essential* if the intersection $I \cap J \neq 0$ for each nonzero ideal $J \subset A$.

Let A be a C^*-categoroid. A C^*-category B is called *categorization* of A if A is contained in B as an essential ideal.

There exists *the smallest categorization* A^+ of A. This C^*-category has the same objects as A, while $\hom_{A^+}(a, a) = \hom_A(a, a)^+$ if the C^*-algebra $\hom(a, a)$ is non-unital and remains such otherwise.

3. Approximate Unit and Grading

Definition 7. Let A be a C^*-categoroid, $u_a = \{u_{a,\lambda}\}_{\lambda\in S(a)}$ be a net in the C^*-algebra $\hom(a,a)$, with $u_{a,\lambda} \geq 0$, $\|u_{a,\lambda}\| \leq 1$, and $\lambda \leq \mu$ imply $u_{a,\lambda} \leq u_{a,\mu}$ (an increasing positive net). We say that u_a is approximate unit in the object a if:
a) $\lim_\lambda \|u_{a,\lambda}\cdot f - f\| = 0$;
b) $\lim_\lambda \|g\cdot u_{a,\lambda} - g\| = 0$.
for all morphisms $f : c \to a$ and $g : a \to b$. The set $U_A = \{u_a\}_{a\in obA} = \{\{u_{a,\lambda}\}_{\lambda\in S(a)}\}_{a\in obA}$ is called approximate unit for the C^*-categoroid A.

Lemma 8. *There exists approximate unit for any C^*-categoroid.*

Proof. Let $g : a \to b$ be a morphism from a C^*-categoroid. Then

$$\|g\cdot u_{a,\lambda} - g\|^2 = \|(u_{a,\lambda}\cdot g^* - g^*)(g\cdot u_{a,\lambda} - g)\| \leq 2\|g^*g\cdot u_{a,\lambda} - g^*g\| \to 0$$

if u_a is approximate unit of the C^*-algebra $\hom(a,a)$. □

Let A be a C^*-categoroid. The Z_2-grading on A is the decomposition into a direct sum $\hom(a,b) = \hom^{(0)}(a,b) \oplus \hom^{(1)}(a,b)$ of two closed linear subspaces $\hom^{(0)}(a,b)$ and $\hom^{(1)}(a,b)$ of $\hom(a,b)$ for any pairs of objects $a,b \in A$ such that:

- if $f \in \hom^{(i)}(a,b)$ and $g \in \hom^{(j)}(b,c)$, then $gf \in \hom^{(i+j)}(a,c)$;
- if $f \in \hom^{(i)}(a,b)$, then $f^* \in \hom^{(i)}(b,a)$.

A morphism of $\hom^{(i)}(a,b)$ is called *homogeneous* of degree i. The degree of a homogeneous element f is denoted by ∂f.

A $*$-functor $\mathcal{F} : A \to B$ of the graded C^*-categoroids A and B is graded if $\mathcal{F}(\hom^{(i)}(a,b)) \subset \hom^{(i)}(\mathcal{F}(a),\mathcal{F}(b))$ holds for any pair objects a,b.

Let $\Gamma : A \to A$ be a $*$-functor which is the identity map on the objects. Then $\Gamma^2 = id_A$,then $\hom^{(0)}(a,b) = \{f : \mathcal{F}(f) = f\}$ and $\hom^{(1)}(a,b) = \{f : \mathcal{F}(f) = -f\}$ give the Z_2-gradation on A. Conversely, given the grading, the corresponding $*$-functor is given by $\mathcal{F}(f^{(0)} + f^{(1)}) = f^{(0)} - f^{(1)}$.

The category of Z_2-graded C^*-categoroids and graded $*$-functors is denoted by C^*_{gr}.

Let $\mathcal{F} : A \to B$ and $\mathcal{G} : A \to B$ be graded $*$-functors. A set $\alpha = \{\alpha_a : \mathcal{F}(a) \to \mathcal{G}(a)\}_{a\in obA}$ of morphisms is called natural transformation of degree i if $\partial\alpha_a = i$ for all α_a and

$$\mathcal{G}(f)\alpha_a - (-1)^{\partial\alpha\partial f}\alpha_b\mathcal{F}(f) = 0$$

for any homogeneous morphism $f : a \to b$ from A.

Natural transformation $\alpha : \mathcal{G} \to \mathcal{F}$ is bounded if $\sup_a \|\alpha_a\| < \infty$.

4. Realization of a C^*-Categoroid

Any $*$-functoroid $\mathcal{F} : A \to \mathcal{H}(B)$ is called *a representation*, where $\mathcal{H}(B)$ is a large C^*-category of right Hilbert B-modules over the C^*-algebra B and B-linear maps which have adjoints [7]. If $\mathcal{F}$ is faithful, then it is called *a faithful representation* or an $*$-*imbedding.*

Theorem 9. *Let A be a C^*-categoroid. There exist a C^*-algebra $\mathcal{A}$ and a faithful representation*

$$\mathcal{F} : A \to \mathcal{H}(\mathcal{A}). \qquad (1)$$

Proof. Let A be a C^*-categoroid. Denote by $\mathcal{A}_i$ the C^*-algebra hom(i, i) and by E_{ij} the Banach space hom(i, j) for any objects $i, j \in obA$. It is easy to check that E_{ij} has the natural structure of a right Hilbert $\mathcal{A}_i$-module for every $j \in obA$ with respect to the $\mathcal{A}_i$-scalar product defined by $< f, g >_i = g^*f$. Note only that the positivity property is a direct consequence of property 4) of the definition of a C^*-categoroid. Let $\mathcal{A} = \oplus_i \mathcal{A}_i$ be a direct sum of C^*-algebras. It easily follows that E_{ij} is a right Hilbert $\mathcal{A}$-module with respect to the action $e \cdot a = e \cdot a_i$, where $a = (a_j)$, and the inner product is defined by $< e, e' >_{\mathcal{A}} = (a_j)$, where $a_j =< e, e' >_i$ for $i = j$, and $a_j = 0$ otherwise. Let E_j be a Hilbert $\mathcal{A}$-module which is the Hilbert sum of $\mathcal{A}$-modules E_{ij}, i.e., $E_j = \oplus_i E_{ij}$. Now we can define the $*$-functoroid $\mathcal{F}$ on the objects of A by $j \mapsto E_j$, and on the morphisms of A in the following way. Let $f : j \to j'$ be a morphism. Then the correspondence $l \mapsto fl$ defines a homomorphism of $\mathcal{A}$-modules $f_i : E_{ij} \to E_{ij'}$ for any object i which has the adjoint map defined by the morphism $f^* : j' \to j$. Therefore f_i is the bounded map. An elementary check shows that

$$(f^*)_i = (f_i)^*, \quad (fg)_i = f_i g_i.$$

Thus we obtain the "diagonal" homomorphism

$$\oplus_i f_i : \oplus_i E_{ij} \to \oplus_i E_{ij'}$$

which defines $\mathcal{F}(f)$. To show that $\mathcal{F}$ is faithful, consider f such that $f_i = 0$ for any object i. Then $f_{j'}(f^*) = ff^* = 0$, which implies that $f = 0$. □

5. Stable C^*-Algebra of a C^*-Categoroid and Its Representation

Let A be a C^*-categoroid and $S^0(A)$ be a $*$-algebra of finite $I \times I$-matrices, i.e., of matrices $(a_{ij})_{i,j \in I}$ with only finite nonzero entries, where $a_{ij} : i \to j$ is a morphism and I is the set of objects of A. By Theorem 9 there exists a faithful $*$-functoroid from A into the C^*-category of Hilbert $\mathcal{A}$-modules and bounded $\mathcal{A}$ -homomorphisms, i.e., there is an injection $*$-functoroid on objects and morphisms. Hence there exists a $*$-monomorphism of $S^0(A)$ into $L_{\mathcal{A}}(\oplus_i E_i)$, where $L_{\mathcal{A}}(\oplus E_i)$ is an C^*-algebra of all bounded $\mathcal{A}$-homomorphisms on $\oplus_i E_i$ which have adjoints. Therefore

$S^0(A)$ has a (unique) C^*-norm induced by $L_{\mathcal{A}}(\oplus E_i)$. A completion of $S^0(A)$ gives a C^*-algebra $S(A)$ called a *stable* C^*-algebra of the C^*-categoroid A.

By definition, $S^0(A)$ is a pre-C^*-algebra of finite matrices over A. Let $L = \{a_1, \ldots, a_n\}$ be a finite subset of objects from A, and A_L be a full C^*-subcategoroid of A with L as a set of objects. If R is another finite subset of obA and $L \subset R$, then we have the canonical $*$-homomorphism $i_{lr} : S^0(A_L) \to S^0(A_R)$ defined by $(f_{ij}) \mapsto (g_{kl})$, where $g_{kl} = f_{ij}$ for $(k,l) = (i,j)$, and is zero otherwise. Thus we obtain the direct system $\{S^0(A_L), i_{LR}\}$ of pre-C^*-algebras.

Lemma 10. a) *Each element from $S(A)$ can be represented as the $I \times I$-matrix (a_{kl}), where $a_{kl} \in \hom(k,l)$.*

b) *Let $L = \{a_1, \ldots, a_n\}$ be a finite subset of objects from A, and A_L be a full C^*-subcategoroid of A with L as a set of objects. Then $S^0(A_L)$ is a C^*-algebra.*

c) *A stable C^*-algebra $S(A)$ is the direct limit of the direct system $\{S^0(A_L), i_{LR}\}$ of C^*-algebras.*

Proof. a) Using the canonical $*$-functoroid $\mathcal{F}$ of (1), we identify the $*$-agebra $S^0(A_L)$ with a subalgebra of $\oplus_{i\in L} E_i$. Let $(e_{kl}) \in \oplus_{i\in L} E_i$ be an element which is a limit of a sequence $(a_{kl}^{(n)})$ of elements from $S^0(A)$. Then $a_{kl}^{(n)}$ converges to e_{kl} for any $k, l \in L$ since $\|a_{kl}^{(n)}\| \leq \|(a_{kl}^{(n)})\|$. But $\mathcal{F}$ is a faithful $*$-functoroid, which implies that it preserves the norm of morphisms, and thus $\mathcal{F}(\hom(k,l))$ is a closed subspace of $Hom(E_k, E_l)$. Therefore $e_{kl} \in \mathcal{F}(\hom(k,l))$.

b) This part of the lemma is an easy consequence of a).

c) This part is a trivial consequence of the definition of $S(A)$ and the definition of the direct limit of C^*-algebras. □

Consider the canonical functoroid $\varsigma : A \to S(A)$ which is the composition of canonical functoroids $A \to S^0(A) \to S(A)$. Let $A \to A^+$ be the canonical faithful imbedding. Then it induces the homomorphism $S(A) \to S(A^+)$. The canonical image of A in $S(A^+)$ is denoted by A^s.

Lemma 11. *Let A be a C^*-categoroid and A^+ be the smallest categorization. Then $S(A)$ is an essential ideal in $S(A^+)$.*

Proof. . Let J be any non-zero ideal in $S(A^+)$. Consider $J_a = J \cap \hom^+(a,a)$ for any object a. If $J_a = 0$ for each $a \in ObA$, then $J \cap \hom^+(a,a') = 0$ and also $J = 0$. This is a contradiction. Therefore $J_a \neq 0$ for some a. But $\hom(a,a)$ is a nonzero essential ideal in $\hom^+(a,a)$, which implies that $J \cap S(A) \neq 0$. □

Theorem 12. *Let A be a C^*-categoroid. There exist a C^*-algebra $\mathcal{A}$ and a faithful representation*

$$\mathcal{L} : A \to \mathcal{H}, \tag{2}$$

where $\mathcal{H}$ is the category of Hilbert spaces and bounded linear maps.

Proof. Consider a stable C^*-algebra $S(A)$ of A as a nondegenerate C^*-subalgebra of $\mathcal{L}(H)$. Then there exists a subalgebra $M(S(A))$ in $\mathcal{L}(A)$, the so-called multiplier algebra of $S(A)$ [8]. Since by Lemma 11 $S(A)$ is an essential ideal in $S(A^+)$, the algebra $S(A^+)$ is contained as a subalgebra in $M(S(A))$. For any object $a \in obA$ the image of the morphism 1_a from A^+ in $M(S(A))$ is also denoted by 1_a. It is clear that 1_a is the projection in $M(S(A))$. The family $\{1_a\}_{obA}$ is the orthogonal family of projections. Thus we get the decomposition $H = \oplus H_a$ where $H_a = 1_a \cdot H$. Define the $*$-functoroid $\mathcal{L}$ by $\mathcal{L}(a) = H_a$ where $a \in obA$, and $\mathcal{L}(f) = 1_{a'} \cdot \varsigma(f) \cdot 1_a$ for the morphism $f : a \to a'$. An elementary check shows that this definition is correct. The faithfulness of $\mathcal{F}$ is an easy consequence of the injectivity of $\varsigma : A \to S(A)$ on morphisms. □

Remark 1. Let A be a Z_2-graded C^*-categoroid. There exists a Z_2-grading on the $S(A)$ extending the Z_2-grading structure of A.

6. Multiplier C^*-Algebra of a C^*-Algebra.

When C^*-categoroid A has one object $\diamond$, we have a situation of C^*-algebras. It is well known that the multiplier algebra of A is the largest unitization (categorization) of A. In this section we give shortly the main definitions and properties of multiplier C^*-algebras which are needed to construct the largest categorization of C^*-categoroid.

Let A be a nondegenerate subalgebra of $B(H)$. By definition,

$$M(A) = \{m \in B(H) : mA \subset A \& Am \subset A\}$$

which is called the multiplier algebra.

We have $M(A) \subset A'' \subset B(H)$ where A'' is the double commutant of A. The algebra $M(A)$ is a unital C^*-algebra containing A as an essential ideal [8], [9].

For the following lemma see [3].

Lemma 13. *Let A be a C^*-algebra and $p \in M(A)$ be a projection. Then the canonical map $pM(A)p \to M(pAp)$ is an isomorphism.*

Proof. Consider the right Hilbert A-module pA. It is easy to check that the algebra of compact A-homomorphisms from pA to pA is pAp. Thus, by Lemma 1.1.14 of [3], $L(pA) = M(pAp)$. Now from the decomposition $A = pA \oplus (1-p)A$ it follows that $pL(A)p = L(pA)$. □

There exists another approach to multipliers. A double-centralizer for A is pair (L, R) of functions $L, R : A \to A$ satisfying the condition $R(a)b =$

$aL(b)$ for all $a, b \in A$. The set of all double-centralizers is denoted by $\mathcal{M}(A)$ which is a C^*-algebra with respect to the following operations:

$$\begin{gathered}(L_1, R_1) + (L_2, R_2) = (L_1 + L_2, R_1 + R_2)\\ (L_1, R_1)(L_2, R_2) = (L_1 L_2, R_2 R_1)\\ \lambda(L, R) = (\lambda L, \lambda R), \lambda \in C\\ (L, R)^* = (R^*, L^*)\\ \|(L, R)\| = \|L\| = \|R\|,\end{gathered}$$

where $T^* : A \to A$ is the map defined by $a \mapsto (T(a^*))^*$, for any map $T : A \to A$ [1]. $\mathcal{M}(A)$ is the largest unitization of A, and the canonical $*$-homomorphism

$$M(A) \to \mathcal{M}(A)$$

is an isomorphism.

Let A be a nondegenerate C^*-subalgebra in $L(\mathcal{H})$. There exists the weakest locally convex topology on the $L(\mathcal{H})$ making the maps $x \mapsto xa$ and $x \mapsto ax$ norm continuous, where $x \in L(\mathcal{H})$ and $a \in A$ [9]. This topology is called *a strict topology.* One has the induced topology on the $M(A) \subset L(\mathcal{H})$, which is called a strict topology, too.

The next lemma will be useful in the sequal (cf. Lemma 2.3.6 of [9]).

Lemma 14. *Let $\{x_i\}_I$ be a norm-bounded net in $L(\mathcal{H})$, $x \in L(\mathcal{H})$ and S be a total subset of A. If $\{x_i s\}$ and $\{s x_i\}$ are norm-convergent to xs and sx, respectively, then $\{x_i\}$ converges strictly to x.*

Proof. Consider the finite linear combination $s = \sum \lambda_k s_k$ of elements from S. We obtain $\|ax_i - ax\| \leq$

$$\begin{gathered}\|ax_i - \sum \lambda_k s_k x_i\| + \|\sum \lambda_k s_k x_i - \sum \lambda_k s_k x\| + \|\sum \lambda_k s_k x - ax\| \leq\\ \leq \|a - \sum \lambda_k s_k\| \cdot (\|x_i\| + \|x\|) + \sum |\lambda_k| \cdot \|s_k x_i - s_k x\|.\end{gathered}$$

The same argument applies to $\{x_i a\}_{i \in I}$. Therefore $\|ax_i - ax\|$ and $\|x_i a - xa\|$ can be made arbitrarily small for any $a \in S(A)$. □

Strict convergence has the following properties (see [9]).

Proposition 15. *A net in A is approximate unit for A if and only if it is strictly convergent to $1 \in M(A)$.*

The $*$-operation is strictly continuous.

If $\{x_i\}_{i \in I}$ and $\{y_j\}_{j \in J}$ are the norm-bounded nets converging to x and y in the strict topology, then $\{x_i \cdot y_j\}_{(i,j) \in I \times J}$ converges strictly to xy.

$M(A)$ is a strict completion of A.

The following is Proposition 1.1.13 from [3].

Proposition 16. *Let A and B be C^*-algebras, and $\phi : A \to M(B)$ be a $*$-homomorphism. Then the following two conditions are equivalent:*

(i) *there is a projection* $p \in M(A)$ *such that* $\overline{\phi(A)B} = pB$*;*
(ii) *there is a unique strictly continuous* $*$*-homomorphism* $\psi : M(A) \to M(B)$ *extending* ϕ*. In particular, if* $\psi : M(A) \to M(B)$ *is a* $*$*-homomorphism for which* $B \subset \psi(A)$*, then* ψ *is strictly continuous.*

7. A Strictly Complete Set of Projections in $M(A)$

To construct a multiplier category, we need some lemmas and definitions.

Definition 17. Let A be a C^*-algebra and $P = \{p_i\}_{i \in I}$ be a set of projections of $M(A)$. We say that P is a strictly complete set of projectons if:

(a) P is orthogonal, i.e., $p_i p_j = 0$ for every $i \neq j$ from I;
(b) the net $\{p_l\}$ converges strictly to 1, where l is a finite subset of I and $p_l = \sum\limits_{a \in l} p_a$; This means that $\sum\limits_{a \in I} p_a = 1$ in the strict topology of $M(A)$.

Definition 18. Let A be a C^*-algebra and $P = \{p_i\}_{i \in I}$ be a complete set of projections. Let $X = (x_{ij})$ be a $I \times I$-matrix, where $x_{ij} = p_j x_{ij} p_i$ is an element from $M(A)$, and $i, j \in I$. We say that X is a P-*a harmonic* matrix over $M(A)$ if in the strict topology of $M(A)$ there exists a sum $\sum\limits_{i,j \in I} x_{ij}$ denoted by x.

Lemma 19. *Let* A *be a* C^**-algebra,* $P = \{p_i\}_{i \in I}$ *be a complete set of projections,* $X = (x_{ij})$ *and* $Y = (y_{ij})$ *be* P*-harmonic matrices,* x *and* y *be the corresponding elements from* $M(A)$*. Then the matrices*

$$X + Y = (x_{ij} + y_{ij}), \quad X^* = (x_{ij}^*), \quad \lambda \cdot X = (\lambda x_{ij}),\ \lambda \in \mathbf{C}$$

and

$$X \cdot Y = Z = (z_{ij})$$

are P*-harmonic over* $M(A)$*, where* $z_{ij} = \sum\limits_{k \in I} x_{ik} \cdot y_{kj}$*, and the corresponding element for* $X \cdot Y$ *is* $x \cdot y$*.*

Proof. The first three statements are easily deduced from the definition of a strict topology on $M(A)$. To prove that Z is P-harmonic, we note that the finite sums $\sum\limits_{k \in R} x_{ik}$ are bounded since $\sum\limits_{k \in R} x_{ik} = 1_R \cdot (\sum\limits_{i,j \in I} x_{ij}) \cdot 1_i$. Now, in a similar manner, we show that the finite sums $\sum\limits_{k \in R} y_{kj}$ are bounded. Hence by Proposition 15, the finite sums

$$\sum_{k \in R} x_{ik} \cdot y_{kj} = \Big(\sum_{k \in R} x_{ik}\Big) \cdot \Big(\sum_{k \in R} y_{kj}\Big)$$

converge strictly to $\sum_{k\in I} x_{ik}\cdot y_{kj}$ in $M(A)$. This means that the elements z_{ij} are correctly defined. Now it is enough to show that the sum $\sum_{ij} z_{ij}$ converges to $x\cdot y$ in $M(A)$. Since $x_{ik}\cdot y_{lj}=0$ for $k\neq l$, we have

$$\sum_{i,j} z_{ij}=\sum_{i,j}\sum_{k} x_{ik}\cdot y_{kj}=\Big(\sum_{i,k} x_{ik}\Big)\cdot\Big(\sum_{l,j} y_{lj}\Big)$$

for $i,\ j,\ k,\ l\in R$. Consider the nets $\{x_R\}_{R\subset obA}$ and $\{y_R\}_{R\subset obA}$, where R is a finite set and $x_R=\sum_{i,k} x_{ik}$, $y_R=\sum_{j,l\in R} y_{lj}$. It is easy to check that these nets are norm-bounded by $\|x\|$ and $\|y\|$, respectively, since $x_R=1_R\cdot x\cdot 1_R$ and $y_R=1_R\cdot y\cdot 1_R$. Applying Proposition 15, we find that $\{z_R=x_R\cdot y_R\}$ converges to $z=xy$. □

Definition 20. Denote by $Mat^P(A)$ the involutive algebra of all P-harmonic matrices over $M(A)$, and by $St^P(A)$ a subalgebra of matrices $X=(x_{ij})$ from $Mat^P(A)$ such that $x_{ij}\in A$ and the net $\{x_R\}_{R\subset obA}$ from Lemma 19 converges to x in the norm topology.

Theorem 21. *Let A be a C^*-algebra, $P=\{p_i\}_{i\in I}$ be a complete set of projections, and $X=(x_{ij})$ be a P-harmonic matrice with $\sum_{i,j\in I} x_{ij}=x$. Then the correspondence $X\mapsto x$ defines an isomorphism $\mu: Mat^P(A)\to M(A)$ which sends the elements of $St^P(A)$ onto A.*

Proof. Injectivity of μ. Let $X=(x_{ij})$ and $X'=(x'_{ij})$ be such that $\{\sum\sum x_{ij}\}=x$ and $\{\sum\sum x'_{ij}\}=x$. Then

$$x_{ab}=p_b\Big(\sum_{i,j} x_{ij}\Big)p_a=p_b x p_a=p_b\Big(\sum_{i,j} x'_{ij}\Big)p_a=x'_{ab}$$

for any $a,b\in I$. This means that $X=X'$.

Surectivity of μ. For each $x\in M(A)$, let $X=(x_{ij})$ with $x_{ij}=p_j x p_i$. Then

$$\sum_i\sum_j x_{ij}=\sum_i\sum_j p_i x p_j=\Big(\sum_i p_i\Big)\cdot x\cdot\Big(\sum_j p_j\Big)=1\cdot x\cdot 1=x.$$

Now let $X\in St(A)$. Since $\{x_R\}\subset A$ and the norm-convergence implies strict convergence, we have $\{x_R\}\to x\in A$. Conversely, if $x\in A$, then $x_{ij}=p_j\cdot x\cdot p_i\in A$ and $\{x_R\}\to x\in A$. □

Corollary 22. *The algebra $Mat^P(A)$ is a C^*-algebra with respect to the norm $\|X\|=\mu\|X\|$.*

Definition 23. Let A be a C^*-algebra and $P=\{p_i\}_{i\in I}$ be a complete set of projections in the $M(A)$. Define the C^*-category $M^P(A)$ with the set I as the set of objects, $p_j\cdot M(A)\cdot p_j$ as the space of morphisms from i to j, and the C^*-structure given by the C^*-structure of $M(A)$. Further, define by A^P the C^*-categoroid with the set I as the set of objects and $p_j \cdot A \cdot p_j$ as the set of morphisms from i to j. The C^*-structure is obtained from the corresponding structure of A.

Theorem 24. *Let A be a C^*-algebra and $P = \{p_i\}_{i\in I}$ be a complete set of projections in $M(A)$. Then A^P is an ideal in $M^P(A)$ and if D is another C^*-categoroid containing A^P as an ideal, then there exists a $*$-functoroid $d : D \to M^P(A)$, which is the identity map on the A^P, such that the diagram*

$$\begin{array}{ccc} A^P & \subset & M^P(A) \\ \cap & \nearrow & \\ D & & \end{array} \tag{3}$$

is commutative, i.e., $M^P(A)$ is a multiplier C^*-category of the C^*-categoroid A^P.

Proof. By virtue of Theorem 21, each element $x \in A$ can be uniquely represented as a matrice $X = (x_{ij})_{i\in I}$ such that the net $\{x_R\}_R$ converges to x in the norm topology. Thus the stable C^*-algebra $S(A^P)$ of the categoroid A^P is naturally isomorphic to the C^*-algebra A. Consider the stable algebra $S(D)$. Since $S(D)$ contains $A = S(A^P)$ as a two-sided closed ideal, there exists a unique $*$-homomorphism $d' : S(D) \to M(A)$ such that the diagram

$$\begin{array}{ccc} A & \subset & M(A) \\ \cap & \nearrow & \\ S(D) & & \end{array} \tag{4}$$

is commutative. Let $y \in S(D)$ be such that $y : i \to j$ is a morphism from D, and $y' = d'(y)$. For an arbitrary $x \in A^P$ we write the identities $y' \cdot x = (p_j \cdot y' \cdot p_i) \cdot x$ and $x \cdot y' = x \cdot (p_j \cdot y' \cdot p_i)$ which can be extended to the identities for each element $x \in A$. Thus $y' = p_j \cdot y' \cdot p_i$ in the $M(A)$. This implies that the correspondence $y \mapsto y'$ correctly defines the $*$-functoroid $d : D \to M(A)$ and the commutativity of diagram 3 is easily obtained from diagram 4. $\square$

Lemma 25. *Let A be a C^*-algebra and $P = \{p_i\}_{i\in I}$ be a strictly complete set of projections. Then there exists $p_R = \sum\limits_{i\in R} p_i$ in the $M(A)$ for an arbitrary $R \subset I$.*

Proof. Consider SA as a nondegenerate subalgebra of $L(\mathcal{H})$. It is easy to show that $p_R \in L(\mathcal{H})$. Let $\{p_S\}_{S\subset R}$ be the net defined by finite subsets of R. We assert that $\{p_S\}_{S\subset R}$ converges to p_R in the strict topology. To show this, note that, by Lemma 14, $\{p_S\}_{S\subset R}$ converges to p_R in the strict topology. But $\{p_S\}_{S\subset R}$ is the net of elements of A, which implies that $p_R \in M(A)$. This completes the proof. $\square$

Corollary 26. *Let A be a C^*-algebra, $P = \{p_i\}_{i\in I}$ be a complete set of projections in $M(A)$, and $P_R = \{p_i\}_{i\in R}$ where R is a subset in I. Let $M_R(A)$ be the subalgebra $1_R \cdot M(A) \cdot 1_R$ in $M(A)$. Then P_R is a complete set of projections in $M_R(A)$ and the C^*-category $M_R^{P_R}(A)$ is naturally isomorphic to $M^P(A)|_R$.*

Proof. By Lemmas 25 and 13 $M(p_R \cdot A \cdot p_R) = M_R(A)$ and P_R is a complete set of projections in $M(p_R \cdot A \cdot p_R)$. Now it is easy to check that $M_R^{P_R}(A) = M^P(A)|_R$. □

8. Multiplier C^*-Category of a C^*-Categoroid

Definition 27. Let A be a C^*-categoroid. A C^*-category $M(A)$ is called *a multiplier C^*-category* of A if A is a closed two-sided ideal in $M(A)$ and has following universal property: let D be a C^*-categoroid containing A as a closed two-sided ideal; then there exists a unique $*$-functoroid $d : D \to M(A)$ which is the identity map on A, such that the diagram

$$\begin{array}{ccc} A & \subset & M(A) \\ \cap & \nearrow & \\ D & & \end{array} \tag{5}$$

is commutative.

From Definition 27 and Theorem 24 it follows that $M^P(A)$ is the multiplier C^*-category of the C^*-categoroid A^P.

Now, we want to construct the multiplier C^*-category for any C^*-categoroid.

Proposition 28. *Let A be a C^*-categoroid. There are an orthogonal strictly complete set of projections $\mathcal{P} = \{p_a\}_{a\in obA}$ in $M(S(A))$ and a natural $*$-isomorphism $\tau : A \to S(A)^{\mathcal{P}}$.*

Proof. Let A^+ be the smallest categorization of A. By Lemma 11 $S(A)$ is an essential ideal in $S(A^+)$. Hence the canonical $*$-homomorphism $S(A^+) \to M(S(A))$ is injective. For each object $a \in A^+$ there is a unital morphism 1_a which gives a set of orthogonal projections $\{p_a\}_{a\in obA} \subset S(A^+)$. (The image of 1_a in $M(S(A))$ is denoted by p_a). The sums $p_l = \sum_{a\in l} p_a$ are projective for any finite set l of objects of A. Thus we have a set of projections $\{p_l\}$ in $S(A^+)$ and $M(S(A))$. It can be easily shown that $\{p_l\}$ is approximate unit for $S(A^+)$ so that $\{1_l\} \to 1$ strictly in $M(S(A^+))$. On the other hand, we have the strictly continuous unital homomorphism $M(S(A^+)) \to M(S(A))$ since $S(A)$ is an ideal in $S(A^+)$ and because of Proposition 16. Thus $\{p_l\} \to 1$ strictly in $M(S(A))$. □

Definition 29. The set of projections $\{p_a\}_{a \in obA}$ defined by Proposition 28 is called *the standard complete set of projections* in $M(S(A))$. Let $M(A)$ be a C^*-category with ObA as the set of objects and the set of elements $1_{a'} \cdot f \cdot 1_a$ from $M(S(A))$ as the set of morphisms from a to a'. The C^*-structure is induced by the corresponding structure of $M(S(A))$.

Lemma 30. *The elements from $S(A)$ with property $x = 1_{a'} \cdot x \cdot 1_a$ form a closed two-sided ideal $\mathcal{A}$ in $M(A)$. The canonical functoroid $\varsigma : A \to S(A)$ induces the isomorphism $A \approx \mathcal{A}$ of C^*-categoroids.*

Proof. The proof of this lemma follows. □

Theorem 31. *Let A be a C^*-categoroid. If D is another C^*-categoroid containing A as a closed two-sided ideal, then there exists a functoroid $d : D \to M(A)$, which is the identity map on A, such that the diagram*

$$\begin{array}{ccc} A & \subset & M(A) \\ \cap & \nearrow_d & \\ D & & \end{array} \tag{6}$$

is commutative.

Proof. This imediatelly follows from Lemma 30 and Theorem 24. □

Let A be a C^*-categoroid and R be a subset of obA. Denote by A_R a full subcategoroid of A which has R as the set of objects.

Theorem 32. *The canonical *-functor of C^*-categories*

$$M(A)_R \to M(A_R)$$

is an isomorphism.

Proof. This is an easy consequence of Lemma 30 and Corollary 26. □

The following theorem is easy to check.

Theorem 33. *Let B be a C^*-category and A be a closed ideal in B. The category B is the multiplier C^*-category of A if and only if for any set with two objects $R = \{a, a'\}$ of obB the canonical $*$-homomorphism $S(B|_R) \approx M(S(A|_R)$ is an isomorphism.*

9. Hilbert Sum of a Family of Objects in a C^*-Category

Lemma 34. *Let A be a C^*-categoroid and I be a set of all objects of A. Let $I = \cup_j I_j$, $j \in J$ be the partition induced by an equivalence relation R. There is a C^*-categoroid A_R with the following properties:*
1) *$obA_R = J$;*
2) *$\hom(j,j) = S(A|_{I_j})$;*
3) *$S(A_R) = S(A)$.*

Proof. Let $\hom^{(0)}(j,j')$ be the linear space of all finite $I_j \times I_j$-matrices of form $\mu_{jj'} = (m_{i_j i_{j'}})$ where $(j,j') \in J \times J$ and $m_{i_j i_{j'}}$ is a morphism from $i_j \in j$ and $i_{j'} \in j'$ (recall that the finiteness means that the matrices have only finitely many nonzero entries). There is a composition low

$$\hom^{(0)}(j',j'') \times \hom^{(0)}(j,j'') \to \hom^{(0)}(j,j'')$$

defined by the usual product of matrices. Thus we get the linear categoroid $A_R^{(0)}$ with the set J as the set of objects and the linear space $\hom^{(0)}(j,j')$ as the space of morphisms from j to j'. Moreover, $A_R^{(0)}$ is an involutive categoroid whose involution is defined by the equation $(a_{kl})^* = (a_{lk}^*)$. Now, assume that $\mathcal{F} : A \to \mathcal{L}(\mathcal{A})$ is the canonical faithful $*$-functoroid defined in Theorem 9. This functoroid induces a faithful $*$-functoroid $\mathcal{F}_R : A_R \to \mathcal{L}(\mathcal{A})$ defined in the following way. On the objects $\mathcal{F}_{S(R)}$, we define $\mathcal{F}_R$ is defined by $\mathcal{F}_R(j) = \oplus_{k \in I_j} E_k$, while on the morphisms $\mathcal{F}_R$ is defined by the rule $\mathcal{F}_R((a_{kl})) = (\mathcal{F}(a_{kl}))$. Hence we obtain the C^*-norm on the morphisms of $A_R^{(0)}$ induced by C^*-category $\mathcal{L}(\mathcal{A})$. The completion of $A_R^{(0)}$ with respect to this C^*-norm gives the C^*-categoroid denoted by A_R. The check of properties 1) and 2) is trivial and thus we omit it. □

Remark ϑ. The construction of the C^*-categoroid A_R implies that there is the canonical $*$-functoroid $\vartheta_R : A \to A_R$ defined by the rule $\vartheta_R(a) = [a]_R$, $a \in obA$. If $f : i_j \to i_{j'}$ is a morphism in A such that $i_j \in I_j$ and $i_{j'} \in I_{j'}$, then $\vartheta_R(f)$ is a $I_j \times I_{j'}$-matrix where all entries are zero except the entry $j \times j'$ which is equal to f. It is easy to check that ϑ is an injective map on the set of morphisms. Moreover, by virtue of property (3) of the preceding lemma there exists an extension $\Theta : M(A) \to M(A_R)$ of ϑ.

Let 1_j be the unital element of $M(S(A|_{I_j})$. According to Theorem 32 one has the canonical $*$-homomorphism $M(S(A|_{I_j}) \to M(S(A))$ induced by the canonical inclusion $A|_{I_j} \to A$. Let p_j, $j \in J$, be the projection in $M(S(A)$ which is the image of $1_j \in M(S(A))$.

Lemma 35. *Let A be a C^*-categoroid and $I = \cup_{j \in J} I_j$ be the partitioning of the set of all objects by the equivalence relation R. Then the set of projections $\{p_j\}_{j \in J}$ is the complete set of projections of $M(S(A))$.*

Proof. Consider the C^*-categoroid A_R which has the properties of Lemma 34. Let 1_{I_j} be the unital morphism of object j of $M(A_R)$. It is easy to show that the image of 1_{I_j}in $M(S(A_R))$ is p_ja complete set of projections, and $\sum_j p_j = 1$ in the strict topology, which completes the proof. □

We need the following constructions.

Construction ι. Let A be a C^*-categoroid, X be a set and $f : X \to obA$ be a map. A C^*-categoroid A_f is given by the set X as the set of objects and $\hom(x, x') = \hom(f(x), f(x'))$ as the Banach space of morphisms. The composition and C^*-structures are inherited from the corresponding structures of A.

Construction $\Delta\Sigma$. Let A be a C^*-categoroid and I be the set of all objects of A. Consider the set $I\times\{0,1\} = I\times\{0\}\cup I\times\{1\}$ and the natural projection $pr_I : I \times \{0, 1\} \to I$. Using the construction ι, we obtain the C^*-categoroid A_{pr_I} denoted by ΔA. Consider the equivalence relation R on the set $I\times\{0\}\cup I\times\{1\}$ of objects of ΔA: the discrete equivalence relation is given on the set $I \times \{0\}$, while the anti-discrete equivalence relation is given on the set $I \times \{1\}$. Thus the set of classes of equivalent elements is given by the set $I\cup\{I\}$. Using Lemma 34, we obtain a C^*-categoroid denoted by A_Σ.

The following lemma is true.

Lemma 36. *A_Σ has the following properties:*

1) *$obA_\Sigma = I \cup \{I\}$;*
2) *$A_\Sigma|_I = A$ and $A_\Sigma|_{\{I\}} = S(A)$;*
3) *$S(A_\Sigma) = M_2(S(A))$;*
4) *There exists a set of morphisms $\{\sigma_i : i \to \{I\}\}_{i\in I}$ in the $M(A_\Sigma)$ such that*

$$\sigma_i^*\sigma_i = 1_{(i,0)} \ \ and \ \sum_{i\in I} \sigma_i\sigma_i^* = 1_{\{I\}} \tag{7}$$

in the strict topology of $Hom(\{I\}, \{I\})$.

Proof. Statements 1) and 2) are trivial. Statement 3) is an easy consequence of the construction $\Delta\Sigma$ and Lemma 34. To prove statment 4), let $\Theta : M(\Delta A) \to M(A_\Sigma)$ be the canonical functor (see Remark ϑ), and let $s_i : (i, 0) \to (i, 1)$ be a morphism in $M(\Delta A)$ defined by the identity morphism $1_i : i \to i$ of $M(A)$. Then, by definition, $\sigma_i = \Theta(s_i) : (i, 0) \to \{I\}$. The first equation of 7 is trivial. To show that the second equation holds, note that the set $\Sigma = \{\sigma_i\sigma_i^*\}_{i\in I}$ is a complete set of projections since $\sigma_i\sigma_i^*$ is the image of the identity morphism 1_i from $M(A)$ in $M(S(A))$. □

Definition 37. Let A be a C^*-categoroid and $M(A)$ be the multiplier C^*-category of A. Let be a set and $\varphi : Q \to obA$ be a map. We say that an object $a \in obA$ is the Hilbert sum of a family of objects $\{\varphi(q)\}_{q\in Q}$ if a family $\{s_q : \varphi(q) \to a\}_{q\in Q}$ of morphisms is given in $M(A)$ such that

$$s_q^*s_q = 1_{\varphi(q)}, \ \ s_{q'}^*s_q = 0 \ \text{ when } \ q \neq= q' \tag{8}$$

and $\sum_{q\in Q} s_q s_q^* = 1_a$ in the strict topology of $Hom(a,a)$.

The proposition below provides as necessary and sufficient condition for an object to be the Hilbert sum of a family of objects.

Proposition 38. *Let A be a C^*-categoroid and $\varphi : Q \to obA$ be a family of objects. An element a is the Hilbert sum of the family $\{\varphi(q)\}_{q\in Q}$ if and only if there exists an isomorphism of C^*-categoroids A_{φ_∞} and $A_{\varphi\Sigma}$, where $\varphi_\infty : Q \cup \{\infty\} \to obA$ is defined by the equations $\varphi_\infty(q) = \varphi(q)$ when $q \in Q$ and $\varphi_\infty(\infty) = a$.*

Proof. From Lemma 36 it immediately follows that the object $\{Q\}$ is the Hilbert sum of the set of objects $\{(q,0\}_{q\in Q}$ in $A_{\varphi\Sigma}$. Thus if $A_{\varphi_\infty} \simeq A_{\varphi\Sigma}$, then one immediately has that $\{Q\}$ is Hilbert sum of family$\{\varphi(q)\}_{q\in Q}$. Conversely, let a be the Hilbert sum of a family $\{\varphi(q)\}_{q\in Q}$Consider the category $\Delta'(A_{\varphi_\infty})$ with the set of objects $Q\cup P$ and with morphisms defined by the rule:

a) $\Delta'(A_{\varphi_\infty})|_Q = A_\varphi$;

b) $\Delta'(A_{\varphi_\infty})|_P = (A_{\varphi_\infty}|_\infty)^P$ (see Theorem 24);

c) $\mathrm{Hom}(q', s_q s_q^*)$ in $\Delta'(A_{\varphi_\infty})$ coincides with all morphisms from $\mathrm{Hom}(q',\infty)$ in A_{φ_∞} which are invariant under the left action of $s_q s_q^*$. It is easy to show that $\Delta'(A_{\varphi_\infty})$ is naturally isomorphic to ΔA_φ. This completes the proof. □

10. Hilbert C^*-Categories

Let A be a C^*-categoroid. One says that A is a C^*-categoroid with zero object o if $\hom(o,o) = 0$. If A has no zero object, then one can add the object o and define hom-spaces as follows: $\hom(o,o) = 0$, $\hom(a,0) = 0$ and $\hom(0,a) = 0$.

Definition 39. Let $\mathcal{A}$ be a C^*-category and $\mathcal{I}$ be a closed ideal. The pair $(\mathcal{A}, I)$ (shortly, $\mathcal{A}$) is called a C^*-category with compact morphisms if the canonical $*$-functor $\mathcal{A} \to M(\mathcal{I})$ is an isomorphism. Any morphism of the C^*-categoroid $\mathcal{I}$ is called *a compact morphism.*

From the definition it immediately follows that for any C^*-category A there exists a C^*-category, unique up to an isomorphism, with compact morphisms for which A is an ideal of all compact morphisms (here we mean the pair $(M(A), A)$).

Definition 40. A C^*-category $\mathcal{A}$ with compact morphisms is called H-additive if there exists the Hilbert sum for any countable family of objects from $\mathcal{A}$. If $\mathcal{A}$ is a pseudo-abelian H-additive C^*-category, then it is called a *Hilbert C^*-category.*

Recall that an additive C^*-category $\mathcal{A}$ is said to be *pseudo-abelian* if every projection has a kernel. Every additive C^*-category is contained in some universal pseudo-abelian C^*-category [4].

Now we have come to the main result of this paper.

Theorem 41. *For a given C^*-categoroid A there exists a natural H-additive C^*-category $M_\infty(A)$ with the following properties:*

a) *The C^*-category $M_\infty(A)$ contains the C^*-category $M(A)$ as a full subcategory;*

b) *each object from $M_\infty(A)$ is the Hilbert sum of a countable set of objects from the subcategory $M(A)$.*

Proof. First we construct a C^*-category $M_\infty(A)$. It is assumed without loss of generality that A has zero object. Let N be the set of natural numbers and N_A be the set of all maps from N into obA. Consider the set $\mathcal{N} = N \times N_A$ and the natural map $\Lambda : \mathcal{N} \to obA$ defined by the formula $\Lambda(n, f) = f(n)$. Using Construction ι, one obtains the C^*-categoroid A_Λ. Consider the equivalence relation R on $\mathcal{N}$: $(n, f) \sim (m, g) \Leftrightarrow f = g$. Now, by applying Lemma 34 to the C^*-categoroid A_Λ and the equivalent relation R we obtain the C^*-categoroid $A_{\Lambda R}$. Denote by $M_\infty(A)$ the C^*-category of multipliers of $A_{\Lambda R}$. By construction, an object in the constructed category is a pair (N, f) and any morphism $\alpha : (N, f) \to (N, g)$ is an $N \times N$-matrix such that the n, m-entry is a morphism from $f(n)$ to $g(m)$ and α can be approximated by matrices of finite type in the norm topology. Now we show that any countable family of objects has the Hilbert sum. Choose and fix a bijection $\Psi : N \to N \times N$. Let $\{(N, f_n)\}_{n \in N}$ be a sequence of objects from $M_\infty(A)$. Consider the object (N, f) defined by the following law: if $n \in N$ and $\Psi(n) = (l, k)$, then $f(n) = f_k(l)$. We need the morphisms $s^k : (N, f_k) \to (N, f)$. For this, let $i_k : N \to N \times N$ be the injection defined by $i_k(n) = (n, k)$, and define the injection $\iota_k : N \to N$ as the composition $\iota_k = \Psi^{-1} \cdot i_k$. Now, define $s^k : (N, f_k) \to (N, f)$ as the $N \times N$-matrix $\sigma^k = (\sigma^k_{nm})$, where $\sigma^k_{nm} = 1_{f_k(n)}$ when $\iota_k(n) = m$, and $\sigma^k_{nm} : f_k(n) \to f(m)$ is zero in other cases. The first part of the latter statement is correct since

$$f(m) = f(\iota_k(n)) = f(\Psi^{-1}(n, k)) = f_k(n).$$

It is easy to show that $s^*_k s_k = 1_{(N, f_k)}$and $s_k s^*$ is a diagonal $N \times N$-matrix $d^k = (d^k_{nm})$ where $d^k_{mm} = 1_{f(m)}$ when $m = \iota_k(n)$ for some n, and $d^k_{nm} = 0$ in other cases. To show that $\sum s_k s^*_k = 1_{(N,f)}$ in the strict topology of $Hom((N, f), (N, f))$, note that the latter algebra is just the multiplier algebra of the stable algebra $S(A_f)$. Consider the equivalence relation R on $obA_f = N$ by

$$n \sim m \Leftrightarrow pr_2(\Psi(n)) = pr_2(\Psi(m)).$$

Using Lemmas 34 and 35, one has the C^*-categoroid A_{fR} and projections $\{p_k\}_{k \in N}$. By construction, p_k is a diagonal $N \times N$-matrix (p^k_{nm}) such that p^k_{nn} is the identity of $f(n)$ if $pr_2\Psi((n)) = pr_2(m, k) = k$. This shows that projections $s_k s_k^*$ are exactly p_k. But Lemma 35 asserts that $\sum p_k = 1$ in the strict topology.

a) For any object $a \in obA$, consider the object (N, φ_1^a) where $\varphi_1^a(1) = a$ and $\varphi_1^a(n) = 0$ when $n > 1$. Any morphism $\alpha : (N, \varphi_1^a) \to (N, \varphi_1^{a'})$ is of finite type $N \times N$-matrix (α_{nm}) such that α_{11} is a morphism from a to a', while the other entries are zero morphisms. Thus one has a natural full and faithful $*$-functoroid $A \subset A_{\Lambda R}$ defined by $a \mapsto \varphi_1^a$ on objects. For a morphism $h : a \to a'$, define the morphism α^h by the $N \times N$-matrix with $\alpha_{11}^h = h$ and $\alpha_{nm}^h = 0$.

b) Let (N, f) be an object in $M_\infty(A)$. Consider the sequence (N, f_i) where $f_i(n) = f(i)$ when $n = i$, and $f_i(n) = 0$ when $n \neq i$. Let $s_i : (N, f_i) \to (N, f)$ be an $N \times N$-matrix defined by the rule: $\sigma_{ii} = 1_{f(i)}$ and $\sigma_{ij} = 0$ in othe cases. It is easy to check that $\{s_i\}_{i \in N}$. □

Corollary 42. *The C^*-category $M_\infty(A)$ is an additive category.*

Proof. From the definition it immediately follows that $M_\infty(A)$ is an additive category with respect to the Hilbert sum of two objects. □

Definition 43. The universal pseudo-abelian C^*-category of $M_\infty(A)$ is denoted by $H(A)$.

By definition, objects of the category $H(A)$ are of the form (a, p), where a is an object From $M_\infty(A)$ and $p \in \hom(a, a)$ is a projection. By definition, a morphism $\phi : (a, p) \to (a', p')$ is given by a morphism $\phi : a \to a'$ from $M_\infty(A)$ with the property: $\phi p = p' \phi = \phi$. Structures of the C^*-category on the $H(A)$ are inherited from the corresponding structures of $M_\infty(A)$.

Definition 44. A morphism $\phi : (a, p) \to (a', p')$ in $H(A)$ is called a compact morphism if the morphism $\phi : a \to a'$ is a compact morphism in $M_\infty(A)$. Denote by $\mathcal{K}(a, a')$ the space of compact morphisms. The C^*-subcategoroid of compact morphisms in $H(A)$ is denoted by $K(A)$.

Proposition 45. *The C^*-category $H(A)$ is a Hilbert C^*-category.*

Proof. Let $\{(a_i, p_i)\}_{i \in N}$ be a countable set of objects on $H(A)$. Since $M_\infty(A)$ is a H-additive C^*-category, there exist an object a and a family of morphisms $\{s_i : a_i \to a\}_{i \in N}$ in $M_\infty(A)$ such that

$$s_i^* s_i = 1_{\varphi(i)}, \;\; s_{i'}^* s_i = 0 \;\text{ when }\; i \neq= i', \tag{9}$$

and $\sum\limits_{i \in N} s_i s_i^* = 1_a$, in the strict topology, i.e. a is the Hilbert sum of the family $\{a_i\}_{i \in N}$. Let $s_i s_i^* = 1_i'$, $s_i p_i s_i^* = p_i'$ and $A_{1_i'} = 1_i' Hom(a, a) 1_i'$. The product of C^*-algebras $\times A_{1_i'}$ is a subalgebra of $Hom(a, a)$. This implies the existence of the element $p = p_1' \times \cdots \times p_n' \times \cdots$ in $Hom(a, a)$. Using Proposition 15, one has that $p = \sum\limits_i s_i p_i s_i^*$ in the strict topology of $Hom(a, a)$. Therefore the family of morphisms $\{s_i p_i : (a_i, p_i) \to (a, p)\}_{i \in N}$ is the realization of a

as the Hilbert sum of the family $\{a_i\}_{i\in N}$. Moreover, using Lemma 13, it is easy to show that $H(A) = M(K(A))$. $\square$

11. On the Hilbert C^*-Category of Hilbert Modules

Let A be a C^*-algebra and $\mathcal{H}(A)$ be a large C^*-category of right Hilbert A-modules and bounded A-linear maps. Let E and E' be right Hilbert A-modules. For given $x \in E'$ and $y \in E$ there is a map $\theta_{x,y} : E \to E'$ defined by $\theta_{x,y}(z) = x\cdot < y, z >$. Since $\theta_{x,y} = \theta^*_{y,x}$, $\theta_{x,y}$ is a bounded linear A-homomorphism [3], [5]. Denote by $\mathcal{K}_A(E, E')$ the closed linear span of $\{\theta_{x,y} | x \in E',\ y \in E\}$, and instead of $\mathcal{K}_A(E, E)$ we will write $\mathcal{K}_A(E)$. An element from $\mathcal{K}_A(E, E')$ is called *a compact A-homomorphism* from E to E'. By virtue of the properties of $\mathcal{K}_A(E, E')$, we conclude that the C^*-categoroid $\mathcal{K}(A)$ of all compact A-homomorphisms is a closed two-sided ideal of $\mathcal{H}(A)$.

Lemma 46. *Let $E = \oplus_{i\in N} E_i$ be the Hilbert sum of Hilbert A-modules and $p_n : E \to E$ be the projection on the submodule $E_n = \oplus_{i=1}^n E_i$. Then the sequence $\{p_n\}_{n\in N}$ converges to 1_E in the strict topology.*

Proof. As is well known, $\mathcal{L}_A(E) = M(\mathcal{K}_A(E))$ and a strict topology on $\mathcal{L}_A(E)$ can be given by the family of semi-norms $\|T\|_\xi$, $\xi \in E$, $T \in \mathcal{L}_A(E)$, where $\|T\|_\xi = \|T\xi\| + \|T^*\xi\|$ [3]. Also, note that any element $\xi \in E$ can be represented as the sequence $(\xi_1, \dots, \xi_n, \dots)$, where $\sum\limits_{i=1}^{\infty} < \xi_i, \xi_i >$ converges in the norm topology of A [3]. To prove the lemma, we have to show that $\|1_E - p_n\|_\xi \to 0$ when $n \to \infty$, where $\xi = (\xi_1, \dots, \xi_n, \dots) \in E$. But $\|1_E - p_n\|_\xi = 2\|\xi - p_n\xi\| = 2\|\sum\limits_{i=n+1}^{\infty} < \xi_i, \xi_i)\| \to 0$ when $n \to 0$. $\square$

Let $E = \oplus_{i\in N} E_i$ and $E' = \oplus_{i\in N} E'_i$ be Hilbert A-modules. It is easy to show that any element $f \in \mathcal{L}_A(E, E')$ can be represented uniquely by a $N \times N$-matrix $m_f = (f_{ij})$, where $f_{ij} \in \mathcal{L}_A(E_i, E'_j)$.

Lemma 47. *Let $E = \oplus_{i\in N} E_i$ and $E' = \oplus_{i\in N} E'_i$ be the sums of Hilbert modules. For any compact A-homomorphism $f \in \mathcal{K}_A(E, E')$ the homomorphisms f_{ij} are compact homomorphisms. If E_i and E'_i are A or 0, then either $f_{ij} \in A$ or $f_{ij} = 0$.*

Proof. To prove the first part, consider the compact homomorphism $\theta_{x,y} : E \to E'$ defined by $\theta_{x,y}(z) = x\cdot < y, z >$, where $y, z \in E$ and $x \in E'$. Let $x = \oplus x_h$, $y = \oplus y_k$ and $z = \oplus z_l$, where $x_h \in E'_h$, $y_k \in E_k$, $z_l \in E_l$. Then $\theta_{x,y}(z_l) = \oplus_h \theta_{x_h,y_k}(z_l)$. Now the $N \times N$-matrix of $\theta_{x,y}$ can be written as

$$\begin{pmatrix} \theta_{x_1,y_1} & \dots & \theta_{x_1,y_k}, & \dots \\ \dots & \dots & \dots & \dots \\ \theta_{x_h,y_1} & \dots & \theta_{x_h,,y_l} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

In the general case, any compact morphism is a limit of compact homomorphisms of the above-indicated type. But the convergence in norm induces the convergence of matrix entries. The second part easily follows from the fact that $\mathcal{K}_A(A, A) = A$, $\mathcal{K}_A(0, A) = 0$, $\mathcal{K}_A(A, 0) = 0$. This completes the proof of the lemma. □

Definition 48. Consider a full subcategory $\mathcal{H}_\infty(A)$ of $\mathcal{H}(A)$ generated by objects of the form $E = \oplus_{i\in N} E_i$, where E_i is the Hilbert module A or the trivial Hilbert module 0. Let E,$E' \in \mathcal{H}_\infty(A)$. Denote by $\mathcal{K}^f_A(E, E')$ the closed linear span of elements f from $\mathcal{L}_A(E, E')$, which has a $N \times N$-matrix with only a finite number of nonzero entries with $f_{ij} \in A$ or $f_{ij} \in 0$. Homomorphisms defined so are called finite type compact homomorphisms.

Proposition 49. *Let A be a σ-unital C^*-algebra. Then the C^*-category $M_\infty(A)$ is naturally isomorphic to the C^*-category $\mathcal{H}_\infty(A)$.*

Proof. Step I. Note that A is a right Hilbert A-module with respect to the inner product $< a, b >= a^*b$. Moreover, it is a countably-generated Hilbert module since there exists a strictly positive element $h \in A$ such that hA is dense in A. Thus for given $a \in A$ there exists a sequence $a_1, \dots, a_n, \dots \in A$ with $\lim ha_i = a$. Now we need to show that $K^f_A(A) = A$. Let $l_a : A \to A$ be the A-linear map defined by $l_a(b) = ab$, and $\lim ha_i = a$. Consider the compact homomorphisms $\theta_{a_i,h} = l_{a_ih}$, $i \in N$. Then

$$\|l_a - \theta_{a_i,h}\| = \|a - a_ih\| \to 0.$$

Thus $A \subset K_A(A)$. To show that $K_A(A) \subset A$, it is sufficient to prove that $\theta_{x,y} \in A$ since A is closed in $\mathcal{L}_A(A, A)$. But the latter fact is trivial because $\theta_{x,y} = l_{xy^*}$.

Step II. Let $E = \oplus_{i\in N} E_i$ and $E' = \oplus_{i\in N} E'_i$ be the sums of Hilbert modules where E_i and E'_i are A or 0. We need to show that $\mathcal{K}^f_A(E, E') = \mathcal{K}_A(E, E')$. For any element $a \in A$, denote by a^{ij} the $N \times N$-matrix with the following property: $a^{ij}_{kl} = a$ when $(k, l) = (i, j)$ and $a^{ij}_{kl} = 0$ when $(k, l) \neq (i, j)$. Let $\iota_j : A \to \oplus_k E'_k$ be an inclusion into the j-summand, and $pr_i : \oplus_k E_k \to A$ be the i-th projection. Then $a^{ij} = \iota_j \cdot a \cdot pr_i$. Therefore a^{ij} is compact by virtue of Step I. This proves that $\mathcal{K}^f_A(E, E') \subset \mathcal{K}_A(E, E')$ because $\mathcal{K}_A(E, E')$ is the closed subspace of $\mathcal{L}_A(E, E')$. To prove that $\mathcal{K}_A(E, E') \subset \mathcal{K}^f_A(E, E')$, it is sufficient to show the existence of a sequence of finite type compact homomorphisms which converges in the norm topology to a given compact homomorphism. Let $p_n : E \to E$ and $p'_n : E' \to E'$ be the projections on the summands of $\oplus^n_{i=1} E_i$ and $\oplus^n_{i=1} E'_i$, respectively. According to Lemma 46, the projections $\{p_n\}$ (resp. $\{p'_n\}$) converge to 1_E (resp. to $1_{E'}$) in the strict topology of $\mathcal{L}$ (E) (resp. $\mathcal{L}_A(E')$). This implies that if $f \in \mathcal{K}_A(E, E')$, then $\|p'_n f p_n - f\| \to 0$ in the norm topology. Thanks to Lemma 47, $p'_n f p_n$ are finite type compact homomorphisms. This completes the proof of the proposition. □

Theorem 50. *Let A be a σ-unital C^*-algebra.*

a) *The C^*-category $\mathcal{H}_c(A)$ of countably-generated Hilbert A-modules is a large Hilbert C^*-category;*

b) *There exists a natural equivalence of C^*-categories $\theta : H(A) \to \mathcal{H}_c(A)$.*

Proof. a) Let E, $E_1, E_2, \dots, E_n, \dots$ be countable generated Hilbert A-modules, and $E = \oplus_{i=1}^{\infty} E_i$ be the usual Hilbert sum of A-modules. The set of morphisms $\{s_i : E_i \to E\}_{i \in N}$, where s_i is an embedding into the i-th summand, is a partial isometry. By Lemma 46 the projections $\{p_n = \sum_{i=1}^{n} s_i^* s_i\}_{n \in N}$ converge to 1 in the strict topology. Thus E is the Hilbert sum of $E_1, E_2, \dots, E_n, \dots$. Conversely, let $\{s_i : E_i \to E\}_{i \in N}$ be a set of partial isometries such that

$$s_i^* s_i = 1_{E_i}, \quad s_{i'}^* s_i = 0 \ \text{ when } \ i \neq= i', \tag{10}$$

and $\sum_{i \in N} s_i s_i^* = 1_E$ in the strict topology of $\mathcal{L}_A(E) = M(\mathcal{K}_A(E))$. Let $q_i = s_i s_i^*$. Note that the usual Hilbert sum $\oplus_{i=1}^{\infty}(q_i E)$ is a submodule of E since $\{q_i = s_i s_i^*\}_{i \in N}$ is the orthogonal set of projections. The latter submodule is a the Hilbert A-submodule of E. Since $\{p_n = \sum_{i=1}^{n} s_i s_i^*\}_{n \in N}$ converges to 1 in the strict topology, we have $\lim p_i \xi = \xi$ in the norm topology of E for any $\xi \in E$. Therefore $E \approx \oplus_{i=1}^{\infty}(q_i E) \approx \oplus_{i=1}^{\infty} E_i$.

Finally, consider the full subcategory (resp. subcategoroid) $\mathcal{H}_c(A; E, E')$ (resp. $\mathcal{K}_c(A; E, E')$)of $\mathcal{H}_c(A)$ with two Hilbert A-modules E, E' as objects. From the identities $\mathcal{L}_A(E \oplus E') = M(\mathcal{K}(E \oplus E'))$, $\mathcal{K}(E \oplus E') = S(\mathcal{K}_c(A; E, E'))$ and Theorem 33 we conclude that $M(\mathcal{K}(\mathcal{H}_c(A))) = \mathcal{H}_c(A)$. Thus $\mathcal{H}_c(A)$ is a H -additive C^*-category. It is easy to show that $\mathcal{H}_c(A)$ is a pseudo-additive category. Therefore $\mathcal{H}_c(A)$ is a Hilbert C^* -category.

b) A Hilbert A-submodule E' of a Hilbert A-module E from $\mathcal{H}_\infty(A)$ is said to be *a fine Hilbert A-module* if there exists a Hilbert A-submodule E'' such that $E' \cap E'' = 0$ and $E' + E'' = E$. Note that any fine Hilbert A-module is countably-generated. Consider the full subcategory $\mathcal{H}_{cf}(A)$ of $\mathcal{H}_c(A)$ with all fine Hilbert modules as objects. It is easy to show that $\mathcal{H}_{cf}(A)$ is a pseudoabelian C^*-category. By Kasparov's Stabilization Theorem (see [5]) each countably-generated A-module is isomorphic to a fine Hilbert A-module. Thus it is sufficient to show the existence of an essential isomorphism of C^*-categories $\theta' : H(A) \to \mathcal{H}_{cf}(A)$. Consider the natural isomorphism $M_\infty(A) \approx \mathcal{H}_\infty(A)$ (Proposition 49) which essentially extends to the isomorphism $H(A) \approx \pi(\mathcal{H}_\infty(A))$, where $\pi(\mathcal{H}_\infty(A))$ is a pseudoabelian C^*-category of $\mathcal{H}_\infty(A)$. But $\pi(\mathcal{H}_\infty(A))$ is naturally isomorphic to $\mathcal{H}_{cf}(A)$ so that:

i) $(E, p) \mapsto \ker p$, for objects;

ii) if $f : (E, p) \to (E', p')$ is a morphism, then $f \mapsto \varphi$, where φ is the composition $\ker p \subset E \xrightarrow{f} E' \xrightarrow{1-p'} \ker p'$. $\square$

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