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FINITE DIFFERENCE SCHEMES FOR SOME MIXED BOUNDARY VALUE PROBLEMS

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Abstract. For the Poisson equation and for a system of equations of the theory of elasticity in a rectangle we consider mixed boundary value problems with integral restrictions, when the Dirichlet conditions are prescribed on a part of the boundary. The corresponding difference schemes are constructed and the estimates for the rate of convergence are obtained under the assumptions that the exact solutions belong to Sobolev spaces.

1. Statement of the Problem

Let $\Omega = \{(x_1, x_2) : 0 < x_\alpha < l_\alpha, \alpha = 1, 2\}$ be a rectangle with the boundary Γ; $\Gamma_* = \{(0, x_2) : 0 < x_2 < l_2\}$, $\Gamma_0 = \Gamma \backslash \Gamma_*$.

Let us consider non-local boundary value problems for the Poisson equation

$$\begin{gathered} \Delta u + f(x) = 0, \;\; x \in \Omega, \; u(x) = 0, \; x \in \Gamma_0, \\ \int\limits_0^{l_1} u(t, x_2)\, dt = 0, \; 0 < x_2 < l_2 \end{gathered} \tag{1}$$

and for a system of equations of the theory of elasticity

$$\begin{gathered} \mu\Delta \boldsymbol{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \boldsymbol{u} + \boldsymbol{f} = 0, \;\; \boldsymbol{u} = (u^1, u^2), \;\; \boldsymbol{f} = (f^1, f^2), \\ u^1(x) = 0, \;\; x \in \Gamma_0, \;\; \int\limits_0^{l_1} u^1(t, x_2)\, dt = 0, \;\; 0 < x_2 < l_2, \;\; u^2(x) = 0, \;\; x \in \Gamma. \end{gathered} \tag{2}$$

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Suppose that $f(x)$, $\boldsymbol{f}(x) \in W_2^{s-2}(\Omega)$ and problems (1)–(2) are uniquely solvable in $W_2^s(\Omega)$, $(s > 1)$, and the Lamé constants λ, μ satisfy the conditions $\mu > 0$, $\lambda + \mu \geq 0$. Under the belonging of the vector function to the space W_2^k it will be meant that each of the components of the vector belongs to that space.

In $\bar{\Omega}$ we introduce the grid domains: $\bar{\omega}_\alpha = \{x_\alpha = i_\alpha h_\alpha : i_\alpha = 0, 1, \ldots, N_\alpha$, $h_\alpha = l_\alpha/N_\alpha\}$, $\omega_\alpha = \bar{\omega}_\alpha \cup (0, l_\alpha)$, $\alpha = 1, 2$, $\omega = \omega_1 \times \omega_2$, $\bar{\omega} = \bar{\omega}_1 \times \bar{\omega}_2$, $\gamma = \bar{\omega} \backslash \omega$, $\gamma_0 = \Gamma_0 \cap \bar{\omega}$, $|h| = (h_1^2 + h_2^2)^{1/2}$, $\hbar_1 = h_1$ for $x_1 \in \omega_1$, $\hbar_1 = h_1/2$ for $x_1 = 0$. For the grid functions and difference ratios the use will be made of the standard notation from [1]. By $c, c_1, c_2, \ldots$ we denote positive constants. A set of grid functions given on the grid domain $\bar{\omega}$ and satisfying the conditions

$$y(x) = 0,\ x \in \gamma_0,\ \sum_{\omega_1} h_1 y + \frac{h_1}{2} y(0, x_2) = 0,\ x_2 \in \omega_2$$

will be denoted by H.

Define the following averaging operators:

$$T_i u(x) =$$
$$= \frac{1}{h_i^2} \int\limits_{x_i - h_i}^{x_i + h_i} (h_i - |x_i - t_i|) u(x_1 + (2-i)(t - x_1), x_2 + (i-1)(t - x_2))\, dt,$$
$$S_i^- u(x) = \frac{1}{h_i} \int\limits_{x_i - h_i}^{x_i} u(x_1 + (2-i)(t - x_1), x_2 + (i-1)(t - x_2))\, dt, \quad i = 1, 2.$$

We approximate problem (1) by the difference scheme

$$\Lambda' y + \varphi = 0, \quad x \in \omega, \quad y \in H, \quad \varphi = T_1 T_2 f, \tag{3}$$

and problem (2) by the difference scheme

$$\Lambda'' \boldsymbol{y} + \boldsymbol{\varphi} = 0, \quad x \in \omega, \quad y^1 \in H, \quad y^2(x) = 0, \quad x \in \gamma, \quad \boldsymbol{\varphi} = T_1 T_2 \boldsymbol{f}, \tag{4}$$

where

$$\Lambda' y = y_{\bar{x}_1 x_1} + y_{\bar{x}_2 x_2},$$
$$(\Lambda'' \boldsymbol{y})^\alpha = (\lambda + 2\mu) y^\alpha_{\bar{x}_\alpha x_\alpha} + 0.5(\lambda + \mu)(y^\beta_{\bar{x}_1 x_2} + y^\beta_{x_1 \bar{x}_2}) + \mu y^\alpha_{\bar{x}_\beta x_\beta},$$
$$\beta = 3 - \alpha, \quad \alpha = 1, 2.$$

On the set H let us introduce the inner product and the norm

$$(y, v) = \sum_{\omega_1^- \times \omega_2} \hbar_1 h_2 y(x) v(x), \quad \|y\| = (y, y)^{1/2}.$$

Let, moreover,

$$(y,v)_0 = \sum_{\omega} h_1 h_2 y(x) v(x), \quad \|y\|_1^2 = \|y\|^2 + \|\nabla y\|^2,$$
$$\|\nabla y\|^2 = \|y_{\bar{x}_1}\|_{(1)}^2 + \|y_{\bar{x}_2}\|_{(2)}^2,$$
$$\|y_{\bar{x}_1}\|_{(1)}^2 = \sum_{\omega_1^+ \times \omega_2} h_1 h_2 r_1 \,(y_{\bar{x}_1})^2, \quad \|y_{\bar{x}_2}\|_{(2)}^2 = \sum_{\omega_1^- \times \omega_2^+} h_1 h_2 r_2 \,(y_{\bar{x}_2})^2,$$
$$r_1 = x_1 - h_1/2; \ \ r_2 = x_1 \ \text{ for } \ x_1 \in \omega_1, \ \ r_2 = h_1/8 \ \text{ for } \ x_1 = 0;$$
$$\|y]|^2 = \sum_{\omega_1^+ \times \omega_2} h_1 h_2 y^2, \quad \|y|]^2 = \sum_{\omega_1 \times \omega_2^+} h_1 h_2 y^2,$$
$$\|y]]^2 = \sum_{\omega_1^+ \times \omega_2^+} h_1 h_2 y^2, \ \ \|y\|_*^2 = \sum_{\omega_2} h_2 y^2, \ \ \|y|]_*^2 = \sum_{\omega_2^+} h_2 y^2,$$

For the two-dimensional grid vector-valued functions $\boldsymbol{y} = (y^1, y^2)$ we introduce the norm $\|\boldsymbol{y}\|_1^2 = \|\boldsymbol{y}\|_{W_2^1(\omega,r)}^2 = \|\nabla \boldsymbol{y}\|^2 + \|\boldsymbol{y}\|^2$, where $\|\nabla \boldsymbol{y}\|^2 = \|\nabla y^1\|^2 + \|\nabla y^2\|^2$, $\|\boldsymbol{y}\|^2 = \|y^1\|^2 + \|y^2\|_0^2$.

2. Investigation of Difference Scheme (3)

The Correctness. Define on the set of grid functions the operator

$$Gy = x_1 y - Py, \quad x \in \bar{\omega},$$

where

$$Py(ih_1, x_2) = \sum_{k=0}^{i} h_1 y(kh_1, x_2) - \frac{h_1}{2}\big(y(ih_1, x_2) + y(0, x_2)\big).$$

Obviously, if $v \in H$,then $Pv(l_1, x_2) = 0$.

Lemma 1. *If the grid function v defined on $\bar{\omega}$ satisfies the conditions $v = 0$, $Pv = 0$ for $x_1 = l_1$, then*

$$\sum_{\omega_1} h_1 (Pv)^2 \le 4 \sum_{\omega_1} h_1 x_1^2 v^2, \tag{5}$$
$$\sum_{\omega_1} h_1 (Gv)^2 \le 9 \sum_{\omega_1} h_1 x_1^2 v^2, \tag{6}$$
$$\sum_{\omega_1} h_1 Pv\, v = -\frac{h_1^2}{8} v^2(0, x_2). \tag{7}$$

Proof. It is not difficult to verify that

$$\sum_{\omega_1} h_1 (Pv)^2 = -\sum_{\omega_1^-} h_1 x_1 (Pv)_{x_1} \big((Pv)^{(+1_1)} + Pv\big),$$

from which, according to

$$(Pv)_{x_1} = \frac{1}{2}(v^{(+1_1)} + v),$$

we get

$$\sum_{\omega_1} h_1 (Pv)^2 = -\frac{1}{2} \sum_{\omega_1^-} h_1 x_1 \big(v^{(+1_1)} + v\big)\big((Pv)^{(+1_1)} + Pv\big) \le$$
$$\le \frac{1}{2}\Big(\sum_{\omega_1^-} h_1 x_1^2 (v^{(+1_1)} + v)^2\Big)^{1/2} \Big(\sum_{\omega_1^-} h_1 ((Pv)^{(+1_1)} + Pv)^2\Big)^{1/2}. \qquad (8)$$

If we notice that

$$\sum_{\omega_1^-} h_1 x_1^2 (v^{(+1_1)} + v)^2 \le 4 \sum_{\omega_1} h_1 x_1^2 \, v^2$$
$$\sum_{\omega_1^-} h_1 ((Pv)^{(+1_1)} + Pv)^2 \le 4 \sum_{\omega_1} h_1 (Pv)^2,$$

then from (8) follows estimate (5).

Further,

$$\sum_{\omega_1} h_1 (Gv)^2 \le \sum_{\omega_1} h_1 x_1^2 v^2 + \sum_{\omega_1} h_1 (Pv)^2 +$$
$$+2\Big(\sum_{\omega_1} h_1 x_1^2 v^2\Big)^{1/2} \Big(\sum_{\omega_1} h_1 (Pv)^2\Big)^{1/2},$$

which together with (5) proves estimate (6).

Relation (7) follows from the easily verifiable identity

$$\sum_{\omega_1} h_1 Pv \, v = \frac{1}{2} \sum_{\omega_1} h_1 \big(\widetilde{P}v + (\widetilde{P}v)^{(-1_1)}\big)\big(\widetilde{P}v - (\widetilde{P}v)^{(-1_1)}\big) =$$
$$= \frac{1}{2}\big((\widetilde{P}v(l_1 - h_1, x_2))^2 - (\widetilde{P}v(0, x_2))^2\big),$$

in which

$$\widetilde{P}v(ih_1, x_2) = \sum_{k=0}^{i} h_1 v(kh_1, x_2) - 0.5 h_1 v(0, x_2).$$

Thus the lemma is proved completely. □

Lemma 2. *For any $y \in H$ the identity*

$$-(\Lambda' y, Gy)_0 = \|\nabla y\|^2$$

holds.

Proof. Owing to (7), we have

$$\sum_{\omega_1} h_1 v\, Gv = \sum_{\omega_1} h_1 x_1 v^2 + \frac{h_1^2}{8} v^2(0, x_2).$$

Substituting here $v = y_{\bar{x}_2}$, we obtain

$$(y_{\bar{x}_2 x_2}, Gy)_0 = -\sum_{\omega_1 \times \omega_2^+} h_1 h_2 y_{\bar{x}_2} G y_{\bar{x}_2} = -\|y_{\bar{x}_2}\|^2_{(2)}. \tag{9}$$

Summation by parts yields

$$(y_{\bar{x}_1 x_1}, Gy)_0 = -\sum_{\omega_1^+ \times \omega_2} h_1 h_2 y_{\bar{x}_1} (Gy)_{\bar{x}_1},$$

and since

$$(Gy)_{\bar{x}_1} = \Big(x_1 - \frac{h_1}{2}\Big) y_{\bar{x}_1}, \tag{10}$$

we have

$$(y_{\bar{x}_1 x_1}, Gy)_0 = -\|y_{\bar{x}_1}\|^2_{(1)}. \tag{11}$$

Equalities (9) and (11) complete the proof of the lemma. □

Lemma 3. *For any $y \in H$ the estimate*

$$\|y\|_1^2 \le (1 + 4l_1)\|\nabla y\|^2$$

is valid.

Proof. We have

$$\sum_{\omega_1^-} \hbar_1 y^2 = -\sum_{\omega_1^+} h_1 r_1\,(y^2)_{\bar{x}_1} = -\sum_{\omega_1^+} h_1 r_1 (y + y^{(-1_1)}) y_{\bar{x}_1} \le 2 \sum_{\omega_1^+} h_1 r_1^2\,(y_{\bar{x}_1})^2 + $$
$$+ \frac{1}{2} \sum_{\omega_1^+} h_1 (y + y^{(-1_1)})^2 \le 2 \sum_{\omega_1^+} h_1 r_1^2\,(y_{\bar{x}_1})^2 + \frac{1}{2} \sum_{\omega_1^-} \hbar_1 y^2.$$

Therefore

$$\sum_{\omega_1^- \times \omega_2} \hbar_1 h_2 y^2 \le 4 \sum_{\omega_1^+ \times \omega_2} h_1 h_2 r_1^2 (y_{\bar{x}_1})^2 \le 4 l_1 \|\nabla y\|^2,$$

which proves the lemma. □

As the consequence of Lemmas 2 and 3 we get the estimate

$$\|y\|_1^2 \le -(1+4l_1)(\Lambda' y, Gy)_0. \tag{12}$$

Thus, if $\Lambda' y \equiv 0$ for $x \in \omega$, then $y(x) \equiv 0$ for $x \in \bar{\omega}$. Consequently, the solution of problem (3) exists and it is unique.

A Priori Esimate of the Method Error. For the error $z = y - u$, where y is a solution of the difference scheme (3) and $u = u(x)$ is the exact solution, we obtain the problem

$$\Lambda' z = \psi, \ \ x \in \omega, \ \ z(x) = 0, \ \ x \in \gamma_0, \ \ Pz(l_1, x_2) = \chi(x_2), \ \ x_2 \in \bar{\omega}_2, \tag{13}$$

where

$$\psi = \eta_{1\bar{x}_1 x_1} + \eta_{2\bar{x}_2 x_2}, \quad \chi(x_2) = \sum_{\omega_1^+} h_1 \eta,$$
$$\eta_\alpha(u) = T_{3-\alpha} u - u, \ \ \alpha = 1, 2,$$
$$\eta(u) = S_1^- u(x) - \frac{1}{2}\big(u(x) + u(x_1 - h_1, x_2)\big). \tag{14}$$

The non-local condition in (13) is already not homogeneous. Therefore we pass in (13) to a new unknown function

$$\widetilde{z}(x) = z(x) - \frac{2}{l_1^2}(l_1 - x_1)\chi(x_2), \tag{15}$$

for which we obtain the problem

$$\Lambda' \widetilde{z} = \psi - \frac{2}{l_1^2}(l_1 - x_1)\chi_{\bar{x}_2 x_2}, \ \ x \in \omega, \ \ \widetilde{z} \in H,$$

for the solution of which on the basis of (12) we will have

$$\|\widetilde{z}\|_1^2 \le (1+4l_1) \times$$
$$\times \Big(|(\eta_{1\bar{x}_1 x_1}, G\widetilde{z})_0| + |(\eta_{2\bar{x}_2 x_2}, G\widetilde{z})_0| + \frac{2}{l_1^2} |((l_1 - x_1)\chi_{\bar{x}_2 x_2}, G\widetilde{z})_0| \Big). \tag{16}$$

Summation by parts, formula (10) and Cauchy-Bunjakovsky's inequality result in

$$|(\eta_{1\bar{x}_1 x_1}, G\widetilde{z})_0| = \Big| \sum_{\omega_1^+ \times \omega_2} h_1 h_2 r_1 \eta_{1\bar{x}_1} \widetilde{z}_{\bar{x}_1} \Big| \le \sqrt{l_1} \|\eta_{1\bar{x}_1}]| \, \|\widetilde{z}_{\bar{x}_1}\|_{(1)}. \tag{17}$$

Applying summation by parts, Cauchy-Bunjakovsky's inequality and estimate (6), we obtain

$$|(\eta_{2\bar{x}_2 x_2}, G\widetilde{z})_0| = \Big| \sum_{\omega_1 \times \omega_2^+} h_1 h_2 \eta_{2\bar{x}_2} G\widetilde{z}_{\bar{x}_2} \Big| \le 3\sqrt{l_1} \|\widetilde{z}_{\bar{x}_2}\|_{(2)} \, \|\eta_{2\bar{x}_2}|] \tag{18}$$

and, analogously,

$$|((l_1 - x_1)\chi_{\bar{x}_2 x_2}, G\widetilde{z})_0| \le l_1^2 \|\chi_{\bar{x}_2}|]_* \, \|\widetilde{z}_{\bar{x}_2}\|_{(2)}. \tag{19}$$

Substituting (17)–(19) in (16), we arrive at

$$\|\widetilde{z}\|_1 \le c_1\big(\|\eta_{1\bar{x}_1}\|_{(1)} + \|\eta_{2\bar{x}_2}\|_{(2)} + x|\chi_{\bar{x}_2}|]_*\big). \tag{20}$$

For the error of the method, according to (14), we can write

$$\|z\|_1 \le \|\widetilde{z}\|_1 + c_2\big(\|\chi\|_* + \|\chi_{\bar{x}_2}|]_*\big). \tag{21}$$

Further, from the definition of χ it directly follows that

$$\|\chi\|_* \le \sqrt{l_1}\|\eta]|, \quad \|\chi_{\bar{x}_2}|]_* \le \sqrt{l_1}\|\eta_{\bar{x}_2}]]. \tag{22}$$

Substituting (20) in (21) and taking into account (22), we get the following statement.

Lemma 4. *For the solution of the difference problem* (13) *the a priori estimate*

$$\|z\|_1 \le c_3\big(\|\eta_{1\bar{x}_1}]| + \|\eta_{2\bar{x}_2}|] + \|\eta]| + \|\eta_{\bar{x}_2}]]\big) \tag{23}$$

is valid.

Esimate of the Rate of Convergence. Lemma 4 shows that in order to obtain an estimate of the rate of convergence for the difference scheme (3) it is sufficient to estimate the norms of summands in the right-hand side of (23).

Note that the linear functional η vanishes on the polynomials of first degree, while $\eta_{\bar{x}_2}$, $\eta_{1\bar{x}_1}$ and $\eta_{2\bar{x}_2}$ vanishe on the polynomials of second degree. Moreover, they are bounded in $W_2^s(\Omega)$, $s > 1$. Therefore, applying to them the well-known technique of estimation [2] and the generalized Bremble-Hilbert lemma [3], we can see that the following statement is valid.

Theorem 1. *The difference scheme* (3) *converges in the grid norm* $W_2^1(\omega{,}r)$, *and the rate of convergence is defined by the estimate*

$$\|y-u\|_1 \le c|h|^{s-1}\|u\|_{W_2^s(\Omega)}, \quad s \in (1;3],$$

where the positive constant c does not depend on $u(x)$ and h.

3. Investigation of the Difference Scheme (4)

The Correctness. Let us investigate the solvability of problem (4). Using summation by parts, we find that

$$(y^1_{\bar{x}_1x_1}, Gy^1)_0 = -\|y^1_{\bar{x}_1}\|^2_{(1)}, \tag{24}$$

$$(y^1_{\bar{x}_2x_2}, Gy^1)_0 = -\|y^1_{\bar{x}_2}\|^2_{(2)}, \tag{25}$$

$$(y^2_{\bar{x}_1x_2}, Gy^1)_0 = -\sum_{\omega} h_1h_2\Big(x_1+\frac{h_1}{2}\Big)y^1_{x_1}y^2_{x_2}, \tag{26}$$

$$(y^2_{x_1\bar{x}_2}, Gy^1)_0 = -\sum_{\omega} h_1h_2\Big(x_1-\frac{h_1}{2}\Big)y^1_{\bar{x}_1}y^2_{\bar{x}_2}, \tag{27}$$

$$(y^2_{\bar{x}_2x_2}, x_1y^2)_0 = -\sum_{\omega^+} h_1h_2x_1|y^2_{\bar{x}_2}|^2, \tag{28}$$

$$(y^1_{\bar{x}_1x_2}, x_1y^2)_0 = -\sum_{\omega^-} h_1h_2x_1y^1_{\bar{x}_1}y^2_{\bar{x}_2}, \tag{29}$$

$$(y^1_{x_1\bar{x}_2}, x_1y^2)_0 = -\sum_{\omega} h_1h_2x_1y^1_{x_1}y^2_{x_2}, \tag{30}$$

$$(y^2_{\bar{x}_1x_1}, x_1y^2)_0 = -\sum_{\omega^+} h_1h_2\Big(x_1-\frac{h_1}{2}\Big)|y^2_{\bar{x}_1}|^2. \tag{31}$$

After several transformations it follows from (27) and (29) that

$$(y^2_{x_1\bar{x}_2}, Gy^1)_0 + (y^1_{\bar{x}_1x_2}, x_1y^2)_0 = -2\sum_{\omega^+} h_1h_2\Big(x_1-\frac{h_1}{4}\Big)y^1_{\bar{x}_1}y^2_{\bar{x}_2}; \tag{32}$$

equalities (26) and (30) imply

$$(y^2_{\bar{x}_1x_2}, Gy^1)_0 + (y^1_{x_1\bar{x}_2}, x_1y^2)_0 = -2\sum_{\omega^-} h_1h_2\Big(x_1+\frac{h_1}{4}\Big)y^1_{x_1}y^2_{x_2}; \tag{33}$$

from (24) we get

$$(y^1_{\bar{x}_1x_1}, Gy^1)_0 = \\ = -\frac{1}{2}\sum_{\omega^+} h_1h_2\Big(x_1-\frac{h_1}{4}\Big)\big|y^1_{\bar{x}_1}\big|^2 - \frac{1}{2}\sum_{\omega^-} h_1h_2\Big(x_1+\frac{h_1}{4}\Big)\big|y^1_{x_1}\big|^2; \tag{34}$$

and equality (28) results in

$$(y^2_{\bar{x}_2x_2}, x_1y^2)_0 = \\ = -\frac{1}{2}\sum_{\omega^+} h_1h_2\Big(x_1-\frac{h_1}{4}\Big)\big|y^2_{\bar{x}_2}\big|^2 - \frac{1}{2}\sum_{\omega^-} h_1h_2\Big(x_1+\frac{h_1}{4}\Big)\big|y^2_{x_2}\big|^2. \tag{35}$$

Taking into account formulas (25) and (31)–(35), from (4) it follows that

$$(\varphi^1, Gy^1)_0 + (\varphi^2, x_1y^2)_0 = 2W_h, \tag{36}$$

where

$$W_h \equiv \frac{\mu}{2}\|\nabla \boldsymbol{y}\|^2 + \frac{\lambda+\mu}{4}\times$$
$$\times\Big(\sum_{\omega^+} h_1h_2\Big(x_1-\frac{h_1}{4}\Big)\Big(y^1_{\bar{x}_1}+y^2_{\bar{x}_2}\Big)^2+\sum_{\omega^-} h_1h_2\Big(x_1+\frac{h_1}{4}\Big)\Big((y^1_{x_1}+y^2_{x_2})^2\Big)$$

is the grid analogue of elastic deformation energy.

Similarly to Lemma 3 we prove that

$$\|\boldsymbol{y}\|_1^2 \le (1+4l_1)\|\nabla \boldsymbol{y}\|^2,$$

therefore (36) implies that

$$\|\boldsymbol{y}\|_1^2 \le (1+4l_1)\big((\varphi^1, Gy^1)_0 + (\varphi^2, x_1y^2)_0\big). \tag{37}$$

Thus, if $\boldsymbol{\varphi}=0$, $x\in\omega$, then $\boldsymbol{y}\equiv 0$. And this means that problem (4) is uniquely solvable.

Estimate of the Rate of Convergence. Let $\boldsymbol{z}=\boldsymbol{y}-\boldsymbol{u}$. Substituting $\boldsymbol{y}=\boldsymbol{z}+\boldsymbol{u}$ in (4), for the method error $\boldsymbol{z}$ we obtain the problem

$$\begin{gathered}\Lambda''\boldsymbol{z}+\boldsymbol{\psi}=0,\quad x\in\omega,\\ z^1(x)=0,\ \ x\in\gamma_0,\ \ Pz^1(l_1,x_2)=\chi(x_2),\\ x_2\in\bar{\omega}_2,\ \ z^2(x)=0,\ \ x\in\gamma,\end{gathered} \tag{38}$$

where $\boldsymbol{\psi}=\boldsymbol{\varphi}+\Lambda''\boldsymbol{u}$ is the approximation error of system (2), and $\chi=-Pu^1(l_1,x_2)$ of the approximation error of the non-local condition.

Let

$$\widetilde{z}^1 = z^1-\frac{2}{l_1^2}(l_1-x_1)\chi(x_2),\ \ \widetilde{z}^2=z^2,\ \ \widetilde{\boldsymbol{z}}=(\widetilde{z}^1,\widetilde{z}^2). \tag{39}$$

Then for $\widetilde{\boldsymbol{z}}$ we obtain the problem with the homogeneous non-local condition

$$\Lambda''\widetilde{\boldsymbol{z}}+\widetilde{\boldsymbol{\psi}}=0,\ \ x\in\omega,\ \ \widetilde{z}^1\in H,\ \ \widetilde{z}^2(x)=0,\ \ x\in\gamma, \tag{40}$$

where

$$\widetilde{\psi}^1=\psi^1+\frac{2\mu}{l_1^2}(l_1-x_1)\chi_{\bar{x}_2x_2},\ \ \widetilde{\psi}^2=\psi^2+\frac{2(\lambda+\mu)}{l_1^2}\chi_{\overset{\circ}{x}_2}.$$

By means of the relations

$$\begin{gathered}T_1T_2\frac{\partial^2u^\beta}{\partial x_\alpha^2}=u^\beta_{\bar{x}_\alpha x_\alpha}+(\eta^\beta_{\alpha\alpha})_{\bar{x}_\alpha x_\alpha},\\ T_1T_2\frac{\partial^2u^\beta}{\partial x_1\partial x_2}=\frac{1}{2}\big(u^\beta_{\bar{x}_1x_2}+u^\beta_{x_1\bar{x}_2}\big)+\eta^\beta_{12x_1x_2},\\ \int_0^{l_1}u^1(t,x_2)\,dt-Pu^1(l_1,x_2)=\chi(x_2)\end{gathered}$$

the components of the approximation error are reduced to the form

$$\psi^\alpha = (\lambda+2\mu)(\eta^\alpha_{\alpha\alpha})_{\bar{x}_\alpha x_\alpha} + \frac{\lambda+\mu}{2}(\eta^\beta_{12})_{x_1x_2} + \mu(\eta^\alpha_{\beta\beta})_{\bar{x}_\beta x_\beta},$$
$$\beta = 3-\alpha, \quad \alpha=1,2, \quad \chi = \sum_{\omega_1^+} h_1\eta^1,$$

where

$$\eta^\beta_{12}(u) = S_1^- S_2^- u^\beta - \frac{1}{2}\big(u^{\beta(-1_1)} + u^{\beta(-1_2)}\big),$$
$$\beta=1,2, \quad \eta^\beta_{\alpha\alpha} = \eta_\alpha(u^\beta), \quad \eta^1 = \eta(u^1),$$

and η_α, η are defined in (14).

Lemma 5. *For the solution of problem* (38) *the a priori estimate*

$$\|\boldsymbol{z}\|_1 \le cJ(\boldsymbol{u})$$

is valid, where

$$J(\boldsymbol{u}) = \|\eta^1_{11\bar{x}_1}]| + \|\eta^1_{12x_1}|] + \|\eta^2_{11\bar{x}_1}]| + \|\eta^2_{12x_2}]| + \|\eta^1_{22\bar{x}_2}|] +$$
$$+\|\eta^2_{22\bar{x}_2}|] + \|\chi\|_* + \|\chi_{\bar{x}_2}|]_*,$$

and the constant c does not depend on $\boldsymbol{u}$ and h.

Proof. By virtue of (37), for the solution of problem (40) we have

$$\|\widetilde{\boldsymbol{z}}\|_1^2 \le (1+4l_1)\big((\widetilde{\psi}^1, G\widetilde{z}^1)_0 + (\widetilde{\psi}^2, x_1\widetilde{z}^2)_0\big). \tag{41}$$

Using summation by parts and Cauchy-Bunjakovsky's inequality, we find that

$$(\widetilde{\psi}^1, G\widetilde{z}^1)_0 \le (\lambda+2\mu)\sqrt{l_1}\|\eta^1_{11\bar{x}_1}]|\,\|\widetilde{z}^1_{\bar{x}_1}\|_{(1)} + \frac{\lambda+\mu}{2}\sqrt{l_1}\|\eta^2_{12x_2}]|\,\|\widetilde{z}^1_{\bar{x}_1}\|_{(1)} +$$
$$+3\mu\sqrt{l_1}\|\eta^1_{22\bar{x}_2}|]\,\|\widetilde{z}^1_{\bar{x}_2}\|_{(2)} + 2\mu\|\chi_{\bar{x}_2}|]_*\,\|\widetilde{z}^1_{\bar{x}_2}\|_{(2)}. \tag{42}$$

Analogously,

$$(\widetilde{\psi}^2, x_1\widetilde{z}^2)_0 \le (\lambda+2\mu)\sqrt{l_1}\|\eta^2_{22\bar{x}_2}|]\,\|\widetilde{z}^2_{\bar{x}_2}\|_{(2)} + \frac{\lambda+\mu}{2}\sqrt{l_1}\|\eta^1_{12x_1}|]\,\|\widetilde{z}^2_{\bar{x}_2}\|_{(2)} +$$
$$+3\mu\sqrt{l_1}\|\widetilde{z}^2_{\bar{x}_1}\|_{(1)}\,\|\eta^2_{11\bar{x}_1}]| + \frac{2(\lambda+\mu)}{l_1}\|\chi\|_*\,\|\widetilde{z}^2_{\bar{x}_2}\|_{(2)}. \tag{43}$$

Note that when deriving inequality (43) we have used the estimate

$$(\eta^2_{11\bar{x}_1x_1}, x_1\widetilde{z}^2)_0 = -\sum_{\omega_1^+\times\omega_2} h_1h_2\Big(x_1 - \frac{h_1}{2}\Big)\eta^2_{11\bar{x}_1}\widetilde{z}^2_{\bar{x}_1} -$$
$$-\frac{1}{2}\sum_{\omega_1\times\omega_2} h_1h_2\eta^2_{11\bar{x}_1}\big(\widetilde{z}^2 + \widetilde{z}^{2(-1_1)}\big) \le$$
$$\le \sqrt{l_1}\|\eta^2_{11\bar{x}_1}]|\,\|\widetilde{z}^2_{\bar{x}_1}\|_{(1)} + \|\eta^2_{11\bar{x}_1}]|\,\|\widetilde{z}^2\| \le 3\sqrt{l_1}\|\eta^2_{11\bar{x}_1}]|\,\|\widetilde{z}^2_{\bar{x}_1}\|_{(1)}.$$

Taking into account (42) and (43), from (41) we obtain

$$\|\widetilde{\boldsymbol{z}}\|_1 \leq cJ(\boldsymbol{u}) \tag{44}$$

Moreover, according to (39) we have

$$\|\boldsymbol{z}\|_1 \leq \|\widetilde{\boldsymbol{z}}\|_1 + \frac{2}{l_1^2}\|(l_1 - x_1)\chi\|_{(1)} \leq \|\widetilde{\boldsymbol{z}}\|_1 + c\big(\|\chi\|_* + \|\chi_{\bar{x}_2}|]_*\big). \tag{45}$$

Inequalities (44) and (45) complete the proof of the lemma. □

Theorem 2. *The difference scheme* (4) *converges in the grid norm* $W_2^1(\omega,r)$, *and the rate of convergence is defined by the estimate*

$$\|\boldsymbol{y} - \boldsymbol{u}\|_1 \leq c|h|^{s-1}\ \|\boldsymbol{u}\|_{W_2^s(\Omega)}, \quad s \in (1;3],$$

where the positive constant c *does not depend on* $\boldsymbol{u}$ *and* h.

The proof follows from Lemma 5 with application of the generalized Bremble-Hilbert lemma.

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