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ON UNIFORM SHAPE THEORY WITH PRECOMPACT SUPPORTS

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Abstract. The uniform shape theory with precompact supports on the category of uniform spaces and uniform maps is defined. This theory is described in terms of direct systems of the uniform shape category of precompact spaces. The invariants, such as the uniform Čech homology and cohomology group with precompact supports, are also defined. The long exact sequences of uniform maps and pro-groups are obtained.

Introduction

The shape theory for uniform spaces which is an extension of the uniform homotopy theory of ANRU-spaces, has been introduced by various mathematicians [1]. Recently, J. Segal, S. Spiez and B. Günter [2] by using the semi-uniform topology have given the construction of a strong shape theory for finitistic uniform spaces. T. Miyata [3] has defined and studied the classical shape theory of such spaces. In the present paper we construct an extension of the Miyata's shape theory from the category of precompact spaces to the category of arbitrary uniform spaces. The extension can be realized in terms of a direct system of uniform shapes of precompact subsets of the uniform spaces. Applications include the definitions of Čech homology and cohomology functors with precompact supports from the category of uniform spaces to the category of abelian groups.

By **Unif** we denote the category of uniform spaces and uniform maps. **pUnif**, **fUnif**, **Upol**, **ANRU** denote the full subcategories of **Unif** consisting of precompact uniform spaces, finitistic uniform spaces, uniform

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polyhedrons and uniform absolute neighborhood retracts, respectively. By **H**(**Unif**) we denote the uniform homotopy category of **Unif**, where the homotopy is understood in the sense of semi-uniform homotopy. **H**(**pUnif**), **H**(**fUnif**), **H**(**pol**), and **H**(**ANRU**) denote the uniform homotopy categories of **pUnif**, **fUnif**, **Upol** and **ANRU**, respectively. Without specific citations, in this setting we use the notions and results from [1], [3] and [4]. Some of the results of this paper were announced in [5].

1. Precompact Uniform Shape Category.

A uniform version of resolution [1] has been defined in [2–3].

Let $X \in$ **Unif**. A uniform resolution of X (with respect to **ANRU**) is an inverse system $\underline{X} = (X_\alpha, p_{\alpha\alpha'}, A)$ in **Unif** together with a morphism $\underline{p} : X \to \underline{X}$ in **pro** − **Unif** with the following conditions:

UR1) Let P be an ANRU-space, ν a uniform covering of P and $h : X \to P$ a uniform map. Then there exist an index $\alpha \in A$ and a uniform map $f : X_\alpha \to P$ such that $f \cdot p_\alpha$ and h are ν-near maps, i.e., $(f \cdot p_\alpha, h) < \nu$.

UR2) Let P be an ANRU-space and ν be a uniform covering of P. Then there exists a uniform covering ν' of P with the following property: if $\alpha \in A$ and f, $f' : X_\alpha \to P$ are uniform maps such that $(f \cdot p_\alpha, f' \cdot p_\alpha) \leq \nu'$, then there exists an index $\alpha' \geq \alpha$ such that $(f \cdot p_{\alpha\alpha'}, f' \cdot p_{\alpha\alpha'}) \leq \nu$.

A uniform polyhedral resolution (precompact polyhedral resolution, **ANRU**-resolution) is a uniform resolution $\underline{p} : X \to \underline{X} = (X_\alpha, p_{\alpha\alpha'}, A)$ such that all X_α are uniform polyhedra (precompact polyhedra, ANRU-spaces).

A uniform resolution $(\underline{p}, \underline{q}, \underline{f})$ of a uniform map $f : X \to Y$ consists of a uniform resolution $\underline{p} : X \to \underline{X}$ of X, a uniform resolution $\underline{q} : Y \to \underline{Y}$ of Y and a system map $\underline{f} = (f_\beta, \varphi) : \underline{X} \to \underline{Y}$ with an increasing function $\varphi : B \to A$ such that $\underline{f} \cdot \underline{p} = \underline{q} \cdot f$.

$(\underline{p}, \underline{q}, \underline{f})$ is a uniform polyhedral (precompact polyhedral, **ANRU**-) resolution of a uniform map $f : X \to Y$ if $\underline{p}$ and $\underline{q}$ are the uniform polyhedral (precompact polyhedral, **ANRU**-) resolutions.

Theorem 1. *Each uniform map $f : X \to Y$ of precompact uniform spaces admits a uniform precompact finite-dimensional polyhedral resolution.*

Proof. Consider the set M of all finite coverings of Y. The nerve $Y_\mu = |N(\mu)|$ of $\mu \in M$ is a finite polyhedron. Let $q_\mu : Y \to Y_\mu$ be a canonical map defined by partition of unity $(\psi_V, V \in \mu)$. It is clear that the family $(\varphi_V, V \in \mu)$, where $\varphi_V = \psi_V \cdot f$, is a partition of unity subordinated to $f^{-1}(\mu) = \{f^{-1}(V),\, V \in \mu\}$ and defines a canonical map $p_\mu : X \to X_\mu = |N(f^{-1}(\mu))|$. There is a simplicial map $f_\mu : X_\mu \to Y_\mu$ sending each vertex $V \in \mu$ of X_μ, $f^{-1}(V) \neq \varnothing$ to the vertex V of Y_μ. The map f_μ satisfies the following relation:

$$f_\mu \cdot p_\mu = q_\mu \cdot f, \;\; \mu \in M.$$

Consider now the set Λ' of all finite uniform coverings of X which are not of the form $f^{-1}(\mu)$, $\mu \in M$. There is a uniform canonical map $p_\lambda : X \to$

$X_\lambda = |N(\lambda)|$ defined by the partition of unity $(\varphi_U, U \in \lambda)$ subordinated to $\lambda \in \Lambda'$. Let $i : M \to \Lambda = \Lambda' \cup M$ be the inclusion map. Consider the set B of all finite subsets of M. Let $Y_\beta = |N(\beta)|$ be the nerve of the covering

$$\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n = \{V_1 \cap V_2 \cap \cdots \cap V_n \mid (V_1, V_2, \ldots, V_n) \in \mu_1 \times \mu_2 \times \cdots \times \mu_n\}$$

for each $\beta = \{\mu_1, \mu_2, \ldots, \mu_n\} \in B$.

For every pair $\beta \leq \beta' = \{\mu_1, \mu_2, \ldots, \mu_n, \ldots, \mu_m\}$ we define the simplicial map $q_{\beta\beta'} : Y_{\beta'} \to Y_\beta$ which takes the vertex $(V_1, V_2, \ldots, V_m) \in N(\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_m)$, $\bigcap_{i=1}^{m} V_i \neq \varnothing$, into the vertex $(V_1, V_2, \ldots, V_n) \in N(\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n)$.

It is clear that

$$q_{\beta\beta'} \cdot q_{\beta'\beta''} = q_{\beta\beta''}, \quad \beta \leq \beta' \leq \beta''.$$

Let $(\psi_{(V_1,V_2,\ldots,V_n)}, (V_1, V_2, \ldots, V_n) \in \mu_1 \times \mu_2 \times \cdots \times \mu_n)$ be a partition of unity subordinated to the covering $\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n$, where $\psi_{(V_1,V_2,\ldots,V_n)} = \psi_{V_1} \cdot \psi_{V_2} \cdot \ldots \cdot \psi_{V_n}$. Let $q_\beta : Y \to Y_\beta$ be the canonical map. Note that

$$q_{\beta\beta'} \cdot q_{\beta'} = q_\beta$$

for each pair $\beta \leq \beta'$.

Consider the set A of all finite subsets $\alpha = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of Λ and order it by the inclusion. Let

$$X_\alpha = |N(\lambda_1 \wedge \lambda_2 \wedge \cdots \wedge \lambda_n)|, \quad \alpha = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \in \Lambda.$$

Analogously to the definition of $q_{\beta\beta'}$, we define $p_{\alpha\alpha'} : X_{\alpha'} \to X_\alpha$. The function $j : B \to A$ defined by formula

$$j(\{\mu_1, \mu_2, \ldots, \mu_n\}) = \{\mu_1, \mu_2, \ldots, \mu_n\}$$

is an increasing function. For every finite subset $\beta = \{\mu_1, \mu_2, \ldots, \mu_n\}$ we define the simplicial map $f_\beta : X_{j(\beta)} \to Y_\beta$ by sending the vertex $(V_1, V_2, \ldots, V_n)$, $f^{-1}(V_1 \cap V_2 \cap \cdots \cap V_n) \neq \varnothing$, of the nerve of $f^{-1}(\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n) = f^{-1}(\mu_1) \wedge f^{-1}(\mu_2) \wedge \cdots \wedge f^{-1}(\mu_n)$ to the vertex $(V_1, V_2, \ldots, V_n)$ of the nerve $N(\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n)$.

Clearly, we have

$$f_\beta \cdot p_{\beta\beta'} = q_{\beta\beta'} \cdot f_{\beta'}.$$

Let $p_\beta : X \to X_\beta$, $\beta = \{\mu_1, \mu_2, \ldots, \mu_n\}$, $n > 1$, be the canonical map defined by the partition $\varphi_{(V_1,V_2,\ldots,V_n)} = \varphi_{V_1} \cdot \varphi_{V_2} \cdot \cdots \cdot \varphi_{V_n}$, where $\varphi_{V_i} = \psi_{V_i} \cdot f$, $i = 1, 2, \ldots, n$. It is clear that

$$f_\beta \cdot p_\beta = q_\beta \cdot f, \quad \beta \in B.$$

Consequently, we have obtained the finite-dimensional precompact inverse systems $\underline{X} = (X_\alpha, p_{\alpha\alpha'}, A)$ and $\underline{Y} = (Y_\beta, q_{\beta\beta'}, B)$ and maps of systems $\underline{p} : X \to \underline{X}$, $\underline{q} : Y \to \underline{Y}$, $\underline{f} : \underline{X} \to \underline{Y}$ such that

$$\underline{q} \cdot f = \underline{f} \cdot \underline{p}.$$

Let u be a uniform covering of Y. It admits a uniform finite refinement u'. Consider the canonical map $q_\mu : Y \to Y_\mu = |N(\mu)|$, where $\mu = u'$. The star-covering w of $N(\mu)$ is a uniform covering and $q_\mu^{-1}(w)$ refines μ. Consequently, $q_\mu^{-1}(w)$ refines u. Hence $\underline{q}$ has property **UB2** [2]. Similarly, we can show that $\underline{p}$ has property **UB2**.

Consider the set B' of all pairs $\nu = (\beta, V)$, where $\beta \in B$ and V is a uniform neighborhood of $q_\beta(Y)$ in Y_β. Order now B':

$$\nu \leq \nu' = (\beta', V') \text{ iff } \beta \leq \beta' \text{ and } q_{\beta\beta'}(V') \subseteq V.$$

Let $Y'_\nu = V$ and $q'_\nu = q_\beta : Y \to V$ for each $\nu = (\beta, V)$ and $q_{\nu\nu'} = q_{\beta\beta'}\,|_{V'}$: $V' \to V$ for every $\nu \leq \nu'$. The system $Y' = (Y'_\nu, q'_{\nu\nu'}, B')$ is a precompact polyhedral inverse system and $\underline{q}' : Y \to \underline{Y}'$ is a map of systems. Similarly, we can define a precompact polyhedral inverse system $\underline{X}' = (X'_\rho, p'_{\rho\rho'}, A')$ and a map of systems $\underline{p}' : X \to \underline{X}'$.

Let $j' : B' \to A'$ be a function defined by the formula

$$j'(\nu) = \big(j(\beta),\, f_\beta^{-1}(V)\big) \in A'.$$

It is clear that j' is the increasing function. For each $\nu \in B'$ we put

$$f'_\nu = f_{\beta|f_\beta^{-1}(V)} : X_{j'(\nu)} = f_\beta^{-1}(V) \to V = Y'_\nu.$$

The family (f'_ν, j') is a map of the system $\underline{X}'$ to the system $\underline{Y}'$ which satisfies the condition $\underline{f}' \cdot \underline{p}' = \underline{q}' \cdot f$.

The condition **UB2** holds for $\underline{q}'$ because it is fulfilled for $\underline{q}$.

Let $\nu = (\beta, V) \in B'$ and let U be a uniform neighborhood of $q'_\nu(Y) = q_\beta(Y)$ in $Y'_\nu = V \subseteq Y_\beta$. The pair $\nu' = (\beta, U) \in B'$ and $\nu \leq \nu'$. The map $q'_{\nu\nu'} = q_{\beta\beta|U} : U \to V$ is the inclusion uniform map. We have

$$q'_{\nu\nu'} \cdot (Y'_{\nu'}) = q'_{\nu\nu'}(U) = U.$$

Hence $\underline{q}'$ has property **UB1** [2]. Similarly, we can show that $\underline{p}'$ has properties **UB1** and **UB2**.

From Theorem 2.3 of [3] we conclude that $\underline{p}' : X \to \underline{X}'$ and $\underline{q}' : Y \to \underline{Y}'$ are uniform resolutions. □

From Theorem V. 15 of [4] and Theorem 1 follows

Corollary 2. *Every uniform map $f : X \to Y$ of precompact uniform spaces admits a precompact* **ANRU***-resolution.*

Corollary 3. *Every precompact uniform space X admits a precompact finite-dimensional polyhedral resolution.*

Corollary 4. *Every precompact uniform space X admits a precompact* **ANRU***-resolution.*

We need the following lemmas.

Lemma 5. *Let X be a precompact space, $P \in$ **ANRU**, and $f : X \to P$ be a uniform map. Then there exist a precompact ANRU-space $\mathcal{Q}$ and uniform maps $h : X \to \mathcal{Q}$, $g : \mathcal{Q} \to P$ such that $g \cdot h = f$.*

Proof. Let $c(f(X))$ be a completion of $f(X)$. This is a compact space because $f(X)$ is a precompact space. There exists an embedding $i : c(f(X)) \to I^m$ in the Tychonoff cube I^m, where $m = w(c(f(X)))$ is the weight of $c(f(X))$. The composition $i \cdot c : f(X) \to I^m$ is a uniform embedding, and we can consider $f(X)$ as a uniform subspace of I^m. By Theorem 9 of ([4],p.40) and Proposition 12 of ([4], p.41) it follows that I^m is an injective space [4]. By Proposition 28 of ([4], p.22) and Proposition 14 of ([4], p.82) the uniform inclusion map $j : f(X) \to P$ has a uniform extension $g : \mathcal{Q} \to P$ on some uniform neighborhood $\mathcal{Q}$ of $f(X)$ in I^m. It is clear that $\mathcal{Q}$ is a precompact ANRU-space. The map $h = i \cdot c \cdot f : X \to \mathcal{Q}$ is the desired one, and $f = g \cdot h$. □

Lemma 6. *Let X be a precompact space, P, $P' \in$ **pUnif** $\cap$ **ANRU** and $f : X \to P'$, $g_1, g_2 : P' \to P$ be uniform maps such that $g_1 \cdot f \approx_u g_2 \cdot f$. Then there exist a precompact ANRU-space P'' and uniform maps $f' : X \to P''$ and $g : P'' \to P$ such that $g \cdot f' = f$ and $g_1 \cdot g \approx_u g_2 \cdot g$.*

Proof. By Lemma 2.7 of [3], there exist an ANRU-space P''_1 and uniform maps $f'_1 : X \to P''_1$ and $g'_1 : P''_1 \to P$ such that $g_1 \cdot f'_1 = f$ and $g_1 \cdot g'_1 \approx_u g_2 \cdot g'_1$.

By Lemma 5, there exist a precompact ANRU-space P'' and uniform maps $f' : X \to P''$ and $r : P'' \to P''_1$ such that $f'_1 = r \cdot f'$. Let $g = g'_1 \cdot r$. Consequently, $f = g'_1 \cdot f'_1 = g'_1 \cdot r \cdot f' = g \cdot f'$ and $g_1 \cdot g \approx_u g_2 \cdot g$. □

We now prove the main theorem of this section.

Theorem 7. *Let $\underline{p} = (p_\alpha) : X \to \underline{X} = (X_\alpha, p_{\alpha\alpha'}, A)$ be a precompact* **ANRU***-resolution of a precompact uniform space X. Then $H\underline{p} = ([p_\alpha]_u) : X \to H\underline{X} = (X_\alpha, [p_{\alpha\alpha'}]_u, A)$ is a* **H**(**pUnif**)*-expansion of X with respect to* **H**(**ANRU** $\cap$ **pUnif**).

Proof. **E1**). Let P be a precompact ANRU-space and $h : X \to P$ be a uniform map. Let ν be a uniform covering of P such that any two ν-near maps f, $g : X \to P$ are u-homotopic (see [3], Lemma 2.5). By condition **UR1**, there exist an index $\alpha \in A$ and a uniform map $f : X_\alpha \to P$ such that $(f \cdot p_\alpha, h) \leq \nu$. Consequently, $f \cdot p_\alpha \approx_u h$.

E2). Let P be a precompact ANRU-space and f_1, $f_2 : X_\alpha \to P$ be uniform maps such that $f_1 \cdot p_\alpha \approx_u f_2 \cdot p_\alpha$. Let ν be a uniform covering of P such that ν-near uniform maps into P are u-homotopic. Choose a covering ν' according to **UR2**. Let $h : X \to P' = P \times P \in$ **ANRU** be the map defined by the formula

$$h(x) = (f_1 \cdot p_\alpha(x),\ f_2 \cdot p_\alpha(x)),\ \ x \in X.$$

The map h is uniform map. Consider projections $g_1 : P \times P \to P$ and $g_2 : P \times P \to P$ on the first and second factors, respectively. The maps g_1 and g_2 are uniform maps. Note that $g_1 \cdot h \approx_u g_2 \cdot h$.

By Lemma 6, there exist a precompact ANRU-space P'' and uniform maps $h' : X \to P$, $g : P'' \to P'$ such that $g \cdot h' = h$ and $g_1 \cdot g \approx_u g_2 \cdot g$. There is a uniform covering ν'' of P'' which refines $(g_1 \cdot g)^{-1}(\nu') \wedge (g_2 \cdot g)^{-1}(\nu')$.

By the condition **UR1**, there exist an index $\alpha'' \in A$ and a uniform map $f : X_{\alpha''} \to P''$ such that $(f \cdot p_{\alpha''}, h') < \nu''$. We can assume that $\alpha'' \geq \alpha$. It is clear that $(g_1 \cdot g \cdot f \cdot p_{\alpha''}, g_1 \cdot g \cdot h') < \nu'$. From $g_1 \cdot g \cdot h' = g_1 \cdot h = f_1 \cdot p_\alpha$ it follows that $(g_1 \cdot g \cdot f \cdot p_{\alpha''},\ f_1 \cdot p_{\alpha\alpha''} \cdot p_{\alpha''}) < \nu'$. Consequently, there is $\alpha_1 \geq \alpha''$ such that $(g_1 \cdot g \cdot f \cdot p_{\alpha''\alpha_1}, f_1 \cdot p_{\alpha\alpha_1}) < \nu$, so that $g_1 \cdot g \cdot f \cdot p_{\alpha''\alpha_1} \approx_u f_1 \cdot p_{\alpha\alpha_1}$.

Analogously, we can prove that there is $\alpha_2 \geq \alpha''$ such that $g_2 \cdot g \cdot f \cdot p_{\alpha''\alpha_2} \approx_u f_2 \cdot p_{\alpha\alpha_2}$. Let $\alpha' \geq \alpha_1, \alpha_2$. It is clear that

$$f_1 \cdot p_{\alpha\alpha'} \approx_u g_1 \cdot g \cdot f \cdot p_{\alpha''\alpha'} \approx_u g_2 \cdot g \cdot f \cdot p_{\alpha''\alpha'} \approx_u f_1 \cdot p_{\alpha\alpha'}. \ \square$$

Using Theorem 7, we can define the precompact uniform shape category **puSH**. By definition **puSH** category is the abstract shape category $\mathbf{SH}_{(\mathcal{T},\mathcal{P})}$, where $\mathcal{T} = \mathbf{H}(\mathbf{pUnif})$ and $\mathcal{P} = \mathbf{H}(\mathbf{ANRU} \cap \mathbf{pUnif})$.

2. Uniform Shape Category with Precompact Supports.

Let $\mathbf{d} - \mathbf{puSH}$ be the category whose objects are direct systems $\overline{X} = (X_\lambda, P_{\lambda\lambda'}, \Lambda)$ of the precompact uniform shape category **puSH** and whose morphisms are systems $(F_\lambda, \Phi) : \overline{X} \to \overline{Y} = (Y_\mu, \mathcal{Q}_{\mu\mu'}, M)$, where $\Phi : \Lambda \to M$ is an increasing function and $F_\lambda : X_\lambda \to Y_{\Phi(\lambda)}$, $\lambda \in \Lambda$ is a family of precompact uniform shape morphisms such that if $\lambda \leq \lambda'$, then $\mathcal{Q}_{\Phi(\lambda)\Phi(\lambda')} \cdot F_\lambda = F_{\lambda'} \cdot P_{\lambda\lambda'}$.

The composition $(H_\lambda, \zeta) = (G_\mu, \Psi) \cdot (F_\lambda, \Phi)$ of morphisms $(F_\lambda, \Phi) : \overline{X} \to \overline{Y}$ and $(G_\mu, \Psi) : \overline{Y} \to \overline{Z} = (Z_\nu, R_{\nu\nu'}, N)$ is defined in the usual way:

$$H_\lambda = G_{\Phi(\lambda)} \cdot F_\lambda, \quad \zeta = \Psi \cdot \Phi.$$

The identity morphism is the pair $(I_{X_\lambda}, 1_\Lambda) : \overline{X} \to \overline{X}$, where $1_\Lambda : \Lambda \to \Lambda$ is the identity function and I_{X_λ}, $\lambda \in \Lambda$ is a family of identity precompact uniform shape morphisms.

We say that two morphisms (F_λ, Φ), $(G_\lambda, \Psi) : \overline{X} \to \overline{Y}$ are homotopy equivalent and we write $(F_\lambda, \Phi) \approx_u (G_\lambda, \Psi)$, if for each index $\lambda \in \Lambda$ there exists an index $\mu \geq \Phi(\lambda)$, $\Psi(\lambda)$ such that $\mathcal{Q}_{\Phi(\lambda)\mu} \cdot F_\lambda = \mathcal{Q}_{\Psi(\lambda)\mu} \cdot G_\lambda$.

This relation is the equivalence relation in the category $\mathbf{d} - \mathbf{puSH}$.

A morphism $(F_\lambda, \Phi) : \overline{X} \to \overline{Y}$ is said to be a uniform homotopy equivalence, provided there is a morphism $(G_\mu, \Psi) : \overline{Y} \to \overline{X}$ such that $(G_\mu, \Psi) \cdot (F_\lambda, \Phi) \approx_u (I_{X_\lambda}, 1_\Lambda)$ and $(F_\lambda, \Phi) \cdot (G_\mu, \Psi) \approx_u (I_{Y_\mu}, 1_M)$.

By $\mathbf{dir} - \mathbf{puSH}$ we denote the quotient category of the category $\mathbf{d} - \mathbf{puSH}$ and by $\overline{(F_\lambda, \Phi)}$ an equivalence class of morphisms (F_λ, Φ).

For each uniform space X consider the family $\Pi = \{A_\mu\}_{\mu \in M}$ of all precompact subspaces of X. We order the set M by putting $\mu \leq \mu'$ whenever

A_μ is a subset of $A_{\mu'}$. Let $\overline{X}_\Pi = (A_\mu, P_{\mu\mu'}, M)$ be a direct system associated with the family Π. Here $P_{\mu\mu'} : A_\mu \to A_{\mu'}$ is the precompact uniform shape morphism induced by the uniform inclusion $p_{\mu\mu'} : A_\mu \to A_{\mu'}$.

A covering $U = (X_\lambda)_{\lambda\in\Lambda}$ of a uniform space X is said to be ΠS-cofinal if there is a function $\Phi : M \to \Lambda$ called ΠS-cofinality function, such that

(i) for every $\mu \in M$, $A_\mu \subset X_{\Phi(\mu)}$;

(ii) if $\mu \leq \mu'$, then $X_{\Phi(\mu)} \subseteq X_{\Phi(\mu')}$.

We say that a covering U is precompact, provided its elements are precompact subsets of X. It is clear that each precompact covering $U = \{X_\lambda\}_{\lambda\in\Lambda}$ of the uniform space X defines a direct system $\overline{X}_U = \{X_\lambda, P_{\lambda\lambda'}, \Lambda\}$ of the category **puSH**, as well. The morphism $P_{\lambda\lambda'}$ is induced by the uniform inclusion $p_{\lambda\lambda'} : X_\lambda \to X_{\lambda'}$.

Theorem 8. *Let $U = \{X_\lambda\}_{\lambda\in\Lambda}$ and $U' = \{X_{\lambda'}\}_{\lambda'\in\Lambda'}$ be precompact $\Pi\mathbf{S}$-cofinal coverings of the uniform space X. Then there is a uniform homotopy equivalence between $\overline{X}_U$ and $\overline{X}_{U'}$.*

Proof. The proof is the same as in T. Sanders [6]. Let $\Theta = \Phi'_{|\Lambda} : \Lambda \to \Lambda'$ and $\Theta' = \Phi_{|\Lambda'} : \Lambda' \to \Lambda$ be restrictions of $\Pi\mathbf{S}$-cofinality functions $\Phi : M \to \Lambda$ and $\Phi' : M \to \Lambda'$.

Consider the family (I_λ, Θ), where $I_\lambda : X_\lambda \to X_{\Theta(\lambda)} = X_{\Phi'(\lambda)}$ is the precompact uniform shape morphism induced by the inclusion $i_\lambda : X_\lambda \to X_{\Theta(\lambda)}$. For every $\lambda_1 \leq \lambda_2$ we have $i_{\lambda_2} \cdot p_{\lambda_1\lambda_2} = p_{\Theta(\lambda_1)\Theta(\lambda_2)} \cdot i_{\lambda_1}$. Consequently, $I_{\lambda_2} \cdot P_{\lambda_1\lambda_2} = P_{\Theta(\lambda_1)\Theta(\lambda_2)} \cdot I_{\lambda_1}$. The family (I_λ, Θ) is the morphism from $\overline{X}_U$ to $\overline{X}_{U'}$. Analogously, the family $(I_{\lambda'}, \Theta')$, where $I_{\lambda'} : X_{\lambda'} \to X_{\Theta'(\lambda')} = X_{\Phi(\lambda')}$ is the precompact uniform shape morphism induced by the inclusion $i_{\lambda'} : X_{\lambda'} \to X_{\Theta'(\lambda')}$, is the morphism from $\overline{X}_{U'}$ to $\overline{X}_U$.

For each $\lambda \in \Lambda$ we have $i_{\Theta(\lambda)} \cdot i_\lambda = p_{\lambda\Theta'(\Theta(\lambda))} \cdot 1_{X_\lambda}$. Note that

$$P_{\Theta'(\Theta(\lambda))\Theta'(\Theta(\lambda))} \cdot I_{\Theta(\lambda)} \cdot I_\lambda = P_{\lambda\Theta'(\Theta(\lambda))} \cdot I_{X_\lambda}.$$

Consequently, $(I_{\lambda'}, \Theta') \cdot (I_\lambda, \Theta) \approx_u (I_{X_\lambda}, 1_\Lambda)$. Similarly, we can prove $(I_\lambda, \Theta) \cdot (I_{\lambda'}, \Theta') \approx_u (I_{X_{\lambda'}}, 1_{\Lambda'})$. □

Definition 9. Let X and Y be uniform spaces, $U = \{X_\lambda\}_{\lambda\in\Lambda}$, $U' = \{X_{\lambda'}\}_{\lambda'\in\Lambda'}$ and $V = \{Y_\mu\}_{\mu\in M}$, $V' = \{Y_{\mu'}\}_{\mu'\in M'}$ be precompact ΠS-cofinal coverings of X and Y, respectively. We say that morphisms $(\overline{F_\lambda, \Phi}) : \overline{X}_U \to \overline{Y}_V$ and $(\overline{F_{\lambda'}, \Phi'}) : \overline{X}_{U'} \to \overline{Y}_{V'}$ are equivalent and write $(F_\lambda, \Phi) \sim (F_{\lambda'}, \Phi')$ if the homotopy equivalences $(I_\lambda, \Theta) : \overline{X}_U \to \overline{X}_{U'}$ and $(I_\mu, \Omega) : \overline{Y}_V \to \overline{Y}_{V'}$, satisfying the condition

$$(\overline{F_{\lambda'}, \Phi'}) \cdot (\overline{I_\lambda, \Theta}) = (\overline{I_\mu, \Omega}) \cdot (\overline{F_\lambda, \Phi}).$$

It is readily seen that this relation is indeed an equivalence relation. The set of all such morphisms is decomposed into classes. The class $[(\overline{F_\lambda, \Phi})]$ of the morphism $(\overline{F_\lambda, \Phi})$ we denote by $F : X \to Y$ and call it a uniform shape morphism with precompact support from X to Y.

Let $F : X \to Y$ and $G : Y \to Z$ be uniform shape morphisms with precompact supports. The composition $G \cdot F : X \to Z$ is defined as a class with the representative $(\overline{G_{\mu'}, \Psi'}) \cdot (\overline{I_\mu, \Omega}) \cdot (\overline{F_\lambda, \Phi})$, where $(\overline{F_\lambda, \Phi}) : \overline{X}_U \to \overline{Y}_V$ and $(\overline{G_{\mu'}, \Psi'}) : \overline{Y}_{V'} \to \overline{Z}_W$ are representatives of F and G, respectively. The class $I_X = [(\overline{I_{X_\lambda}, 1_\Lambda})]$ is called the identity uniform shape morphism with precompact support. It satisfies the condition

$$F \cdot I_X = I_Y \cdot F = F.$$

For each of the uniform shape morphisms with precompact supports $F : X \to Y$, $G : Y \to Z$ and $H : Z \to W$ we have

$$H \cdot (G \cdot F) = (H \cdot G) \cdot F.$$

We define now the category $\mathbf{uSH^p}$, called the uniform shape category with precompact supports. The objects of the category $\mathbf{uSH^p}$ are all the objects of the category **Unif** and the morphisms of $\mathbf{uSH^p}$ are the uniform shape morphisms with precompact supports.

We say that two uniform spaces X and Y have the same uniform shape with precompact supports, and write $\mathrm{ush}^p(X) = \mathrm{ush}^p(Y)$, provided they are isomorphic objects of the category $\mathbf{uSH^p}$.

Theorem 10. *Let $f : X \to Y$ be a uniform map, and let $U = \{X_\lambda\}_{\lambda\in\Lambda}$ and $V = \{Y_\mu\}_{\mu\in M}$ be precompact $\Pi\mathbf{S}$-cofinal coverings of X and Y, respectively. Then f induces morphism $(F_\lambda, \Phi) : \overline{X}_U \to \overline{Y}_V$. If $f \approx_u g : X \to Y$, then $(F_\lambda, \Phi) \approx_u (G_\lambda, \Psi)$, where (G_λ, Ψ) is induced by the uniform map g.*

Proof. The image $f(X_\lambda)$ of precompact subspace X_λ is a precompact subspace of Y. There exists an index μ such that $f(X_\lambda) \subset Y_\mu$. In this way we have defined an increasing function $\Phi : \Lambda \to M$ such that $f(X_\lambda) \subset Y_{\Phi(\lambda)}$.

The restriction $f_{|X_\lambda} : X_\lambda \to Y_{\Phi(\lambda)}$ defines a precompact uniform shape morphism $F_\lambda : X_\lambda \to Y_{\Phi(\lambda)}$. The family (F_λ, Φ) is the morphism from $\overline{X}_U$ to $\overline{Y}_V$. Indeed, let $P_{\lambda\lambda'} : X_\lambda \to X_{\lambda'}$ be the precompact uniform shape morphism induced by the uniform inclusion $p_{\lambda\lambda'} : X_\lambda \to X_{\lambda'}$. Note that $f_{|X_{\lambda'}} \cdot p_{\lambda\lambda'} = q_{\Phi(\lambda)\Phi(\lambda')} \cdot f_{|X_\lambda}$ in the category **Unif**. Hence $F_{\lambda'} \cdot P_{\lambda\lambda'} = \mathcal{Q}_{\Phi(\lambda)\Phi(\lambda')} \cdot F_\lambda$, where $\mathcal{Q}_{\Phi(\lambda)\Phi(\lambda')} : Y_{\Phi(\lambda)} \to Y_{\Phi(\lambda')}$ is the precompact uniform shape morphism induced by the inclusion $q_{\Phi(\lambda)\Phi(\lambda')} : Y_{\Phi(\lambda)} \to Y_{\Phi(\lambda')}$.

Let $H : X * I \to Y$ be a semi-uniform homotopy between f and g. For every precompact subspace X_λ of X the set $X_\lambda * I$ is precompact. The image $H(X_\lambda * I)$ is precompact and contains precompact subspaces $f(X_\lambda)$ and $g(X_\lambda)$. There is an increasing function $\Psi : \Lambda \to M$ such that $g(X_\lambda) \subset Y_{\Psi(\lambda)}$. There exists an index $\mu \geq \Phi(\lambda)$, $\Psi(\lambda)$ such that $Y_{\Phi(\lambda)}$, $Y_{\Psi(\lambda)}, H(X_\lambda * I) \subset Y_\mu$. It is clear that $q_{\Psi(\lambda')\mu} \cdot g_{|X_\lambda} = q_{\Phi(\lambda)\mu} \cdot f_{|X_\lambda}$ in the category $\mathbf{H(pUnif)}$. Hence $\mathcal{Q}_{\Phi(\lambda)\mu} \cdot F_\lambda = \mathcal{Q}_{\Psi(\lambda)\mu} \cdot G_\lambda$, i.e., $(F_\lambda, \Phi) \approx_u (G_\lambda, \Psi)$. □

Theorem 11. *Let U, U' and V, V' be precompact ΠS-cofinal coverings of uniform spaces X and Y, respectively. If $(F_\lambda, \Phi) : \overline{X}_U \to \overline{Y}_V$ and $(F_{\lambda'}, \Phi') : \overline{X}_{U'} \to \overline{Y}_{V'}$ are morphisms induced by a uniform map $f : X \to Y$, then $(F_\lambda, \Phi) \approx_u (F_{\lambda'}, \Phi')$.*

Proof. The proof is analogous to that of Proposition 3.3 from [6]. □

From Theorems 10 and 11 it follows that there exists a covariant functor $uS^p : \mathbf{H}(\mathbf{Unif}) \to \mathbf{uSH^p}$ such that $uS^p(X) = X$ for every uniform space X, and $uS^p([f]_u) = F$ for every uniform homotopy class $[f]_u$. Here F is the uniform shape morphism with precompact support induced by the uniform map f.

Corollary 12. *If $X \approx_u Y$, then* $\mathrm{ush}^p(X) = \mathrm{ush}^p(Y)$.

Corollary 13. *Let X and Y be precompact uniform spaces. Then* $\mathrm{ush}^p(X) = \mathrm{ush}^p(Y)$ *iff* $\mathrm{push}(X) = \mathrm{push}(Y)$.

3. Homology with Precompact Supports.

In [3,7] T. Miyata has studied the homology theory of uniform spaces. He defined Čech's type homology (cohomology) functor $\check{H}_k(-;G)$ $(\check{H}^k(-;G))$ from the uniform shape category **uSH** to the category **Ab** of abelian groups. For every uniform space X the k-th Čech homology (cohomology) group $\check{H}^k(X;G)$ $(\check{H}^k(X;G))$ is based on the uniform coverings of X, i.e.,

$$\check{H}_k(X;G) = \varprojlim\{H_k(N(u);G),\ p_{uu'k},\ U\operatorname{cov}(X)\}$$
$$\check{H}^k(X;G) = \varinjlim\{H^k(N(u);G)\ ,p^k_{uu'},\ U\operatorname{cov}(X)\}$$

where $H_k(N(u);G)$ $(H^k(N(u);G))$ denotes the k-th simplicial homology (cohomology) group of the nerve $N(u)$ of uniform covering u of X.

It is clear that the k-th Čech homology (cohomology) group of the precompact space X is the inverse (direct) limit of the inverse (direct) system consisting of k-th simplicial homology (cohomology) groups of nerves of finite uniform coverings of the precompact space X.

Let X be a uniform space and $U = \{X_\lambda\}_{\lambda\in\Lambda}$ be a precompact ΠS-cofinal covering of X. Consider a direct system $\overline{X}_U = (X_\lambda, P_{\lambda\lambda'}, \Lambda)$ of the category **puSH**. Then $\check{H}_k(\overline{X}_U;G) = (\check{H}_k(X_\lambda;G), P_{\lambda\lambda'k}, \Lambda)$ $(\check{H}^k(\overline{X}_U;G) = (\check{H}^k(X_\lambda;G), P^k_{\lambda\lambda'}, \Lambda))$ is a direct (inverse) system of the category **Ab**. Here $P_{\lambda\lambda'k}(P^k_{\lambda\lambda'})$ is the homomorphism induced by the precompact uniform shape morphism $P_{\lambda\lambda'} : X_\lambda \to X_{\lambda'}$.

For each morphism $(F_\lambda, \Phi) : \overline{X}_U \to \overline{Y}_V = (Y_\mu, \mathcal{Q}_{\mu\mu'}, M)$ the family $(F_{\lambda k}, \Phi) : \check{H}_k(\overline{X}_U;G) \to \check{H}_k(\overline{Y}_V;G)$ $((F^k_\lambda, \Phi) : \check{H}^k(\overline{Y}_V;G) \to \check{H}^k(\overline{X}_U;G))$ is a morphism of $\mathbf{dir} - \mathbf{Ab}(\mathbf{pro} - \mathbf{Ab})$. If $(F_\lambda, \Phi) \approx_u (G_\lambda, \Psi)$, then $(F_{\lambda k}, \Phi) \sim (G_{\lambda k}, \Psi)$ $((F^k_\lambda, \Phi) \sim (G^k_\lambda, \Psi))$ equivalence holds in the category $\mathbf{dir} - \mathbf{Ab}(\mathbf{pro} - \mathbf{Ab})$. It is clear that for each of the morphisms

$(F_\lambda, \Phi) : \overline{X}_U \to \overline{Y}_V$ and $(G_\mu, \Psi) : \overline{Y}_V \to \overline{Z}_W$ we have the following relations:

$$((G_{\Phi(\lambda)} \cdot F_\lambda)_k, \Psi \cdot \Phi) \sim (G_{\mu k}, \Psi) \cdot (F_{\lambda k}, \Phi)$$
$$((G_{\Phi(\lambda)} \cdot F_\lambda)^k, \Psi \cdot \Phi) \sim (F_\lambda^k, \Phi) \cdot (G_\mu^k, \Psi)).$$

Furthermore, $((I_{X_\lambda})_k, 1_\Lambda)$ $(((I_{X_\lambda})^k, 1_\Lambda))$ is the identity morphism

$$\check{H}_k(\overline{X}_U; G) \to \check{H}_k(\overline{X}_U; G)(\check{H}^k(\overline{X}_U; G) \to \check{H}^k(\overline{X}_U; G)).$$

Consequently, the functor

$$\check{H}_k(-; G) : \mathbf{puSH} \to \mathbf{Ab}(\check{H}^k(-; G) : \mathbf{puSH} \to \mathbf{Ab})$$

induces the functor

$$\mathrm{dir} - \check{H}_k(-; G) : \mathbf{dir} - \mathbf{puSH} \to \mathbf{dir} - \mathbf{Ab}$$
$$(\mathrm{dir} - \check{H}^k(-; G) : \mathbf{dir} - \mathbf{puSH} \to \mathbf{pro} - \mathbf{Ab})$$

by the formulas

$$\mathrm{dir} - \check{H}_k(\overline{X}_U; G) = \check{H}_k(\overline{X}_U; G)(\mathrm{dir} - \check{H}^k(\overline{X}_U; G) = \check{H}^k(\overline{X}_U; G))$$
$$\mathrm{dir} - \check{H}_k((\overline{F_\lambda, \Phi})) = (\overline{F_{\lambda k}, \Phi})(\mathrm{dir} - \check{H}^k((\overline{F_\lambda, \Phi})) = (\underline{F_\lambda^k, \Phi})).$$

Define now homology (cohomology) dir-group (pro-group) of the uniform space X. Consider precompact ΠS-cofinal covering U of X and dir-group $\check{H}_k(\overline{X}_U; G)$ (pro-proup $\check{H}^k(\overline{X}_U; G)$). If U' is the other precompact ΠS-cofinal covering of X, then there is a uniform homotopy equivalence $(I_\lambda, \Theta) : \overline{X}_U \to \overline{X}_{U'}$ (see Theorem 8). Consequently, (I_λ, Θ) induces an isomorphism of dir-groups $(I_{\lambda k}, \Theta) : \check{H}_k(\overline{X}_u; G) \to \check{H}_k(\overline{X}_{U'}; G)$ (pro-groups $(I_\lambda^k, \Theta) : \check{H}^k(\overline{X}_{u'}; G) \to \check{H}^k(\overline{X}_U; G)$). We denote by $\mathrm{dir} - \check{H}_k(X; G)$ $(\mathrm{dir} - \check{H}^k(X; G))$ the equivalence class of dir-groups (pro-groups) which contains $\check{H}_k(\overline{X}_U; G)$ $(\check{H}^k(\overline{X}_U; G))$ and call the k-th homology (cohomology) dir-group (pro-group) of the uniform space X with coefficients in the Abelian group G.

Let $F : X \to Y$ be a uniform shape morphism with precompact support defined by a morphism $(F_\lambda, \Phi) : \overline{X}_U \to \overline{Y}_V$. For the representative $(F_{\lambda'}, \Phi') : \overline{X}_{U'} \to \overline{Y}_{V'}$ we have $(F_{\lambda'}, \Phi') \cdot (I_\lambda, \Theta) \approx_u (I_\mu, \Omega) \cdot (F_\lambda, \Phi)$. Consequently, $(F_{\lambda' k}, \Phi') \cdot (I_{\lambda k}, \Theta) = (I_{\mu_k}, \Omega) \cdot (F_{\lambda k}, \Phi)$ $((I_\lambda^k, \Theta) \cdot (F_{\lambda'}^k, \Phi') = (F_\lambda^k, \Phi) \cdot (I_\mu^k, \Omega))$. Hence $(F_{\lambda k}, \Phi)$ and $(F_{\lambda' k}, \Phi')$ $((F_\lambda^k, \Phi)$ and $(F_{\lambda'}^k, \Phi'))$ coincide. Clearly, $F : X \to Y$ defines the morphism

$$\mathrm{dir} - \check{H}_k(F) : \mathrm{dir} - \check{H}_k(X; G) \to \mathrm{dir} - \check{H}_k(Y; G)$$
$$(\mathrm{dir} - \check{H}^k(F) : \mathrm{dir} - \check{H}^k(Y; G) \to \mathrm{dir} - \check{H}^k(X; G)).$$

This shows that we have the following

Theorem 14. *The k-th homology* (*cohomology*) dir-*group* $\operatorname{dir}-\check{H}_k(-;G)$ (pro-*group* $\operatorname{dir}-\check{H}^k(-;G)$) *is a functor* $\mathbf{uSH^p} \to \operatorname{dir}-\mathbf{Ab}(\mathbf{uSH^p} \to \mathbf{pro}-\mathbf{Ab})$.

Corollary 15. *Let X and Y be uniform spaces. If* $\operatorname{ush}^p(X) = \operatorname{ush}^p(Y)$, *then* $\operatorname{dir}-\check{H}_k(X;G) = \operatorname{dir}-\check{H}_k(Y;G)$ *and* $\operatorname{dir}-\check{H}^k(X;G) = \operatorname{dir}-\check{H}^k(Y;G)$.

For each uniform space X we now define Čech homology (cohomology) groups with precompact supports

$$_{ps}\check{H}_k(X;G) = \lim_{\rightarrow} \operatorname{dir}-\check{H}_k(X;G)$$

$$(_{ps}\check{H}^k(X;G) = \lim_{\leftarrow} \operatorname{dir}-\check{H}^k(X;G)).$$

Let $\check{F}_k = \lim\limits_{\rightarrow} \operatorname{dir}-\check{H}_k(F)$ ($\check{F}^k = \lim\limits_{\leftarrow} \operatorname{dir}-\check{H}^k(F)$) for each uniform shape morphism wich precompact support $F: X \to Y$.

Corollary 16. *The k-th Čech homology (cohomology) group with precompact supports* $_{ps}\check{H}_k(-;G)$ ($_{ps}\check{H}^k(-;G)$) *is a functor* $\mathbf{uSH^p} \to \mathbf{Ab}$.

Corollary 17. *Let X and Y be uniform spaces. If* $\operatorname{ush}^p(X) = \operatorname{ush}^p(Y)$, *then* $_{ps}\check{H}_k(X;G) = {}_{ps}\check{H}_k(Y;G)$ *and* $_{ps}\check{H}^k(X;G) = {}_{ps}\check{H}^k(Y;G)$.

4. The Homology Exact Sequence of a Uniform Map.

Let $_f\varphi$ be the category of finite simplicial sets and simplicial maps, and let $\mathbf{K}$ the extended homotopy category associated with $_f\varphi$ [8].

It is known that each category $\mathbf{L}$ yields the mapping category $\mathbf{L_{maps}}$ whose objects are the morphisms $f: X \to Y$ of $\mathbf{L}$ and whose morphisms are the pairs $g = (g_1, g_2) : f \to f' : X' \to Y'$ such that $f' \cdot g_1 = g_2 \cdot f$.

We have associated the "homotopy" category $\mathbf{K}$-$_f\varphi_{\mathbf{maps}}$ whose objects are the those of $_f\varphi_{\mathbf{maps}}$ and whose morphisms are the "homotopy" classes of morphisms in $_f\varphi_{\mathbf{maps}}$ [8]. Applying pro, we obtain the following categories: $\mathbf{pro} - {}_f\varphi_{\mathbf{maps}}$, $\mathbf{pro} - \mathbf{K} - {}_f\varphi_{\mathbf{maps}}$.

According to [8], we now define the functor

$$\check{C} : \mathbf{pUnif_{maps}} \to \mathbf{pro} - \mathbf{K} - {}_f\varphi_{\mathbf{maps}}.$$

Consider the set $_fU\operatorname{cov}(f)$ of all triples (α, β, ν), where α and β are finite uniform coverings of Y and X, respectively, and $\nu : \beta \to f^{-1}\alpha$ is a refining map. Let $f_{\alpha\beta\nu} : N(\beta) \to N(\alpha)$ be a simplicial map induced by ν. By the definition,

$$f_{\alpha\beta\nu}(U) = f \cdot \nu(U), \quad U \in \beta.$$

A refining map $(\mu_1, \mu_2) : (\alpha', \beta', \nu') \to (\alpha, \beta, \nu)$ consists of refining maps $\mu_1 : \beta' \to \beta$ and $\mu_2 : \alpha' \to \alpha$ such that $\nu \cdot \mu_1 = \mu_2 \cdot \nu'$. For each refining map (μ_1, μ_2) we have

$$\mu_{2*} \cdot f_{\alpha'\beta'\nu'} = f_{\alpha\beta\nu} \cdot \mu_{1*},$$

where μ_{2*} and μ_{1*} are, respectively, the simplicial maps $\mu_{2*} : Y_{\alpha'} \to Y_\alpha$ and $\mu_{1*} : X_{\beta'} \to X_\beta$. Consequently, $(\mu_{1*}, \mu_{2*}) : f_{\alpha'\beta'\nu'} \to f_{\alpha\beta\nu}$ is a morphism in ${}_f\varphi_{\mathbf{maps}}$. For the other refining map $(\mu'_{1*}, \mu'_{2*}) : (\alpha', \beta', \nu') \to (\alpha, \beta, \nu)$ the pairs (μ_{1*}, μ_{2*}) and (μ'_{1*}, μ'_{2*}) are contiguous in ${}_f\varphi_{\mathbf{maps}}$, and hence $[(\mu_{1*}, \mu_{2*})] = [(\mu'_{1*}, \mu'_{2*})]$.

The family $\check{C}(f) = (f_{\alpha\beta\nu} : N(\beta) \to N(\alpha))_{(\alpha,\beta,\nu)\in {}_fU\,\mathrm{cov}(f)}$ is an element of the category $\mathbf{pro} - \mathbf{K} - {}_f\varphi_{\mathbf{maps}}$. Let $g = (g_1, g_2) : f' \to f$ be a morphism of ${}_f\varphi_{\mathbf{maps}}$. Let $\check{C}(g) : \check{C}(f') \to \check{C}(f)$ be a morphism given by the family $(g_{\alpha\beta\nu}, \Theta)$. Note that $\Theta(\alpha, \beta, \nu) = (g_2^{-1}\alpha, g_1^{-1}\beta, g_*(\nu))$ for each $(\alpha, \beta, \nu) \in {}_fU\,\mathrm{cov}(f)$, where $g_*(\nu)(G) = f^{'-1} \cdot g_2^{-1} \cdot f \cdot \nu \cdot g_1(G)$, $G \in g_1^{-1}\beta$. The map $g_{\alpha\beta\nu} : f_{\Theta(\alpha,\beta,\nu)} \to f_{\alpha\beta\nu}$ is defined by the formula $g_{\alpha\beta\nu} = ((g_1)_\beta, (g_2)_\alpha)$.

Consider the composition of the functor $\check{C}$ and of the uniform realization functor $|\cdot|_u : {}_f\varphi \to \mathbf{Upol}$ [3]. Consequently, we can consider $\check{C}$ as the functor from $\mathbf{pUnif_{maps}}$ into $\mathbf{pro} - \mathbf{Upol_{maps}}$, $\mathbf{pro} - \mathbf{H} - \mathbf{Upol_{maps}}$ or $\mathbf{pro} - \mathbf{H} - \mathbf{ANRU_{maps}}$.

Let $f : K \to L$ be a map in ${}_f\varphi$ and M_f be a mapping cylinder (see [9]). It is known that $f = r \cdot i$, where $i : K \to M_f$ is the inclusion map and $r : M_j \to L$ is a retraction. There is the inclusion map $j : L \to M_f$ such that $r \cdot j = 1_L$ and $j \cdot r$ and 1_M are contiguous. Let $H_n(f; G) = H_n(M_f, K; G)$.

The sequence

$$\cdots \to H_n(K; G) \to H_n(L; G) \to H_n(f; G) \to \cdots$$

is the long exact sequence. Hence we have the functor

$$\mathbf{H_n} : \mathbf{K} - {}_f\varphi_{\mathbf{maps}} \to \mathbf{LES}(\mathbf{Ab})$$

from the category $\mathbf{K} - {}_f\varphi_{\mathbf{maps}}$ to the category of long exact sequences in $\mathbf{Ab}$. Consider the composition

$$\mathrm{pro}-H_n \cdot \check{C} : \mathbf{pUnif_{maps}} \to \mathbf{pro} - (\mathbf{LES}(\mathbf{Ab}))$$

of functors $\check{C}$ and $\mathrm{pro}-H_n : \mathbf{pro} - \mathbf{K} - {}_f\varphi_{\mathbf{maps}} \to \mathbf{pro} - (\mathbf{LES}(\mathbf{Ab}))$.

There is a natural functor [2]

$$\gamma : \mathbf{pro} - (\mathbf{LES}(\mathbf{Ab})) \to \mathbf{LES}(\mathbf{pro} - \mathbf{Ab}).$$

Consequently, we have the functor

$$\gamma \cdot \mathrm{pro}-H_n \cdot \check{C} : \mathbf{pUnif_{maps}} \to \mathbf{LES}(\mathbf{pro} - \mathbf{Ab}).$$

Omitting γ and $\check{C}$, we write

$$\mathrm{pro}-H_n(f; G) = (H_n(f_{\alpha\beta\nu}; G))_{(\alpha,\beta,\nu)} \in {}_fU\,\mathrm{cov}(f)$$

and obtain the following

Theorem 18. *For each uniform map $f : X \to Y$ of precompact uniform spaces X and Y there is a long exact sequence*

$$\cdots \to \mathrm{pro}-H_n(X; G) \to \mathrm{pro}-H_n(Y; G) \to \mathrm{pro}-H_n(f; G) \to \cdots$$

According to [8], we say that the uniform map $f : X \to Y$ is movable iff $\check{C}(f)$ is movable in $\mathbf{pro} - \mathbf{K} - {}_f\varphi_{\mathbf{maps}}$.

Theorem 19. *Let $f : X \to Y$ be a movable uniform map of precompact spaces with countable weight. Then there is a long exact sequence*

$$\cdots \to \check{H}_n(X;G) \to \check{H}_n(Y;G) \to \check{H}_n(f;G) \to \cdots ,$$

where $\check{H}_n(f;G) = \varprojlim \operatorname{pro} - H_n(f;G)$.

Proof. By the condition of the theorem, ${}_fU \operatorname{cov}(f)$ has a countable cofinal subset. Hence each pro-group in the long exact sequence

$$\cdots \to \operatorname{pro} - H_n(X;G) \to \operatorname{pro} - H_n(Y;G) \to \operatorname{pro} - H_n(f;G) \to \cdots$$

is indexed by a countable set. Each term of this sequence satisfies the Mittag-Leffler condition. From Proposition IV.1.2 of [8] it follows that the limit sequence

$$\cdots \to \check{H}_n(X;G) \to \check{H}_n(Y;G) \to \check{H}_n(f;G) \to \cdots$$

is exact. □

References

1. S. Mardešic and J. Segal, Shape theory. *North Holland, Amsterdam etc.,* 1982.
2. J. Segal, S. Spiez and B. Günter, Strong shape of uniform spaces. *Topology and Appl.* **49**(1993), 237–249.
3. T. Miyata, Uniform shape theory. *Glasnik Mat.* **49**(1994), No. 29, 123–168.
4. J. Isbell, Uniform spaces. *Mathematical Surveys* 12*, AMS,* 1964.
5. V. Baladze, A characterization of precompact shape and (co)homology groups of remainder. *Inter. Math. Congr., Abstracts, Zagreb,* 2000.
6. T. Sanders, On the generalized and the H-shape theory. *Duke Math. J.* **40**(1973), 743–754.
7. T. Miyata, Homology, cohomology, and uniform shape. *Glas. Mat. Ser.* III **50**(1995), No. 30, 85–109.
8. D. Edwards and P. Tulley Mc Auley, The shape of a map. *Fund. Math.* **96**(1977), 195–210.
9. E. Spanier, Algebraic topology. *McGrow-Mill, New York,* 1966.

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