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ON THE LIMIT DISTRIBUTION OF SQUARE DEVIATION OF A WIDE CLASS OF ESTIMATORS OF FUNCTIONAL CHARACTERISTICS OF THE OBSERVATION DISTRIBUTION LAW

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Abstract. We consider the nonparametric estimators of functional characteristics of the distribution, from which the well known estimators are separately obtained for density, distribution, hazard and regression functions both in the one- and the two-dimensional case. The theorem is proved, which makes it possible to study in the unified manner the limit distribution of square deviation for the abovementioned functional characteristics.

1. Let $X, X_1, X_2, \dots, X_n$ be a sequence of independent, identically distributed random variables with values in a p-dimensional Euclidean space R_p, $p \geq 1$, which have the density $f(x)$, $x = (x_1, \dots, x_p)$. Let $g(x), x \in R_p$, be some functional characteristic of the distribution of a random variable X and we only know that $g(x)$ belongs to some sufficiently wide set of functions. We want to construct a random function $g_n(x) = \mathbf{T}(x, X_1, X_2, \dots, X_n)$ which could be used as an approximation for $g(x)$ (the nonparametric case). The method of nonparametric estimation of $g(x)$ is usually defined as a method which is valid irrespective of the distribution from which the sample is taken. Following R. Rosenblatt's, this area of investigation could be called "curve estimation". Using the notation from J. Marron and W. Härdle [1], some important special cases of curve estimation are as follows:

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D is the density estimation: $g(x) \equiv f(x)$;

H is the hazard function estimation:

$$g(x) \equiv h(x) = \frac{f(x)}{1 - F(x)},$$

where $F(x)$ is the distribution function of a random variable X;

R is the regression function estimation:

$$g(x) \equiv m(x) = E(Y|\overline{X} = x),$$

where $\overline{X} = (X^{(1)}, \dots, X^{(p)})$, $X = (\overline{X}, Y)$.

There are various classes of nonparametric estimators of the above-mentioned functional characteristics: density distributions – Rosenblatt–Parzen kernel estimators, histograms, Chentsov projective estimators; regression functions – kernel estimators, regressograms and hazard functions – Wat–son-Leadbetter estimators.

In the literature there are a number of studies dealing with the limit distribution of an integral square deviation of estimators of the distribution density and a regression curve. For instance, the limit distribution of a square deviation $\|f_n - f\|^2_{L_2(r)}$ of estimators f_n of density f constructed by means of "kernel" was studied by P. Bickel and M. Rosenblatt [2]. An analogous question for estimators constructed with the aid of systems of orthonormalized functions was studied by E. Nadaraya [3]. The main technique of these studies was to use the Breumann–Brillinger approximation of the empirical process by a sequence of Brownian bridges.

In 1975, Y. Komlos, P. Major and G. Tusnady [5] succeeded in improving the approximation order in Brullinger's theorem. Using this result, E. Nadaraya obtained the limit distribution of square distribution of two nonparametric estimators of the kernel type density constructed with respect to two independent samples and applied this result to obtain a homogenity test [6]. In the two-dimensional case, the limit distribution of a quadratic measure of deviation was obtained by P. Revesz [7] and M. Mirzakhmedov [8] by using the Breumann–Brillinger multidimensional analysis of approximation [9], and in the case of the kernel estimator of a regression curve by V.D. Konakov [10].

The second method of investigation of the limit distribution of square deviation is an additional randomization, namely, instead of n observations we take a random number of observations which is distributed by the Poisson law with a mean value n and does not depend on observations. This method was used by E. Nadaraya to estimate a regression function [11], and by Rosenblatt in the case of distribution density [12].

However it should be said that the above-mentined methods have the following disadvantages: the limit distribution of square deviation of estimators is obtained for a narrow class of kernels, orthonormalized functions and functional characteristics, and, besides, the conditions imposed on the parameter λ_n contained in these estimators are rather strict ([2], [3], [6], [8], [10], [11], [12]).

Problems of the limit distribution of square deviation of kernel type estimators of functional characteristics were later studied in greater detail. P. Hall [13] and E. Nadaraya [14] nearly simultaneously proposed the third method of investigation, namely the martingale method. E. Nadaraya [14] and Sh. Khashimov [15] used the central limit theorem obtained by R. Liptser and A. Shiryaev for semimartingales [16]. As compared with the previously available methods ([2], [6], [10], [11]), this method made it possible to get a better understanding of the accumulated facts, as well as to simplify many theorems, obtain new results and weaken a number of conditions.[1]

The aim we pursue in this paper is to obtain the limit distribution of square deviation for general type estimators $g_n(x)$ of the functional characteristic $g(x)$ of the distribution law of the random variable X. These estimators actually include many well known estimators such as density estimators, regression functions and so on. The proposed method enables one to study consistently the limit distribution of square deviation both in the one- and the two-dimensional case. The proof of the basic theorem mainly follows the same scheme as used in proving Theorem 1 in [14].

2. Let us consider nonparametric estimators of the general form [1]:

$$g_n(x) = \frac{\sum\limits_{i=1}^{n} \delta_{0\lambda}(x, X_i)}{\sum\limits_{i=1}^{n} \delta_{1\lambda}(x, X_i)}, \tag{2.1}$$

where $\delta_{0\lambda}(x,y)$ and $\delta_{1\lambda}(x,y)$ are Borel-measurable functions on $R_p \times R_p$, and $\lambda = \lambda(n)$ is a sequence of positive integers converging to infinity and satisfying the condition $\lambda(n) = o(n)$ for $n \to \infty$.

Let us first consider the important examples constructed from (2.1):

D–1. **Kernel estimator.** Let $\delta_{0\lambda}(x, X_i) = \lambda^p K(\lambda(x - X_i))$, $K : R_p \to R$ and $\delta_{1\lambda}(x, X_i) \equiv 1$. Then we come to the following class of estimators:

$$g_n(x) = n^{-1}\lambda^p \sum_{i=1}^{n} K\big(\lambda(x - X_i)\big).$$

They are called estimators of the Rosenblatt–Parzen type for the probability density $f(x)$.

D–2. **Projective estimator.** Let $\{\varphi_j\}$ be an orthonormal basis in the space $L_2(r)$. Let $\delta_{0\lambda}(x, X_i) = \sum\limits_{j=1}^{\lambda} \varphi_j(x)\varphi_j(X_i)r(X_i)$ and $\delta_{1\lambda}(x, X_i) \equiv 1$.

[1] This survey by no means claims to be complete.

Then

$$g_n(x)=\sum_{j=1}^{\lambda}\widehat{a}_j\varphi_j(x),\quad \widehat{a}_j=\frac{1}{n}\sum_{i=1}^{n}\varphi_j(X_i)r(X_i).$$

We have obtained the Chentsov estimators (projective estimators).

D–3**. Histogram.** Let the interval $[0,1]$ be partitioned into λ equal intervals A_i, $i=\overline{1,\lambda}$, of length $h=\frac{1}{\lambda}$. Let $\delta_{0\lambda}(x,X_i)=\lambda\sum\limits_{k=1}^{\lambda}I_k(x)I_k(X_i)$, where $I_k(\cdot)$ is the indicator of the interval A_k, and $\delta_{1\lambda}(x,X_i)\equiv 1$. Then from (2.1) we obtain the histogram

$$g_n(x)=\frac{\nu_i}{nh},\quad x\in A_i,\quad i=\overline{1,\lambda},$$

where ν_i, $i=\overline{1,\lambda}$, are the numbers of observations occurring respectively in A_i, $i=\overline{1,\lambda}$.

D–4**. Frequency polygon.** Let A_i be defined as in *D*–3. Let

$$\delta_{0\lambda}(x,X_i)=\overline{\delta}_{0\lambda}(x_k,X_i)+\lambda\big(\overline{\delta}_{0\lambda}(x_{k+1},X_i)-\overline{\delta}_{0\lambda}(x_k,X_i)\big)(x-x_k),\\ x\in[x_k,x_{k+1}],$$

and

$$\delta_{1\lambda}(x,X_i)\equiv 1,$$

where x_k is the middle point of the interval A_k and

$$\overline{\delta}_{0\lambda}(x,X_i)=\lambda\sum_{j=1}^{\lambda}I_j(x)I_j(X_i).$$

Then by (2.1) we define the function $g_n(x)$ which is usually called “the frequency polygon”:

$$g_n(x)=\frac{\nu_k}{nh}+\frac{\nu_{k+1}-\nu_k}{nh^2}\,(x-x_k),\quad x\in[x_k,x_{k+1}],\quad k=\overline{1,\lambda-1},\quad h=\frac{1}{\lambda}\,.$$

R–1**. Kernel estimator.** Let $\delta_{0\lambda}(x,X_i)=\lambda^pK(\lambda(x-\overline{X}_i))Y_i$ and $\delta_{1\lambda}(x,X_i)=\lambda^pK(\lambda(x-\overline{X}_i))$, where $X_i=(\overline{X}_i,Y_i)$, $\overline{X}_i=(X_i^{(1)},\dots,X_i^{(p)})$. Then we come to the class of estimators:

$$g_n(x)=\frac{\sum\limits_{i=1}^{n}Y_iK(\lambda(x-\overline{X}_i))}{\sum\limits_{i=1}^{n}K(\lambda(x-\overline{X}_i))}\,.$$

which are called kernel type estimators for the regression function $m(x)$ ([16], [17]).

R–2. **Regressogram.** Let A_i and $I_i(x)$, $i=\overline{1,\lambda}$, be defined as in $D-3$. Let $\delta_{0\lambda}(x,X_i)=\lambda\sum\limits_{i=1}^{\lambda}I_j(x)I_j(\overline{X}_i)\cdot Y_i$, $\delta_{1\lambda}(x,X_i)=\lambda\sum\limits_{j=1}^{\lambda}I_j(x)I_j(\overline{X}_i)$, $X_i=(\overline{X}_i,Y_i)$. Then by (2.1) we obtain the regressogram

$$g_n(x)=\frac{1}{N_j}\sum_{(\overline{X}_i\in A_j)}Y_i,\quad x\in A_j,\quad j=\overline{1,\lambda},$$

where N_j is the number of Xs occurring in A_j.

R–3. **Kernel estimator of the regression function connected with the convolution problem.** Let $X=Y+Z$, Y and Z be independent random variables, Z having a normal distribution $N(0,\sigma^2)$ and σ being assumed to be known. Let $g_0(x)=E(Y|X=x)$ and $g(x)=g_0(x)-x$. Let, further, $\delta_{0\lambda}(x,X_i)=\lambda^2\sigma^2K^{(1)}(\lambda(x-X_i))$, $\delta_{1\lambda}(x,X_i)=\lambda K(\lambda(x-X_i))$. Then by (2.1) we obtain [28]

$$g_n(x)=\frac{\lambda\sigma^2\sum\limits_{i=1}^{n}K^{(1)}(\lambda(x-X_i))}{\sum\limits_{i=1}^{n}K(\lambda(x-X_i))}.$$

Here $K(x)$ is the same as in D–1.

H–1. **Kernel estimator.** Let $\delta_{0\lambda}(x,X_i)=\lambda K(\lambda(x-X_i))$, $\delta_{1\lambda}(x,X_i)=1-\mathcal{K}(\lambda(x-X_i))$, where

$$\mathcal{K}(u)=\int\limits_{-\infty}^{u}K(t)\,dt.$$

Then we come to the following Watson-Leadbetter class of estimators [18]:

$$g_n(x)=\frac{\frac{\lambda}{n}\sum\limits_{i=1}^{n}\mathcal{K}(\lambda(x-X_i))}{1-\frac{1}{n}\sum\limits_{i=1}^{n}\mathcal{K}(\lambda(x-X_i))}.$$

H–2. **Sequence of delta estimators.** This case is a direct generalization of H–1. Define $\delta_{0\lambda}(x,X_i)$ as in (2.1) and

$$\delta_{1\lambda}(x,X_i)=1-\int\limits_{-\infty}^{x}\delta_{0\lambda}(t,X_i)\,dt.$$

Let us now proceed to investigating the limit behavior of the square deviation distribution of estimators (2.1).

Notation. We need the following notation:

$$\delta_\lambda(x,X_i)=\delta_{0\lambda}(x,X_i)-\delta_{1\lambda}(x,X_i)g(x),$$

$$\widehat{g}_n(x) = \frac{1}{n}\sum_{i=1}^{n}\delta_\lambda(x,X_i),$$

$$U_n = n\int\big(\widehat{g}_n(x) - E\widehat{g}_n(x)\big)^2 r(x)\,dx, \quad \Delta_n = EU_n,$$

$$\alpha_\lambda(x,y) = \delta_\lambda(x,y) - E\delta_\lambda(x,X_1),$$

$$\delta_{ij}(u,v,\lambda) = \int\delta_{i\lambda}(x,u)\delta_{j\lambda}(x,v)g^{i+j}(x)r(x)\,dx, \quad i,j=0,1,$$

$$\sigma_n^2 = 2\iint\big(E\alpha_\lambda(u_1,X_1)\alpha_\lambda(u_2,X_1)\big)^2 r(u_1)r(u_2)\,du_1\,du_2,$$

$$L_n = E\bigg[\iint\alpha_\lambda(u_1,X_1)\alpha_\lambda(u_2,X_2)E\alpha_\lambda(u_1,X_1)\alpha_\lambda(u_2,X_1)\cdot \cdot r(u_1)r(u_2)\,du_1\,du_2\bigg]^2,$$

$$\eta_{ij}^{(n)} = \sqrt{\frac{n}{n-1}}\,\frac{2}{n\sigma_n}\int\alpha_\lambda(x,X_i)\alpha_\lambda(x,X_j)r(x)\,dx,$$

$$\xi_k^{(n)} = \sum_{i=1}^{k-1}\eta_{ik}^{(n)}, \quad k=\overline{2,n}, \quad \xi_1^{(n)}=0, \quad \xi_k^{(n)}=0, \quad k>n,$$

$$Y_k^{(n)} = \sum_{i=1}^{k}\xi_i^{(n)}, \quad \mathcal{F}_k^{(n)} = \sigma(\omega:\ X_1,\dots,X_k),$$

where $\mathcal{F}_k^{(n)}$ is the σ-algebra generated by the random variables $X_1,\dots,X_k$ and $\mathcal{F}_0^{(n)} = \sigma(\varnothing,\Omega)$.

We make the assumptions:

1°. $E\bigg(\displaystyle\int\delta_{0\lambda}^2(x,X_1)r(x)\,dx\bigg)^\gamma \le c_1\lambda^{\gamma\alpha}$,

$E\bigg(\displaystyle\int\delta_{1\lambda}^2(x,X_1)g^2(x)\,dx\bigg)^\gamma \le c_2\lambda^{\gamma\alpha}, \quad \alpha>0, \quad \gamma=1,2,$

2°. $E\big(\delta_{ij}^s(X_1,X_2,\lambda)\big) \le c_3\lambda^{(s-\beta)\alpha}, \quad \alpha>0, \quad 0<\beta\le 1, \quad s=2,4, \quad i,j=0,1,$

3°. $\sigma_n\to\infty$ and $L_n = o(\sigma_n^4)$,

4°. $n^{-1}\sigma_n^{-4}\cdot\lambda^{(4-\beta)\alpha}\to 0$ as $n\to\infty$.

Lemma 2.1. *The stochastic sequence* $(Y_j^{(n)},\mathcal{F}_j^{(n)})_{j\ge1}$ *is a martingale, and* $(\xi_j^{(n)},\mathcal{F}_j^{(n)})_{j\ge1}$ *a martingale-difference.*

Indeed, condition 1° implies that $E|Y_j^{(n)}| < \infty$ for all $j \ge 1$,

$E(\xi_{j+1}^{(n)}|\mathcal{F}_j^{(n)}) = 0$ a. s. so that

$$E\big(Y_{j+1}^{(n)}|\mathcal{F}_j^{(n)}\big) = E\bigg(\sum_{i=1}^{j+1}\xi_i^{(n)}\big|\mathcal{F}_j^{(n)}\bigg) = E\bigg(\sum_{i=1}^{j}\xi_i^{(n)}\big|\mathcal{F}_j^{(n)}\bigg) + \\ + E\big(\xi_{j+1}^{(n)}\big|\mathcal{F}_j^{(n)}\big) = Y_j^{(n)} \ \text{(a.s.)}. \qquad \square$$

Theorem 2.1. *Let conditions* 1°–4° *be fulfilled. Then*

$$\sigma_n^{-1}(U_n - \Delta_n) \xrightarrow{d} N(0,1) \quad as \quad n \to \infty,$$

where d *denotes the convergence in distribution, and* $N(0,1)$ *a random variable having normal distribution with zero mean value and dispersion* 1.

Proof. We have

$$\frac{U_n - \Delta_n}{\sigma_n} = n^{-1}\sigma_n^{-1}\sum_{i\neq j}\int \alpha_\lambda(x,X_i)\alpha_\lambda(x,X_j)r(x)\,dx + \\ + (n\sigma_n)^{-1}\sum_{j=1}^{n}\Big(\int \alpha_\lambda^2(x,X_j)r(x)\,dx - \Delta_n\Big) = \\ = \sqrt{\frac{n-1}{n}}\,H_n^{(1)} + H_n^{(2)}, \tag{2.2}$$

where

$$H_n^{(1)} = \sum_{j=1}^{n}\xi_j^{(n)},$$

$$H_n^{(2)} = \frac{1}{n\sigma_n}\sum_{j=1}^{n}\Big[\int \alpha_\lambda^2(x,X_j)r(x)\,dx - E\int \alpha_\lambda^2(x,X_j)r(x)\,dx\Big].$$

Let us first show that $H_n^{(2)}$ converges to zero in probability.

By conditions 1° and 2° we obtain the inequalities:

$$E\Big(\int \alpha_\lambda^2(x,X_1)r(x)\,dx\Big)^\gamma \le c_4\Big[E\Big(\int \delta_{0\lambda}^2(x,X_1)r(x)\,dx\Big)^\gamma + \\ +E\Big(\int \delta_{1\lambda}^2(x,X_1)g^2(x)\,dx\Big)^\gamma\Big] \le c_5\lambda^{\gamma\alpha}, \tag{2.3}$$

$$\sigma_n^2 \le c_6\big[E\delta_{00}^2(X_1,X_2,\lambda) + E\delta_{01}^2(X_1,X_2,\lambda) + \\ +E\delta_{10}^2(X_1,X_2,\lambda) + E\delta_{11}^2(X_1,X_2,\lambda)\big] \le c_7\lambda^{(2-\beta)\alpha}, \tag{2.4}$$

$$E\Big(\int \alpha_\lambda(u,X_1)\alpha_\lambda(u,X_2)r(u)\,du\Big)^s \le \\ \le c_8\big(E\delta_{00}^s(X_1,X_2,\lambda) + E\delta_{01}^s(X_1,X_2,\lambda) + E\delta_{10}^s(X_1,X_2,\lambda) + \\ +E\delta_{11}^s(X_1,X_2,\lambda)\big) \le c_9\lambda^{(s-\beta)\alpha}, \quad s \ge 2. \tag{2.5}$$

By (2.3) we have

$$DH_n^{(2)} \le \frac{1}{n\sigma_n^2}\, E\Big(\int \alpha_\lambda^2(x,X_1)r(x)\,dx\Big)^2 = O\Big(\frac{\lambda^{2\alpha}}{n\sigma_n^2}\Big).$$

Hence, by virtue of condition 4° and (2.4), $H_n^{(2)} \xrightarrow{P} 0$ (here and below the letter P over the arrow denotes convergence in probability).

Next we are to show that $H_n^{(1)} \xrightarrow{d} N(0,1)$. This can be done by using Corollaries 2 and 6 of Theorem 2 from [19] which contain the conditions of asymptotic normality of a square-integrable martingale-difference. Let us show that our sequence $(\xi_k,\mathcal{F}_k)_{k\ge 1}$ satisfies these conditions (for simplicity, in what follows we will write ξ_k and η_{ij} instead of $\xi_k^{(n)}$ and $\eta_{ij}^{(n)}$).

Note that $\sum\limits_{k=1}^n E\xi_k^2 = 1$. Asymptotic normality takes place if for $n\to\infty$

$$\sum_{k=1}^n E\big[\xi_k^2 I(|\xi_k|\ge\varepsilon)/\mathcal{F}_{k-1})\big] \xrightarrow{P} 0 \tag{2.6}$$

and

$$\sum_{k=1}^n \xi_k^2 \xrightarrow{P} 1. \tag{2.7}$$

In [19] it is proved that if (2.7) and the condition $\sup\limits_{1\le k\le n}|\xi_k| \xrightarrow{P} 0$ are fulfilled, then condition (2.6) holds too. Since for $\varepsilon>0$

$$P\Big\{\sup_{1\le k\le n}|\xi_k|\ge\varepsilon\Big\} \le \varepsilon^{-4}\sum_{k=1}^n E\xi_k^4,$$

to prove the convergence $H_n^{(1)} \xrightarrow{d} N(0,1)$, by virtue of relation (2.12) below it remains to explain only (2.7), i.e., that $\sum\limits_{k=1}^n \xi_k^2 \xrightarrow{d} 1$. For this it suffices to show that $E\big(\sum\limits_{k=1}^n \xi_k^2 - 1\big)^2 \to 0$ as $n\to\infty$, i.e., that (by virtue of $\sum\limits_{k=1}^n E\xi_k^2 = 1$)

$$E\Big(\sum_{k=1}^n \xi_k^2\Big)^2 = \sum_{k=1}^n E\xi_k^4 + 2\sum_{1\le k_1<k_2\le n} E\xi_{k_1}^2\xi_{k_2}^2 \to 1 \tag{2.8}$$

as $n\to\infty$. Let us prove (2.8). By the definitions of η_{ik} and ξ_k we obtain

$$\sum_{k=1}^n E\xi_k^4 = \frac{16}{n^4\sigma_n^4}\,(M_n^{(1)} + M_n^{(2)}), \tag{2.9}$$

where

$$M_k^{(1)} = \sum_{k=1}^{n}(k-1)\iiiint E\prod_{j=1}^{4}\alpha_\lambda(x_j,X_1)E\prod_{j=1}^{4}\alpha_\lambda(x_j,X_k)\prod_{j=1}^{4}r(x_j)dx_j,$$

$$M_k^{(2)} = 3\sum_{k=1}^{n}(k-1)(k-2)\iiiint E\alpha_\lambda(x_1,X_1)\alpha_\lambda(x_2,X_1)\cdot$$

$$\cdot E\alpha_\lambda(x_3,X_2)\alpha\lambda(x_4,X_2)\cdot E\prod_{j=1}^{4}\alpha_\lambda(x_j,X_k)r(x_j)dx_j.$$

Let us estimate $M_n^{(1)}$ and $M_n^{(2)}$. By condition 2° we have

$$\begin{aligned}M_n^{(1)} &= \sum_{k=1}^{n}(k-1)\iiint\Big(\int\alpha_\lambda(x,u)\alpha_\lambda(x,v)r(x)\,dx\Big)^4 f(u)f(v)\,du\,dv \le\\ &\le c_{10}n^2\big(E\delta_{00}^4(X_1,X_2,\lambda)+E\delta_{01}^4(X_1,X_2,\lambda)+\\ &\quad+E\delta_{10}^4(X_1,X_2,\lambda)+E\delta_{11}^4(X_1,X_2,\lambda)\big)\le\\ &\le c_{11}n^2\lambda^{(4-\beta)\alpha},\end{aligned} \tag{2.10}$$

and also

$$\begin{aligned}M_n^{(2)} &= 3\sum_{k=1}^{n}(k-1)(k-2)E\Big(\int\alpha_\lambda(x_1,X_1)\alpha_\lambda(x_1,X_3)r(x)\,dx\Big)^2\cdot\\ &\quad\cdot\Big(\int\alpha_\lambda(x_2,X_2)\alpha_\lambda(x_2,X_3)r(x_2)\,dx_2\Big)^2\le\\ &\le c_{12}n^3E\big(\delta_{00}^2(X_1,X_3,\lambda)+\delta_{01}^2(X_1,X_3,\lambda)+\delta_{10}^2(X_1,X_3,\lambda)+\\ &\quad+\delta_{11}^2(X_1,X_3,\lambda)\big)\big(\delta_{00}^2(X_2,X_3,\lambda)+\delta_{01}^2(X_2,X_3,\lambda)+\\ &\quad+\delta_{10}^2(X_2,X_3,\lambda)+\delta_{11}^2(X_2,X_3,\lambda)\big)\le\\ &\le c_{14}n^3\lambda^{(4-\beta)\alpha}.\end{aligned} \tag{2.11}$$

From (2.9), (2.10) and (2.11) it follows that

$$\sum_{k=1}^{n}E\xi_k^4 \le c_{15}\,\frac{\lambda^{(4-\beta)\alpha}}{n\sigma_n^4}\,.$$

But by virtue of condition 4° this implies

$$\sum_{k=1}^{n}E\xi_k^4\to 0 \quad\text{as}\quad n\to\infty. \tag{2.12}$$

Next, by the definition of ξ_i it follows that

$$\xi_{k_1}^2\xi_{k_2}^2 = B_{k_1k_2}^{(1)}+B_{k_1k_2}^{(2)}+B_{k_1k_2}^{(3)}+B_{k_1k_2}^{(4)},$$

where

$$B^{(1)}_{k_1k_2}=\sigma_2(k_1)\sigma_2(k_2),\quad B^{(2)}_{k_1k_2}=\sigma_2(k_1)\sigma_1(k_2),$$
$$B^{(3)}_{k_1k_2}=\sigma_1(k_1)\sigma_2(k_2),\quad B^{(4)}_{k_1k_2}=\sigma_1(k_1)\sigma_1(k_2),$$
$$\sigma_1(k)=\sum_{i,j=1}^{k-1}\eta_{ik}\eta_{jk},\quad \sigma_2(k)=\sum_{i=1}^{k-1}\eta^2_{ik}.$$

Therefore

$$2\sum_{k_1<k_2}E\xi^2_{k_1}\xi^2_{k_2}=\sum_{i=1}^{4}A^{(i)}_n,$$

where

$$A^{(i)}_n=2\sum_{1\le k_1<k_2\le n}EB^{(i)}_{k_1k_2},\quad i=\overline{1,4}.$$

In the first place we will consider the sum $A^{(3)}_n$. By the definitions of η_{ij} and $\alpha_\lambda(x,y)$ we have

$$E\eta^2_{ik_2}\eta_{sk_1}\eta_{tk_1}=0,\quad s\neq t,\quad k_1<k_2.$$

Therefore

$$A^{(3)}_n=0. \tag{2.13}$$

Let us estimate $A^{(2)}_n$. For this, we write $EB^{(2)}_{k_1k_2}$ in the form

$$EB^{(2)}_{k_1k_2}=\sum_{i=1}^{k_1-1}\sum_{r\neq s=1}^{k_1}E\eta^2_{ik_1}\eta_{rk_2}\eta_{sk_2}+\sum_{i=1}^{k_1-1}\sum_{r\neq s=k_1+1}^{k_2-1}E\eta^2_{ik_1}\eta_{rk_2}\eta_{sk_2}. \tag{2.14}$$

The second sum in (2.14) is equal to zero since i, r and s are pairwise different and $E\eta^2_{ik_1}\eta_{rk_2}\eta_{sk_2}=0$. In the first sum $E\eta^2_{ik_1}\eta_{rk_2}\eta_{sk_2}$ can be different from zero if and only if $s=k_1$ or $r=k_1$. Thus

$$EB^{(2)}_{k_1k_2}=2\sum_{i=1}^{k_1-1}E(\eta^2_{ik_1}\eta_{ik_2}\eta_{k_1k_2}). \tag{2.15}$$

Using (2.3), (2.4) and (2.5), from (2.15) we find

$$|EB^{(2)}_{k_1k_2}|\le\frac{32(k_1-1)}{n^4\sigma^4_n}\Big(E\Big(\int\alpha_\lambda(u,X_1)\alpha_\lambda(u,X_2)r(u)\,du\Big)^4\Big)^{\frac12}\cdot$$
$$\cdot E\Big(\int\alpha^2_\lambda(v,X_1)r(v)\,dv\Big)\sigma_n\le$$
$$\le\frac{c_{16}(k_1-1)}{n^4\sigma^3_n}\lambda^{\frac{(4-\beta)\alpha}{2}}\cdot\lambda^\alpha.$$

Therefore

$$|A_n^{(2)}| \le c_{16}(n\sigma_n)^{-4}\lambda^{\frac{(4-\beta)\alpha}{2}+\alpha} \sum_{k_1<k_2} (k_1-1)\cdot\sigma_n \le c_{17}\,\frac{\lambda^{(4-\beta)\alpha}}{n\sigma_n^4}\,. \tag{2.16}$$

Now let us consider $A_n^{(4)}$. We have

$$EB_{k_1k_2}^{(4)} = E\sum_{s\ne t=1}^{k_1-1}\eta_{sk_1}\eta_{tk_1}\sum_{k\ne r=1}^{k_2-1}\eta_{kk_2}\eta_{rk_2} = 4\sum_{s<t}^{k_1-1}\sum_{k<r}^{k_2-1}E\eta_{sk_1}\eta_{tk_1}\eta_{kk_2}\eta_{rk_2}.$$

In this sum, the terms of form $E\eta_{sk_1}\eta_{tk_1}\eta_{sk_2}\eta_{tk_2}$ can be different from zero, while all other terms are equal to zero. Therefore

$$EB_{k_1k_2}^{(4)} = 4\sum_{s<t}^{k_1-1}E\eta_{sk_1}\eta_{tk_1}\eta_{sk_2}\eta_{tk_2}. \tag{2.17}$$

Clearly,

$$\begin{aligned} E\eta_{sk_1}\eta_{tk_1}\eta_{sk_2}\eta_{tk_2} &= E\big(E(\eta_{sk_1}\eta_{tk_1}\eta_{sk_2}\eta_{tk_2}\big|X_t,X_{k_1},X_{k_2})\big) = \\ &= E\big(\eta_{tk_1}\eta_{tk_2}E(\eta_{sk_1}\eta_{sk_2}\big|X_t,X_{k_1},X_{k_2})\big). \end{aligned}$$

But it is easy to see that

$$\begin{aligned} E(\eta_{sk_1}\eta_{sk_2}|X_t,X_{k_1},X_{k_2}) &= \frac{n}{n-1}\,\frac{4}{n^2\sigma_n^2}\iint\alpha_\lambda(x,X_{k_1})\alpha_\lambda(y,X_{k_2})\,\cdot \\ &\cdot E(\alpha_\lambda(x,X_s)\alpha_\lambda(y,X_s)|X_t,X_{k_1},X_{k_2})r(x)r(y)\,dx\,dy = \\ = \frac{n}{n-1}\,\frac{4}{n^2\sigma_n^2}&\iint\alpha_\lambda(x,X_{k_1})\alpha_\lambda(y,X_{k_2})E\alpha_\lambda(x,X_1)\alpha_\lambda(y,X_1)r(x)r(y)\,dx\,dy. \end{aligned}$$

Therefore

$$\begin{aligned} E\eta_{sk_1}\eta_{tk_1}\eta_{sk_2}\eta_{tk_2} &= \frac{4n}{n-1}\,(n\sigma_n)^{-1}E\eta_{tk_1}\eta_{tk_2}\,\cdot \\ &\cdot\iint\alpha_\lambda(x,X_{k_1})\alpha_\lambda(y,X_{k_2})E\alpha_\lambda(x,X_1)\alpha_\lambda(y,X_1)r(x)r(y)\,dx\,dy = \\ &= \Big(\frac{n}{n-1}\Big)^2 16(n\sigma_n)^{-4}E\iint\alpha_\lambda(u,X_{k_1})\alpha_\lambda(v,X_{k_2})\,\cdot \\ &\cdot E\alpha_\lambda(u,X_t)\alpha_\lambda(v,X_t)r(u)r(v)\,du\,dv\,\cdot \\ &\cdot\iint\alpha_\lambda(x,X_{k_1})\alpha_\lambda(y,X_{k_2})E\alpha_\lambda(x,X_1)\alpha_\lambda(y,X_1)r(x)r(y)\,dx\,dy = \\ &= \Big(\frac{n}{n-1}\Big)^2 16(n\sigma_n)^{-4}E\Big[\iint\alpha_\lambda(u,X_1)\alpha_\lambda(v,X_2)\,\cdot \\ &\cdot\alpha_\lambda(u,X_1)\alpha_\lambda(v,X_1)r(u)r(v)\,du\,dv\Big]^2 = \\ &= \Big(\frac{n}{n-1}\Big)^2\frac{16}{n^4\sigma_n^4}\,L_n. \end{aligned} \tag{2.18}$$

Since $\sum\limits_{k_1<k_2}(k_1-1)(k_1-2)=O(n^4)$ and, by condition, $L_n=o(\sigma_n^4)$, from (2.17) and (2.18) we have

$$A_n^{(4)}\to 0 \quad \text{as} \quad n\to\infty. \tag{2.19}$$

Finally, let us show that $A_n^{(1)}\to 1$ as $n\to\infty$. We write $EB_{k_1k_2}^{(1)}$ in the form

$$\begin{aligned}
EB_{k_1k_2}^{(1)} &= \sum_{k=1}^{k_1-1} E\eta_{kk_1}^2\eta_{kk_2}^2+\sum_{k\neq s}E\eta_{kk_1}^2\eta_{sk_2}^2+\\
&\quad+\sum_{k=1}^{k_1-1}E\eta_{kk_1}^2\eta_{k_1k_2}^2+\sum_{k=1}^{k_1-1}\sum_{s=k_1+1}^{k_2-1}E\eta_{kk_1}^2\eta_{sk_1}^2=\\
&=2(k_1-1)E\eta_{12}^2\eta_{13}^2+(k_1-1)(k_1-2)(E\eta_{12}^2)^2+\\
&\quad+(k_1-1)(k_2-k_1-1)(E\eta_{12}^2)^2=\\
&=2(k_1-1)E\eta_{12}^2\eta_{13}^2+(k_1-1)(k_2-3)(E\eta_{12}^2)^2.
\end{aligned}$$

Thus

$$A_n^{(1)}=4E\eta_{12}^2\eta_{13}^2\sum_{k_1<k_2}(k_1-1)+2(E\eta_{12}^2)^2\sum_{k_1<k_2}(k_1-1)(k_2-3). \tag{2.20}$$

Taking into account the relations

$$\begin{aligned}
&\sum_{k_1<k_2}(k_1-1)(k_2-3)=\frac{n(n-1)(n-2)(n-3)}{8},\\
&\sum_{k_1<k_2}(k_1-1)=\frac{n(n-1)(n-2)}{6},\\
&E\eta_{12}^2\eta_{13}^2=\Big(\frac{n}{n-1}\Big)^2\frac{16}{n^4\sigma_n^4}\,E\Big(\int\alpha_\lambda(u,X_1)\alpha_\lambda(u,X_2)r(u)\,du\Big)^2\cdot\\
&\qquad\cdot\Big(\int\alpha_\lambda(v,X_1)\alpha_\lambda(v,X_3)r(v)\,dv\Big)^2\le c_{18}\,\frac{\lambda^{(4-\beta)\alpha}}{n^4\sigma_n^4}\,,\\
&(E\eta_{12}^2)^2=\Big(\frac{n}{n-1}\Big)^2\frac{16}{n^4\sigma_n^4}\Big(E\Big(\int\alpha_\lambda(u,X_1)\alpha_\lambda(u,X_2)r(u)\,du\Big)^2\Big)^2=\\
&=\Big(\frac{n}{n-1}\Big)^2\frac{4}{n^4\sigma_n^4}\Big(2\iint\big(E\alpha_\lambda(u,X_1)\alpha_\lambda(v,X_1)\big)^2r(u)r(v)\,du\,dv\Big)^2=\\
&\qquad=\Big(\frac{n}{n-1}\Big)^2\frac{4}{n^4\sigma_n^4}\,\sigma_n^4=\Big(\frac{n}{n-1}\Big)^2 4n^{-4},
\end{aligned}$$

from (20) we obtain

$$A_n^{(1)}=O\Big(\frac{\lambda^{(4-\beta)\alpha}}{n\sigma_n^4}\Big)+1+o(1).$$

Hence

$$A_n^{(1)} \to 1 \quad \text{as} \quad n \to \infty. \tag{2.21}$$

Combining relations (2.13), (2.16), (2.19) and (2.21), we eventually obtain

$$E\Big(\sum_{k=1}^{n} \xi_k^2 - 1\Big)^2 \to 0 \quad \text{as} \quad n \to \infty. \quad \square$$

3. D–1 – **Kernel estimator.** Let again $X_i = (X_i^{(1)}, \dots, X_i^{(p)})$, $i = \overline{1, n}$, be a sequence of independent, identically distributed random variables with density $f(x)$, $x = (x_1, \dots, x_p)$. As said in Subsection 2, with an appropriate choice of functions $\delta_{0\lambda}(x, y)$ and $\delta_{1\lambda}(x, y)$, from (2.1) we obtain kernel estimators of the distribution density $f(x)$ (Rosenblatt–Parzen estimators) of the form

$$g_n(x) = n^{-1} \lambda^p \sum_{i=1}^{n} K(\lambda(x - X_i)).$$

The function $K(x)$, $x \in R_p$, is said to belong to the class H_ν ($\nu \geq 2$), if it satisfies the following conditions:

$$\int K(x)\, dx = 1, \quad \sup_{x \in R_p} |K(x)| < \infty,$$

$$\int x_1^{i_1} x_2^{i_2} \cdots x_p^{i_p} K(x)\, dx = 0 \quad \text{if} \quad \sum_{j=1}^{p} i_j < \nu,$$

$$\int |x_1^{i_1} x_2^{i_2} \cdots x_p^{i_p}|\, |K(x)|\, dx < \infty \quad \text{if} \quad \sum_{j=1}^{p} i_j = \nu.$$

Let $r(x) = r_1(x_1) \cdots r_p(x_p)$, $r_j(x_j) \geq 0$, $x_j \in R_1$, be piecewise-continuous and bounded functions.

Let $K \in H_2$ and $f(x)$ be bounded. In that case, conditions 1° and 2° of Theorem 2.1 are fulfilled for $\alpha = p$ and $\beta = 1$. Indeed, since $\delta_{0\lambda}(x, X_i) = \lambda^p K((x - X_i)\lambda)$ and $\delta_{1\lambda}(x, X_i) \equiv 1$, we have

$$1^\circ.\ E\Big(\int \delta_{0\lambda}^2(x, X_1) r(x)\, dx\Big)^\gamma = \lambda^{\gamma p} E\Big(\int \lambda^p K^2(\lambda(x - X_i)) r(x)\, dx\Big)^\gamma \leq$$
$$\leq c_1 \lambda^{\gamma p} \int K^2(u)\, du. \tag{3.1}$$

$$2^\circ.\ E\big(\delta_{00}^s(X_1, X_2, \lambda)\big) = E\Big(\int \lambda^{2p} K((x - X_1)\lambda) K(\lambda(x - X_2)) r(x)\, dx\Big)^s \leq$$
$$\leq c_2 \lambda^{(s-1)p} \int K_1^s(u)\, du, \quad s = 2, 4, \tag{3.2}$$

where

$$K_1(u) = \int |K(t)|\, |K(u - t)|\, dt.$$

Further, it is easy to check that for $\delta_{1\lambda}(x, X_i)$, $\delta_{10}(X_1, X_2, \lambda)$, $\delta_{01}(X_1, X_2, \lambda)$ and $\delta_{11}(X_1, X_2, \lambda)$, conditions 1° and 2° are fulfilled trivially.

Let us now show that condition 3° is valid. Since $\alpha_\lambda(u, v) = \lambda^p[K(\lambda(u - v)) - EK(\lambda(u - X_1))]$, we have

$$\sigma_n^2 = 2 \iint \big(E\alpha_\lambda(u_1, X_1)\alpha_\lambda(u_2, X_1)\big)^2 r(u_1) r(u_2)\, du_1\, du_2 =$$
$$= 2\lambda^{4p} \iint \Big[\lambda^{-p} \int K(t) K(\lambda(u_2 - u_1) - t) f\Big(u_1 - \frac{t}{\lambda}\Big) dt -$$
$$-\lambda^{-2p} \int K(t) f\Big(u_1 - \frac{t}{\lambda}\Big)\, dt \int K(t) f\Big(u_2 - \frac{t}{\lambda}\Big) dt\Big]^2 r(u_1) r(u_2)\, du_1\, du_2.$$

Hence, after performing the change of variables, we obtain

$$\sigma_n^2 = 2\lambda^{3p} \iint \Big[\lambda^{-p} \int K(t) K(z - t) f\Big(u_1 - \frac{t}{\lambda}\Big) dt -$$
$$-\lambda^{-2p} \int K(t) f\Big(u_1 - \frac{t}{\lambda}\Big)\, dt \int K(t) f\Big(u_1 + \frac{z}{\lambda} - \frac{t}{\lambda}\Big) dt\Big]^2 \cdot$$
$$\cdot r(u_1) r\Big(u_1 + \frac{z}{\lambda}\Big)\, du_1\, dz \sim \lambda^p A_n(K, f, r) \equiv$$
$$\equiv 2\lambda^p \iint \Big[\int K(t) K(z - t) f\Big(u_1 - \frac{t}{\lambda}\Big)\, dt\Big]^2 r(u_1) r\Big(u_1 + \frac{z}{\lambda}\Big)\, du_1\, dz,$$

and at that

$$A_n(K, f, r) = 2 \iint f^2(u_1) r(u_1) K_0^2(z) r\Big(u_1 + \frac{z}{\lambda}\Big)\, du_1\, dz +$$
$$+ I_{1n} + I_{2n}, \tag{3.3}$$

where

$$I_{1n} = 2 \iint \Big[\int K(t) K(z - t)\Big(f\Big(u_1 - \frac{t}{\lambda}\Big) - f(u_1)\Big) dt\Big]^2 \cdot$$
$$\cdot r(u_1) r\Big(u_1 + \frac{z}{\lambda}\Big) du_1\, dz,$$
$$I_{2n} = 4 \iint \Big[\int K(t) K(z - t)\Big(f\Big(u_1 - \frac{t}{\lambda}\Big) - f(u_1)\Big) dt \cdot$$
$$\cdot \int K(t) K(z - t) f(u_1) dt\Big] r(u_1) r\Big(u_1 + \frac{z}{\lambda}\Big) du_1\, dz,$$
$$K_0(x) = \int K(u - x) K(u)\, du.$$

By virtue of the generalized Minkovski integral inequality we have

$$I_{1n} \le c_3 \int |K(t)| \omega_2\Big(-\frac{t}{\lambda}\Big)\, dt, \quad I_{2n} \le c_4 \int |K(t)| \omega_1\Big(-\frac{t}{\lambda}\Big)\, dt, \tag{3.4}$$

where

$$\omega_q(h) = \Big(\int |f(x + h) - f(x)|^q dx\Big)^{1/q}, \quad q = 1, 2.$$

The expression $\omega_p(h)$ is called the continuity L_q-modulus of the function f. It is supposedly bounded like the function h because $\omega_q(h) \le 2\|f\|_q$. Moreover, $\omega_q(h) \to 0$ as $|h| \to \infty$. Therefore, by the Lebesgue theorem on majorized convergence, the integrals in the right-hand part of (3.4) tend to zero as $n \to \infty$.

Hence (3.3) implies that

$$\lambda^{-p}\sigma_n^2 \to \sigma^2 = 2\int f^2(u)r^2(u)du \int K_0^2(u)du \quad \text{as} \quad n \to \infty. \tag{3.5}$$

The first part of condition 3° is fulfilled. It remains to check the second part of condition 3°, i.e., to show that

$$L_n = o(\sigma_n^4).$$

We have

$$\begin{aligned}
L_n = E\Big[\iint & \alpha_\lambda(u_1, X_1)\alpha_\lambda(u_2, X_2)E\alpha_\lambda(u_1, X_1)\alpha_\lambda(u_2, X_2)\cdot \\
& \cdot r(u_1)r(u_2)\,du_1\,du_2\Big]^2 = \\
= \lambda^{8p} & \iiiint \operatorname{cov}\big(K(\lambda(u_1 - X_1)), K(\lambda(v_1 - X_1))\big)\cdot \\
& \cdot \operatorname{cov}\big(K(\lambda(u_2 - X_1)), K(\lambda(v_2 - X_1))\big)\cdot \\
& \cdot \operatorname{cov}\big(K(\lambda(u_1 - X_1)), K(\lambda(u_2 - X_1))\big)\cdot \\
& \cdot \operatorname{cov}\big(K(\lambda(u_1 - X_1)), K(\lambda(v_2 - X_1))\big)\cdot \\
& \cdot r(u_1)r(u_2)r(v_1)r(v_2)du_1\,du_2\,dv_1\,dv_2 = \\
= \lambda^{p} & \iiiint \Big[\int K(t_1)K(z_1 - t_1)f\Big(u_1 - \frac{t_1}{\lambda}\Big)dt_1 - \\
& \quad - EK(\lambda(u_1 - X_1))EK(\lambda(u_1 - X_1) + z_1)\Big]\cdot \\
& \cdot \Big[\int K(t_2)K(z_2 - t_2)f\Big(u_1 - \frac{z_3}{\lambda} - \frac{t_2}{\lambda}\Big)dt_2 - \\
& \quad - EK(\lambda(u_1 - X_1 + z_2 - z_3)EK(\lambda(u_1 - X_1) - z_3)\Big]\cdot \\
& \cdot \Big[\int K(t_3)K(z_3 - z_1 - t_3)f\Big(u_1 + \frac{z_1}{\lambda} - \frac{t_3}{\lambda}\Big)dt_3 - \\
& \quad - EK(\lambda(u_1 - X_1) + z_1)EK(\lambda(u_1 - X_1) - z_3)\Big]\cdot \\
& \cdot \Big[\int K(t_4)K(z_2 - z_3 - t_4)f\Big(u_1 - \frac{t_4}{\lambda}\Big)dt_4 - \\
& \quad - EK(\lambda(u_1 - X_1))EK(\lambda(u_1 - X_1) + z_2 - z_3)\Big]\cdot \\
& \cdot r(u_1)r\Big(u_1 + \frac{z_1}{\lambda}\Big)r\Big(u_1 + \frac{1}{\lambda}(z_2 - z_3)\Big)\cdot
\end{aligned}$$

$$\cdot\, r\Big(u_1-\frac{z_3}{\lambda}\Big)du_1\,dz_1\,dz_2\,dz_3. \tag{3.6}$$

The integral in (3.6) can be represented as a sum of 16 integrals which coincide with one of possible integrals of the form

$$\begin{aligned}\widetilde{L}_1=\lambda^p\iiiint\Big[\iiiint K(t_1)K(t_2)K(t_3)K(t_4)K(z_1-t_1)K(z_2-t_2)\,\cdot\\ \cdot\, K(z_3-t_3)K(z_2-z_3-t_4)f\Big(u_1-\frac{t_1}{\lambda}\Big)f\Big(u_1-\frac{z_3}{\lambda}-\frac{t_2}{\lambda}\Big)\,\cdot\\ \cdot\, f\Big(u_1+\frac{z_1}{\lambda}-\frac{t_3}{\lambda}\Big)f\Big(u_1-\frac{t_4}{\lambda}\Big)dt_1\,dt_2\,dt_3\,dt_4\Big]r(u_1)r\Big(u_1+\frac{z_1}{\lambda}\Big)\,\cdot\\ \cdot\, r\Big(u_1+\frac{1}{\lambda}(z_2-z_3)\Big)r\Big(u_1-\frac{z_3}{\lambda}\Big)du_1\,dz_1\,dz_2\,dz_3,\end{aligned}$$

$$\begin{aligned}\widetilde{L}_2=\lambda^p\iiiint\Big[\iiint K(t_1)K(t_2)K(t_3)K(z_1-t_1)K(z_2-t_2)K(z_3-t_3)\,\cdot\\ \cdot\, f\Big(u_1-\frac{t_1}{\lambda}\Big)f\Big(u_1-\frac{z_3}{\lambda}-\frac{t_2}{\lambda}\Big)f\Big(u_1+\frac{z_1}{\lambda}-\frac{t_3}{\lambda}\Big)dt_1\,dt_2\,dt_3\Big]\,\cdot\\ \cdot\, E(\lambda(u_1-X_1))EK(\lambda(u_1-X_1)+z_2-z_3)r(u_1)r\Big(u_1+\frac{z_1}{\lambda}\Big)\,\cdot\\ \cdot\, r\Big(u_1+\frac{1}{\lambda}(z_2-z_3)\Big)r\Big(u_1-\frac{z_3}{\lambda}\Big)du_1\,dz_1\,dz_2\,dz_3,\end{aligned}$$

$$\begin{aligned}\widetilde{L}_3=\lambda^p\iiiint\Big[\iint K(t_1)K(t_2)K(z_1-t_1)K(z_2-t_2)\,\cdot\\ \cdot\, f\Big(u_1-\frac{t_1}{\lambda}\Big)f\Big(u_1-\frac{z_3}{\lambda}-\frac{t_2}{\lambda}\Big)dt_1\,dt_2\Big]\,\cdot\\ \cdot\, EK(\lambda(u_1-X_1)+z_1)EK(\lambda(u_1-X_1)-z_3)EK(\lambda(u_1-X_1))\,\cdot\\ \cdot\, EK(\lambda(u_1-X_1)+z_2-z_3)r(u_1)r\Big(u_1+\frac{z_1}{\lambda}\Big)\,\cdot\\ \cdot\, r\Big(u_1+\frac{1}{\lambda}(z_2-z_3)\Big)r\Big(u_1-\frac{z_3}{\lambda}\Big)du_1\,dz_1\,dz_2\,dz_3,\end{aligned}$$

$$\begin{aligned}\widetilde{L}_4=\lambda^p\iiiint\Big[\int K(t_1)K(z_1-t_1)f\Big(u_1-\frac{t_1}{\lambda}\Big)dt_1\Big]\,\cdot\\ \cdot\, EK(\lambda(u_1-X_1)+z_2-z_3)EK(\lambda(u_1-X_1)-z_3)\,\cdot\\ \cdot\, EK(\lambda(u_1-X_1)+z_1)EK(\lambda(u_1-X_1)+z_2-z_3)\,\cdot\\ \cdot\, r(u_1)r\Big(u_1+\frac{z_1}{\lambda}\Big)r\Big(u_1+\frac{1}{\lambda}(z_2-z_3)\Big)\,\cdot\\ \cdot\, r\Big(u_1-\frac{z_3}{\lambda}\Big)du_1\,dz_1\,dz_2\,dz_3,\end{aligned}$$

$$\widetilde{L}_5=\lambda^p\iiiint\big(EK(\lambda(u_1-X_1))\big)^2\big(EK(\lambda(u_1-X_1)+z_1)\big)^2\,\cdot$$

$$\cdot \big(EK(\lambda(u_1 - X_1) + z_2 - z_3)\big)^2 \big(EK(\lambda(u_1 - X_1) - z_3)\big)^2 r(u_1) \cdot$$
$$\cdot\, r\Big(u_1 + \frac{z_1}{\lambda}\Big) r\Big(u_1 + \frac{1}{\lambda}(z_2 - z_3)\Big) \cdot$$
$$\cdot\, r\Big(u_1 - \frac{z_3}{\lambda}\Big) du_1\, dz_1\, dz_2\, dz_3, \tag{3.7}$$

Simple calculations show that

$$\widetilde{L}_1 = O(\lambda^p), \quad \widetilde{L}_i = O(1), \quad i = \overline{2,5}. \tag{3.8}$$

Now from (3.6), (3.7) and (3.8) we have

$$L_n \le c_5 \lambda^p. \tag{3.8$'$}$$

This and (3.5) imply

$$\frac{L_n}{\sigma_n^4} \le c_5 \frac{1}{(\lambda^{-p}\sigma_n^2)^2 \lambda^p} \le c_6 \lambda^{-p} \to 0. \tag{3.9}$$

Condition 3° is completely fulfilled. Further, by virtue of (3.5), condition 4° takes the form

$$\frac{\lambda^p}{n} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.10}$$

Thus, combining (3.1), (3.2), (3.5), (3.9) and (3.10), we come to

Theorem 3.1. *Let $K \in H_2$ and $f(x)$ be bounded. If $n^{-1}\lambda^p \to 0$ as $n \to \infty$, then*

$$\lambda^{p/2}\sigma^{-1}(\overline{U}_n - \overline{\Delta}_n) \xrightarrow{d} N(0,1),$$

where

$$\overline{U}_n = \frac{n}{\lambda^p} \int \big(g_n(x) - Eg_n(x)\big)^2 r(x)\, dx,$$
$$\overline{\Delta}_n = E\overline{U}_n.$$

Our next problem is to replace the "averaged" density $Eg_n(x)$ in the expression of $\overline{U}_n$ by the density $f(x)$. For this we assume that the function $f(x)$, $x \in R_p$, belongs to the set W_ν of all bounded functions having arbitrary derivatives up to ν-th, $\nu \ge 2$, order inclusive.

Denote

$$T_n = \frac{n}{\lambda^p} \int \big(g_n(x) - f(x)\big)^2 r(x)\, dx.$$

The following lemma is valid.

Lemma 3.1. *Let $K \in H_\nu$ and $f \in W_\nu$. If $n\lambda^{-p-2\nu} \to 0$ as $n \to \infty$, then*

$$\lambda^{p/2}(T_n - \overline{U}_n - \theta_n^{(1)}) = o_p(1),$$

where

$$\theta_n^{(1)} = \frac{n}{\lambda^p} \int \big(Eg_n(x) - f(x)\big)^2 r(x)\, dx.$$

Proof. We have

$$T_n - \overline{U}_n - \theta_n^{(1)} \equiv R_n = 2\,\frac{n}{\lambda^p}\int \big(Eg_n(x) - f(x)\big)\big(g_n(x) - Eg_n(x)\big) r(x)\,dx.$$

Further, since $f \in W_\nu$ and $K \in H_\nu$, we can write

$$Eg_n(x) - f(x) = \lambda^{-\nu} \sum_{|\ell|=s} \int \frac{t^\ell}{\ell!} K(t) f^{(\ell)}\Big(x - \theta\,\frac{t}{\lambda}\Big) dt = O(\lambda^{-\nu}), \quad (3.12)$$

where

$$|\ell| = \sum_{j=1}^{p} \ell_j, \quad \ell! = \ell_1! \cdots \ell_p!, \quad t^\ell = t_1^{\ell_1} \cdots t_p^{\ell_p}, \quad f^{(\ell)}(x) = \frac{\partial^{|\ell|} f}{\partial x_1^{\ell_1} \cdots \partial x_p^{\ell_p}}\,.$$

The estimator $O(\cdot)$ in (3.12) is uniform with respect to $x \in R_p$.

Let us estimate $E|R_n|$. Since

$$\operatorname{cov}\big(g_n(x), g_n(y)\big) = \frac{\lambda^{2p}}{n}\Big\{ \int K(\lambda(x-u))K(\lambda(y-u)) f(u)\,du - \\ - \int K(\lambda(x-u)) f(u)\,du \int K(\lambda(y-u)) f(u)\,du \Big\},$$

we obtain

$$E|R_n| \le n\lambda^{-p} E\Big\{ \frac{\lambda^{2p}}{n}\Big[\int K(\lambda(x_1 - X_1))\big(Eg_n(x_1) - f(x_1)\big) r(x_1) dx_1 - \\ - E \int K(\lambda(x_1 - X_1))\big(Eg_n(x) - f(x_1)\big) r(x_1)\,dx_1 \Big]^2 \Big\}^{1/2} \le \\ \le 2\sqrt{n} \Big\{ \int f(x)\,dx \Big(\int K(\lambda(x-u))\big(Eg_n(x) - f(x)\big) r(x)\,dx \Big)^2 \Big\}^{1/2} \le \\ \le c_7 \sqrt{n} \lambda^{-p-\nu}. \quad \square$$

Lemma 3.2. *$\overline{\Delta}_n = E\overline{U}_n = \theta_n^{(2)} + O(\lambda^{-p})$, where*

$$\theta_n^{(2)} = \lambda^p \iint K^2(\lambda(x-u)) f(u) r(x)\,du\,dx.$$

Theorem 3.2. *Let $K(x) \in H_\nu$ and $f(x) \in W_\nu$. Then*

a) *If $n^{-1}\lambda^p \to 0$ and $n\lambda^{-p-2\nu} \to 0$ as $n \to \infty$, then*

$$\lambda^{p/2} \sigma^{-1} (T_n - \theta_n^{(1)} - \theta_n^{(2)}) \xrightarrow{d} N(0,1);$$

b) *If $n^{-1}\lambda^p \to 0$ and $n\lambda^{-p/2-2\nu} \to 0$ as $n \to \infty$, then*

$$\lambda^{p/2} \sigma^{-1} (T_n - \theta_n^{(2)}) \xrightarrow{d} N(0,1);$$

c) *Let the condition* $K(u) = K(-u)$ *be added to the conditions we have for* $K(x)$, $x \in R_p$. *If* $n^{-1}\lambda^p \to 0$, $n\lambda^{-p/2-2\nu} \to 0$ *and* $\lambda^{p/2-2} \to 0$ $(1 \le p \le 3)$ *as* $n \to \infty$, *then*

$$\lambda^{p/2}\sigma^{-1}\Big(T_n - \int f(x)r(x)\,dx \int K^2(u)\,du\Big) \xrightarrow{d} N(0,1).$$

Proof. The case a) obviously follows from Theorem 3.1 and Lemmas 3.1 and 3.2. The statement b) immediately follows from (3.11), since

$$\lambda^{p/2}\theta_n^{(1)} = \frac{n}{\lambda^{p/2}} \int \big(Eg_n(x) - f(x)\big)^2 r(x)\,dx = O(n \cdot \lambda^{-p/2-2\nu}).$$

Finally, the statement c) follows from the above statement and the following representation of $\theta_n^{(2)}$:

$$\theta_n^{(2)} = \int f(x)r(x)\,dx \int K^2(u)\,du + O(\lambda^{-2}). \qquad \square$$

Assume that $\lambda = n^\alpha$, $\alpha > 0$. The conditions of the case a) concerning n and λ are fulfilled if $\frac{1}{p+2\nu} < \alpha < \frac{1}{p}$; the conditions of the case b) are valid if $\frac{2}{p+4\nu} < \alpha < \frac{1}{p}$ and $p < 4\nu$; the conditions of the statement c) are compatible if $\frac{2}{p+4\nu} < \alpha < \frac{1}{p}$ and $1 \le p \le 3$.

Note that in [12] there is a statement analogous to the case c), where the kernel $K(x)$, $x \in R_p$, belongs to some narrow class of functions; the normalizing variable $\int f(x)r(x)dx \cdot \int K^2(x)dx$ in this statement is given without justification since the dimension of the Euclidean space is not taken into account.

As mentioned above, P. Hall in [13] and E. Nadaraya in [14] simultaneously introduced the martingale method of investigation of the limit distribution of square deviation of Rosenblatt–Parzen estimators. In [14] a one-dimensional analogue of Theorem 3.2 is considered (and it is also noted that this theorem can be generalized to the multi-dimensional case), while in [13] the statement a) of Theorem 3.2 is given for the case $\nu = 2$.

Further we will consider the application of Theorem 3.2 to the two-dimensional case to verify the statistical hypotheses on fitness and independence.

Let $X_i = (X_i^{(1)}, X_i^{(2)})$, $i = 1, 2, \dots$, be a sequence of independent, identically distributed variables with density $f(x)$, $x = (x_1, x_2)$.

The statement c) of Theorem 3.2 enables us to construct a test of asymptotic level α for checking hypothesis H_0 by which $f(x) = f_0(x)$. For this, we have to calculate T_n and neglect H_0 provided that

$$T_n \ge d_n(\alpha) = \int f_0(x)r(x)\,dx \int K^2(u)\,du + \lambda^{-1}\varepsilon_\alpha\sigma,$$

where ε_α is the quantile of level α of a standard normal distribution.

Let us assume that the tested Hypothesis H_0 is not true and, actually, the following hypothesis is valid:

$$H^{(n)} \; : \; f^{(n)}(x) = f_0(x) + \gamma_n \varphi(x), \quad x \in R_2,$$
$$\int \varphi(x)\,dx = 0, \quad \gamma \downarrow 0.$$

The next theorem is a generalization of the theorem from [12] for a wide class of kernels K.

Theorem 3.3. *Let $K(x)$ and $f^{(n(}(x)$ satisfy the conditions of the case c) of Theorem 3.2. Let $T_n = n\lambda^{-2}\int(g_n(x) - f_0(x))^2 r(x)dx$. If $\lambda = n^\delta$ and $\gamma_n = n^{-1/2}\lambda^{1/2}$, $\frac{1}{1+2\nu} < \delta < \frac{1}{2}$, then*

$$\lambda\Big(T_n - \int f_0(x) r(x)\,dx \int K^2(x)\,dx\Big) \xrightarrow{d} N(a, \sigma^2),$$
$$a = \int \varphi^2(u) r(u)\,du, \quad \sigma^2 = 2\int f_0^2(u) r^2(u)\,du \int K_0^2(u)\,du.$$

Proof. Represent T_n as the sum

$$\begin{aligned} T_n &= n\lambda^{-2}\int \big(g_n(x) - f_0(x)\big)^2 r(x)\,dx = \\ &= n\lambda^{-2}\int \big(g_n(x) - f^{(n)}(x)\big)^2 r(x)\,dx + \\ &\quad + n\lambda^{-2}\int \big(f_0(x) - f^{(n)}(x)\big)^2 r(x)\,dx + \\ &\quad + 2n\lambda^{-2}\int \big(f^{(n)}(x) - f_0(x)\big)\big(g_n(x) - f^{(n)}(x)\big) r(x)\,dx = \\ &= A_{1n} + A_{2n} + A_{3n}. \end{aligned} \tag{3.13}$$

By the proof of Theorem 3.2 it is not difficult to see that

$$\lambda\Big(A_{1n} - \int f_0(x) r(x)\,dx \int K^2(u)\,du\Big) \xrightarrow{d} N(0, \sigma^2). \tag{3.14}$$

By analogy with the proof of Lemma 3.1 we find

$$\begin{aligned} \lambda E|A_{3n}| &\le 2n\lambda^{-1} E\Big|\int \big(f^{(n)}(x) - f_0(x)\big)\big(g_n(x) - E_n g_n(x)\big) r(x)\,dx\Big| + \\ &\quad + 2\lambda^{-1} n\Big|\int \big(f^{(n)}(x) - f_0(x)\big)\big(E_n g_n(x) - f^{(n)}(x)\big) r(x)\,dx\Big| \le \\ &\le c_8\big(\sqrt{n}\gamma_n\lambda^{-1} + \gamma_n\lambda^{-\nu-1}\big) = \\ &= c_9\big(n^{-\delta/2} + n^{\frac{1}{2} - \frac{\delta(1+2\nu)}{2}}\big) \to 0, \end{aligned} \tag{3.15}$$

where $E_n g_n$ denotes mathematical expectation with respect to $f^{(n)}(x)$.

Further,

$$n\lambda^{-1}A_{2n} = n\lambda^{-1}\gamma_n^2 \int \varphi^2(x)r(x)\,dx = \int \varphi^2(x)r(x)\,dx. \tag{3.16}$$

Hence (3.13)–(3.16) imply the statement of our theorem. □

Corollary. *The local behavior of the power $P_{H^{(n)}}(T_n \geq d_n(\alpha))$ is as follows:*

$$P_{H^{(n)}}(T_n \geq d_n(\alpha)) \to 1 - \Phi\Big(\varepsilon_\alpha - \frac{1}{\sigma}\int \varphi^2(x)r(x)\,dx\Big).$$

Note that for positive weight functions $r(x)$, $x \in R_2$, the test of checking Hypothesis H_0 against alternatives of the form $H^{(n)}$ is asymptotically strictry unbiased since the mathematical expectation $\int \varphi^2(x)r(x)dx > 0$ and is equal to zero if and only if $\varphi(x) = 0$, $x \in R_2$.

Now we introduce into consideration the alternatives called "singular" [20], [21] (see also [12]):

$$H^{(n)} : f^{(n)}(x_1, x_2) = f_0(x_1, x_2) + \alpha_n\varphi\Big(\frac{x_1 - \ell_1}{\gamma_n}, \frac{x_2 - \ell_2}{\gamma_n}\Big),$$

where $\alpha_n \downarrow 0$, $\gamma_n \downarrow 0$, the function $\varphi(u_1, u_2)$ is bounded and has bounded partial derivatives up to second order, and

$$\int \varphi(u_1, u_2)\,du_1\,du_2 = 0,$$

$\ell = (\ell_1, \ell_2)$ is some fixed point of continuity $r(x)$ such that $r(\ell) \neq 0$.

Theorem 3.4. *Let $K(x)$ and $f_0(x)$, $x \in R_2$, satisfy the conditions of Theorem 3.3 for $s = 2$. Let $\lambda = n^\gamma$, $\frac{1}{5} < \gamma < \frac{1}{2\alpha}$, $\alpha_n = \lambda^\alpha n^{-1/2}$, $\gamma_n = \lambda^{-\beta}$, $2\beta > \alpha$, $\alpha - \beta = \frac{1}{2}$, $1 < \alpha < \frac{7}{6}$, $\frac{1}{2} < \beta < \frac{2}{3}$.*

Then

$$\lambda\Big(T_n - \int f_0(x)r(x)\,dx \int K^2(u)\,du\Big) \xrightarrow{d} N(a, \sigma^2),$$

$$a = r(\ell)\int \varphi^2(u)\,du, \quad \sigma^2 = 2\int f_0^2(x)r^2(x)\,dx \int K_0^2(u)\,du.$$

The proof of Theorem 3.4 repeats that of Theorem 4 from [20].

The conditions on α and β are fulfilled if, for instance, we set: $\alpha = \frac{11}{10}$, $\beta = \frac{3}{5}$; $\alpha = \frac{8}{7}$, $\beta = \frac{9}{14}$; $\alpha = \frac{10}{9}$, $\beta = \frac{11}{18}$; $\alpha = \frac{9}{8}$, $\beta = \frac{5}{8}$ and so on.

Remark. By integrating $f^{(n)}(x, y)$ we establish that the alternatives differ from the zero hypothesis by a value of order $\alpha_n\gamma_n^2 = o(n^{-1/2})$. This means that the tests based on a deviation between the two-dimensional empirical function and the hypothetic function of the distribution cannot distinguish "singular" hypotheses from the basic one. Therefore tests based on T_n for "singular" alternatives are more powerful than tests of the above-mentioned kind.

Assume now that the components of the vector $X_i = (X_i^{(1)}, X_i^{(2)})$ are independent random variables with marginal density $f_1(x_1)$ and $f_2(x_2)$, respectively (Hypothesis H_0). Let, further, $g_{n1}(x)$ and $g_{n2}(x)$ denote the kernel estimators of the densities $f_1(x_1)$ and $f_2(x_2)$ based on $K_1(x_1)$ and $K_2(x_2)$, $x_1, x_2 \in R_1$, and $K(x_1, x_2) = K_1(x_1)K_2(x_2) \in H_2$. To test Hypothesis H_0 in [12] and [21] for a narrow class of kernels, we studied the limit distribution of the functional

$$S_n = n\lambda^{-2} \int \big(g_n(x_1, x_2) - g_{n1}(x_1)g_{n2}(x_2)\big)^2 r(x)\, dx.$$

Using the case c) of Theorem 3.2 and the technique described in proving Theorem 2 in [12], it can be easily shown that

$$\lambda\sigma^{-1}\big(S_n - A(f, n)\big) \xrightarrow{d} N(0, 1)$$

if we use the conditions of the case c) of Theorem 3.2, where

$$\begin{aligned}
A(f, u) &= \int f(x)r(x)\, dx \int K^2(u)\, du - \\
&\quad - \lambda^{-1} \int K_1^2(u)\, du \int f_1(x_1)f_2^2(x_2)r(x)\, dx - \\
&\quad - \lambda^{-1} \int K_2^2(u)\, du \int f_2(x_2)f_1^2(x_1)r(x)\, dx, \\
\sigma^2(f) &= 2 \int f^2(u)r^2(u)\, du \int (K_1 * K_1(x))^2 dx \int (K_2 * K_2(x))^2 dx.
\end{aligned}$$

If we replace the normalizing values $A(f_1, f_2, n)$ and $\sigma^2(f_1, f_2)$ by the estimators $A_n = A(g_{n1}, g_{n2})$ and $\sigma_n^2 = \sigma^2(g_{n1}, g_{n2})$, then, as proved in [21], we obtain

$$\lambda\sigma_n^{-1}(S_n - A_n) \xrightarrow{d} N(0, 1).$$

This statement enables us to construct a test of asymptotic level α for checking Hypothesis $H_0 : f(x_1, x_2) = f_1(x_1)f_2(x_2)$; critical domains for testing H_0 are established by the inequality

$$S_n \geq \widetilde{d}_n(\varepsilon),$$

where

$$\widetilde{d}_n(\varepsilon) = A(g_{n1}, g_{n2}, n) + \lambda^{-1}\varepsilon_\alpha \cdot \sigma_n.$$

4. *D*–2 – **Projective estimator.** Let $L_2(R_p, r)$ be the space of functions defined on R_p and square-integrable with respect to to the measure μ, $d\mu = r(x)dx$. In the space $L_2(R_p, r)$, choose some orthonormalized basis $\{\varphi_j(x)\}$. Then, with an appropriate choice of functions $\delta_{0\lambda}(x, y)$ and $\delta_{1\lambda}(x, y)$ (see

§ 2), from (2.1) we obtain projective estimators of the distribution density $f(x)$ (Chentsov estimators) of the form

$$g_n(x) = \sum_{j=1}^{\lambda} \widehat{a}_j \varphi_j(x), \quad \widehat{a}_j = \frac{1}{n} \sum_{k=1}^{n} \varphi_j(X_k) r(X_k).$$

The values $\Delta_n = EU_n$ and σ_n^2 can be represented through the basis $\{\varphi_j(x)\}$ in explicit form.

We have

$$\begin{aligned}\Delta_n &= En \int \big(g_n(x) - Eg_n(x)\big)^2 r(x)\,dx = \\ &= \int E\big[\delta_{0\lambda}(u, X_1) - E\delta_{0\lambda}(u, X_1)\big]^2 r(u)\,du = \\ &= \sum_{j=1}^{\lambda} \Big[\int \varphi_j^2(u) r^2(u) f(u)\,du - a_j^2\Big],\end{aligned} \tag{4.1}$$

where

$$a_j = \int \varphi_j(x) f(x) r(x)\,dx.$$

$$\begin{aligned}\sigma_n^2 &= 2 \iint \big(E\alpha_\lambda(u_1, X_1)\alpha_\lambda(u_2, X_1)\big)^2 r(u_1) r(u_2)\,du_1\,du_2 = \\ &= 2 \sum_{i=1}^{\lambda} \sum_{j=1}^{\lambda} \big[\operatorname{cov}\big(\varphi_i(X_1) r(X_1), \varphi_j(X_1) r(X_1)\big)\big]^2 = \\ &= 2 \sum_{i=1}^{\lambda} \sum_{j=1}^{\lambda} \Big(\int \varphi_j(u)\varphi_i(u) r^2(u) f(u)\,du - a_i a_j\Big)^2.\end{aligned} \tag{4.2}$$

Let now $\sup\limits_{x \in R_p} f(x) r(x) dx < \infty$. We will show that

$$L_n = O(\sigma_n^2). \tag{4.3}$$

Indeed,

$$\begin{aligned}L_n &= E\Big[\iint \alpha_\lambda(u_1, X_1)\alpha_\lambda(u_2, X_2) E\alpha_\lambda(u_1, X_1)\alpha_\lambda(u_2, X_2) \cdot \\ &\qquad \cdot r(u_1) r(u_2)\,du_1\,du_2\Big]^2 = \\ &= E\Big[\sum_{i,j}^{\lambda} \varphi_i^*(X_1)\varphi_j^*(X_2) a_{ij}\Big]^2,\end{aligned} \tag{4.4}$$

where

$$\varphi_j^*(X_1) = \varphi_i(X_1) r(X_1) - E\varphi_i(X_1) r(X_1),$$

$$a_{ij} = \operatorname{cov}\big(\varphi_i(X_1)r(X_1), \varphi_j(X_1)r(X_1)\big).$$

From (4.4) we have

$$\begin{aligned}
L_n \le 4\bigg[E\bigg(\sum_{i,j}^{\lambda} a_{ij}\varphi_i(X_1)\varphi_j(X_2)r(X_1)r(X_2)\bigg)^2 + \\
+ 2E\bigg(\sum_{i,j}^{\lambda} a_{ij}a_i\varphi_j(X_1)r(X_1)\bigg)^2 + \bigg(\sum_{i,j}^{\lambda} a_{ij}a_ia_j\bigg)^2\bigg] = \\
= \Sigma_1 + \Sigma_2 + \Sigma_3. \qquad (4.5)
\end{aligned}$$

Let us estimate Σ_1. By virtue of $\sup\limits_{x\in R_p} f(x)r(x) < \infty$ and (4.2) we obtain

$$\begin{aligned}
\Sigma_1 = 4\iint\bigg(\sum_{i,j}^{\lambda} a_{ij}\varphi_i(u)\varphi_j(v)r(u)r(v)\bigg)^2 f(u)f(v)\,du\,dv \le \\
\le c_1\iint\bigg(\sum_{i,j}^{\lambda} a_{ij}\varphi_i(u)\varphi_i(v)\bigg)^2 r(u)r(v)\,du\,dv = c_1\sum_{i,j}^{\lambda} a_{ij}^2 = \\
= c_1\sigma_n^2. \qquad (4.6)
\end{aligned}$$

It is not difficult to estimate the value Σ_2 because

$$\begin{aligned}
\Sigma_2 = 8E\bigg(\sum_{i,j}^{\lambda} a_{ij}a_i\varphi_j(X_1)r(X_1)\bigg)^2 = \\
= 8\int\bigg(\sum_{i,j}^{\lambda} a_{ij}a_i\varphi_j(u)r(u)\bigg)^2 f(u)\,du \le \\
\le c_2\int\bigg(\sum_{i,j}^{\lambda} a_{ij}a_i\varphi_j(u)\bigg)^2 r(u)\,du = \\
= c_2\int\bigg[\sum_{j}^{\lambda}\bigg(\sum_{i}^{\lambda} a_{ij}a_i\bigg)\varphi_j(x)\bigg]^2 r(x)\,dx = \\
= c_3\sum_{j}^{\lambda}\bigg(\sum_{i}^{\lambda} a_{ij}a_i\bigg)^2 \le c_4\sum_{j=1}^{\lambda} a_j^2\sum_{i,j}^{\lambda} a_{ij}^2 \le c_5\sum_{i,j}^{\lambda} a_{ij}^2 = \\
= c_5\sigma_n^2. \qquad (4.7)
\end{aligned}$$

And, eventually,

$$\Sigma_3 = 4\bigg(\sum_{i,j}^{\lambda} a_{ij}a_ia_j\bigg)^2 \le 4\bigg(\sum_{i=1}^{\lambda} a_i^2\bigg)^2\bigg(\sum_{i,j}^{\lambda} a_{ij}^2\bigg) \le c_6\sigma_n^2. \qquad (4.8)$$

Combining the estimators for all terms Σ_1, Σ_2 and Σ_3 given in (4.6), (4.7) and (4.8), we obtain (4.3) from (4.5).

Assume further that

$$\max_{1\le k\le\lambda}\sup_{x\in R_p}|\varphi_k(x)r(x)|\le c_7\lambda^{\varepsilon},\quad \varepsilon\ge 0.$$

In that case conditions 1° and 2° of Theorem 2.1 are fulfilled for $\alpha=2\varepsilon+1$ and $\beta=1$. Indeed, since $\delta_{0\lambda}(x,X_i)=\sum\limits_{j=1}^{\lambda}\varphi_j(x)\varphi_j(X_i)r(X_i)$ and $\delta_{1\lambda}(x,X_i)\equiv 1$, we have

$$1^\circ.\ E\Big(\int\delta_{01}^2(x,X_1)r(x)\,dx\Big)^{\gamma}=E\bigg(\sum_{j}^{\lambda}\varphi_j^2(X_1)r^2(X_1)\bigg)^{\gamma}\le$$
$$\le c_8\lambda^{\gamma(2\varepsilon+1)}. \tag{4.9}$$

2°. Let us consider the case with $i=j=0$. Then

$$\delta_{00}(u,v,\lambda)=\int\delta_{0\lambda}(x,u)\delta_{0\lambda}(x,v)r(x)\,dx=\sum_{j=1}^{\lambda}\varphi_j(u)\varphi_j(v)r(u)r(v).$$

Therefore

$$E\big(\delta_{00}^s(X_1,X_2,\lambda)\big)=E\bigg(\sum_{j=1}^{\lambda}\varphi_j(X_1)\varphi_j(X_2)r(X_1)r(X_2)\bigg)^s=$$
$$=\int\bigg(\sum_{j=1}^{\lambda}\varphi_j(u)\varphi_j(v)r(u)r(v)\bigg)^{s-2}\cdot$$
$$\cdot\bigg(\sum_{j=1}^{\lambda}\varphi_j(u)\varphi_j(v)r(u)r(v)\bigg)^2 f(u)f(v)\,du\,dv\le$$
$$\le c_9\lambda^{(2\varepsilon+1)(s-2)}\int\bigg(\sum_{j=1}^{\lambda}\varphi_j(u)\varphi_j(v)r(u)r(v)\bigg)^2 f(u)f(v)\,du\,dv\le$$
$$\le c_{10}\lambda^{(2\varepsilon+1)(s-2)}\sum_{j}^{\lambda}\int\varphi_j^2(u)r(u)\,du\int\varphi_j^2(v)r^2(v)f(v)\,dv=$$
$$=c_{10}\lambda^{(2\varepsilon+1)(s-2)}\sum_{j=1}^{\lambda}\int\varphi_j^2(v)r^2(v)f(v)\,dv\le c_{11}\lambda^{(s-1)(2\varepsilon+1)}.$$

Let now $i=0$, $j=1$. We have

$$E\big(\delta_{01}^s(X_1,X_2,\lambda)\big)=E\bigg(\int\bigg(\sum_{j=1}^{\lambda}\varphi_j(x)\varphi_j(X_1)r(X_1)\bigg)f(x)r(x)\,dx\bigg)^s\le$$

$$\leq c_{12}\lambda^{(2\varepsilon+1)(s-2)} E\biggl(\int \sum_{j}^{\lambda} \varphi_j(x) r(x) \varphi_j(X_1) r(X_1) f(x)\, dx\biggr)^2 =$$

$$= c_{12}\lambda^{(2\varepsilon+1)(s-2)} E\biggl(\sum_{j=1}^{\lambda} a_j \varphi_j(X_1) r(X_j)\biggr)^2 \leq$$

$$\leq c_{13}\lambda^{(2\varepsilon+1)(s-2)} \sum_{j=1}^{\lambda} a_j^2 \sum_{j=1}^{\lambda} \int \varphi_j^2(u) r^2(u) f(u)\, du \leq$$

$$\leq c_{14}\lambda^{(s-1)(2\varepsilon+1)}.$$

Hence

$$E\bigl(\delta_{ij}^s(X_1, X_2, \lambda)\bigr) \leq c_{15}\lambda^{(s-1)(2\varepsilon+1)}, \quad s = 2, 4. \tag{4.10}$$

Further, we rewrite (4.1) and (4.2) as

$$\Delta_n = A_n + A_n^*, \quad \sigma_n^2 = B_n^2 + B_n^*, \tag{4.11}$$

where

$$A_n = \sum_{j=1}^{\lambda} \int \varphi_j^2(u) r^2(u) f(u)\, du,$$

$$B_n^2 = 2 \sum_{i,j} \Bigl[\int \varphi_i(u) \varphi_j(u) r^2(u) f(u)\, du\Bigr]^2,$$

and

$$|A_n^*| \leq \|f\|^2 \quad \text{and} \quad |B_n^*| \leq 2 \int \biggl(\sum_{i=1}^{\lambda} a_i \varphi_i(u)\biggr)^2 r^2(u) f(u)\, du \leq c_{16}\|f\|^2.$$

Thus, after combining (4.3), (4.9) (4.10) and (4.11), we come to

Theorem 4.1. *Let*

$$\max_{1\leq k\leq\lambda} \sup_{x\in R_p} |\varphi_j(x) r(x)| \leq c_7\lambda^{\varepsilon}, \ \ \varepsilon \geq 0 \quad \textit{and} \quad \sup_{x\in R_p} f(x) r(x) < \infty.$$

If $\sigma_n \to \infty$ *and* $n^{-1}\sigma_n^{-4}\lambda^{3\alpha} \to 0$, $\alpha = 2\varepsilon + 1$, *as* $n \to \infty$, *then*

$$\sqrt{\lambda}\Bigl(\frac{n}{\lambda}\|g_n - Eg_n\|^2 - A_{0n}\Bigr) B_{0n}^{-1} \xrightarrow{d} N(0,1),$$

where

$$A_{0n} = \lambda^{-1} A_n, \quad B_{0n}^2 = \lambda^{-1} B_n^2.$$

The conditions of Theorem 4.1 are simpler than those of Theorem 5.2.1 from [21] and also than those of Theorem 5.1 from [22].

Remark. E. Nadaraya was the first to study in [3] the limit distribution of square distribution of projectve estimators.

Applications. Let us now apply this theorem to some concrete orthonormalized systems which were individually studied by many authors.

a) Let $-\pi \le X \le \pi$ and $\varphi_j(x)$, $j = 1, 2, \dots$, be a system of trigonometric functions on $E = [-\pi, \pi]$. In that case $\varepsilon = 0$, i.e., $\alpha = 1$ and

$$A_{0n} = \frac{1}{2\pi} \quad \text{and} \quad B_{0n}^2 = \frac{1}{\pi} \int\limits_E \Big[\int\limits_E \Phi_\lambda(u) f(x + 2u) du \Big] f(x)\, dx \to$$

$$\to \sigma^2 = \frac{1}{\pi} \|f\|^2 \quad \text{as} \quad n \to \infty,$$

where $\Phi_\lambda(u) = \frac{1}{(2\lambda+1)\pi} \frac{\sin^2(2\lambda+1)u}{\sin^2 u}$ is the Fayer kernel.

Thus, if $\frac{\lambda}{n} \to 0$, then by Theorem 4.1 we conclude that

$$(2\lambda + 1)^{1/2} \Big(\frac{n}{2\lambda + 1} \|g_n - Eg_n\|^2 - \frac{1}{2\pi} \Big) \xrightarrow{d} N(0, \sigma).$$

Let now X be a random vector with values from $E = [-\pi, \pi] \times [-\pi, \pi]$, of which we know that it has density $f(x)$, $x \in E$. Let $\{\varphi_j\}$ be a double trigonometric system of functions on E. It is easy to see that

$$A_{0n} = \frac{1}{4\pi^2},$$

$$B_{0n}^2 = \frac{1}{2\pi^2} \int\limits_E \int\limits_E \big[\Phi_\lambda(u) \Phi_\lambda(v) f(x + u, y + v) dx\, dy \big] f(u, v)\, du\, dv \to$$

$$\to \sigma^2 = \frac{1}{2\pi^2} \|f\|^2.$$

Thus, if $\frac{\lambda^2}{n} \to 0$ for $n \to \infty$, then Theorem 4.1 implies (see [21–22]) that

$$(2\lambda + 1) \Big(\frac{n}{(2\lambda + 1)^2} \|g_n - Eg_n\|^2 - \frac{1}{4\pi^2} \Big) \xrightarrow{d} N(0, \sigma).$$

b) Let $\{\varphi_j\}$ be an orthonormalized Legendre basis. Let $d(x) > 0$ be some bounded function on $[-1, 1]$. Assume that $f(x)$ and $d(x)$ have bounded derivatives in $[-1, 1]$ and $d(x) = O((1 - x^2)^\gamma)$, $\gamma \ge \frac{3}{4}$, for $x \uparrow 1$ or $x \downarrow -1$. Let $g_n(x)$ be a projective estimator for $f(x)d(x)$. Then, as shown in [21], we have

$$A_{0n} = \int\limits_{-1}^{1} f(x) d^2(x)\, \frac{1}{\sqrt{1 - x^2}}\, dx + O\Big(\frac{\ln \lambda}{\lambda} \Big),$$

$$B_{0n}^2 \to \sigma^2 = \frac{2}{\pi} \int\limits_{-1}^{1} f^2(x) d^4(x)\, \frac{1}{\sqrt{1 - x^2}}\, dx.$$

In that case $\varepsilon=\frac{1}{2}$, i.e., $\alpha=2$. Thus if $\frac{\lambda^4}{n}\to 0$, then Theorem 4.1 implies (see [23]) that

$$\lambda^{1/2}\bigg(\frac{n}{\lambda}\,\|g_n-Eg_n\|^2-\int\limits_{-1}^{1}f(x)d^2(x)\,\frac{dx}{\sqrt{1-x^2}}\bigg)\xrightarrow{d}N(0,\sigma).$$

5. *D*–3 – **Histogram.** Let X be a bounded random value, $0\le X\le 1$ and the interval $[0,1]$ be partitioned into λ equal intervals A_i, $i=\overline{1,\lambda}$, of length $h=\frac{1}{\lambda}$. Let $\delta_{0\lambda}(x,X_i)=\lambda\sum\limits_{k=1}^{\lambda}I_k(x)I_k(X_i)$, where $I_k(\cdot)$ is the indicator of the interval A_k, and $\delta_{1\lambda}(x,X_i)\equiv 1$. Then from (2.1) we obtain the standard histogram

$$g_n(x)=\frac{\nu_i}{nh},\quad x\in A_i,\quad i=\overline{1,\lambda},$$

where ν_i, $i=\overline{1,\lambda}$ are the numbers of observations occurring in A_i, $i=\overline{1,\lambda}$. Let $f(x)$ and $r(x)$ be continuous on $[0,1]$.

In that case conditions 1° and 2° of Theorem 2.1 are fulfilled for $\beta=1$ and $\alpha=1$, since $\delta_{0\lambda}(x,y)$ is majorized by the Parzen kernel, i.e.,

$$\delta_{0\lambda}(x,y)\le\lambda K(\lambda(x-y)),$$

where $K(z)$ is the indicator of the interval $[-1,1]$.

We will now show that condition 3° is valid. We have

$$\begin{aligned}
\sigma_n^2&=2\iint\big(E\alpha_\lambda(u_1,X_1)\alpha_\lambda(u_2,X_1)\big)^2r(u_1)r(u_2)\,du_1\,du_2=\\
&=2\iint\big(\operatorname{cov}(\delta_{0\lambda}(u_1,X_1),\delta_{0\lambda}(u_2,X_1))\big)^2r(u_1)r(u_2)\,du_1\,du_2=\\
&=2\lambda^4\iint\bigg(\sum_{i=1}^{\lambda}I_i(u_1)I_i(u_2)\int\limits_{\Delta_i}f(u)\,du\bigg)^2r(u_1)r(u_2)\,du_1\,du_2+O(1)=\\
&=2\lambda^4\sum_{i=1}^{\lambda}\bigg(\int\limits_{\Delta_i}r(u)\,du\bigg)^2\bigg(\int\limits_{\Delta_i}f(u)\,du\bigg)^2+O(1)=\\
&=2\sum_{i=1}^{\lambda}f^2(\xi_i)r^2(\eta_i)+O(1),
\end{aligned}\tag{5.1}$$

where ξ_i, η_i are some points of the interval A_i.

Further, we rewrite the sum $\sum\limits_{i=1}^{\lambda}f^2(\xi_i)r^2(\eta_i)$ as

$$\sum_{i=1}^{\lambda}f^2(\xi_i)r^2(\eta_i)=\sum_{i=1}^{\lambda}f^2(\xi_i)r^2(\xi_i)+H_n$$

and at that we have

$$\lambda^{-1}|H_n| = \lambda^{-1}\left|\sum_{i=1}^{\lambda} r^2(\eta_i)f^2(\xi_i) - r^2(\xi_i)f^2(\xi_i)\right| \leq$$
$$\leq c_1 \sum_{i=1}^{\lambda} |r(\eta_i) - r(\xi_i)|\lambda^{-1} \leq c_2\biggl(\sum_{i=1}^{\lambda} M_i\lambda^{-1} - \sum_{i=1}^{\lambda} m_i\lambda^{-1}\biggr) =$$
$$= c_2(\overline{S}_\lambda - \underline{S}_\lambda) \to 0 \quad \text{as} \quad n \to \infty,$$

where $M_i = \sup\limits_{x\in A_i} r(x)$, $m_i = \inf\limits_{x\in A_i} r(x)$, while $\overline{S}_\lambda$ and $\underline{S}_\lambda$ are the upper and lower Darboux sums $r(x)$ corresponding to this partitioning of the interval $[0,1]$.

Hence we obtain

$$\lambda^{-1}\sum_{i=1}^{\lambda} f^2(\xi_i)r^2(\eta_i) \to \int_0^1 f^2(x)r^2(x)\,dx. \tag{5.2}$$

Thus, by (5.1) and (5.2), we establish that

$$\lambda^{-1}\sigma_n^2 \to \sigma^2 = 2\int_0^1 f^2(x)r^2(x)\,dx.$$

Further, since $\delta_{0\lambda}(x,y)$ is majorized by the kernel $\lambda K(x-y))$, where $K(x)$ is the indicator of the interval $[-1,1]$, we establish similarly to relation (3.9) that

$$L_n = o(\sigma_n^4).$$

Therefore conditions 3° are also satisfied. Condition 4° takes the form $\frac{\lambda}{n} \to 0$ as $n \to \infty$. Further it is not difficult to calculate that

$$\Delta_n = \lambda^2 \sum_{i=1}^{\lambda} \int_{A_i} r(u)\,du \int_{A_i} f(x)\,dx + O(1). \tag{5.3}$$

If $r(u) \equiv 1$, $u \in [0,1]$, then

$$\lambda^{-1}\Delta_n = 1 + O\Bigl(\frac{1}{\lambda}\Bigr).$$

But if $f(x)$ and $r(x)$ satisfy the Lipshitz condition, then from (5.3) it readily follows that

$$\lambda^{-1}\Delta_n = \int_0^1 f(x)r(x)\,dx + O\Bigl(\frac{1}{\lambda}\Bigr).$$

Hence we formulate

Theorem 5.1. a) *If $f(x)$ is continuous, $r(x)\equiv 1$, $x\in[0,1]$, and $\frac{\lambda}{n}\to 0$ as $n\to\infty$, then*

$$\lambda^{1/2}(U_n^{(1)}-1)\Big(2\int f^2(x)\,dx\Big)^{-1/2}\xrightarrow{d} N(0,1),$$

where

$$U_n^{(1)}=\frac{n}{\lambda}\int\big(g_n(x)-Eg_n(x)\big)^2dx.$$

b) *If $f(x)$ and $r(x)$ satisfy the Lipshitz condition and $\frac{\lambda}{n}\to 0$, then*

$$\lambda^{1/2}\Big(U_n^{(2)}-\int f(x)r(x)\,dx\Big)\Big(2\int f^2(x)r^2(x)\,dx\Big)^{-1/2}\xrightarrow{d} N(0,1),$$

where

$$U_n^{(2)}=\frac{n}{\lambda}\int\big(g_n(x)-Eg_n(x)\big)^2r(x)\,dx.$$

As is known, the test χ_n^2, defining a deviation measure between the sampled n hypothetic distribution, is defined as follows:

$$\chi_n^2=\sum_{i=1}^{\lambda}\frac{n}{p_i}\Big(\frac{\nu_i}{n}-p_i\Big)^2,\quad p_i=\int\limits_{A_i}f(x)\,dx.$$

χ^2 can also be represented in the form

$$\chi_n^2=n\int\limits_0^1\big(g_n(x)-Eg_n(x)\big)^2\big(Eg_n(x)\big)^{-1}\,dx.$$

Corollary of Theorem 5.1 (S. Tumanyan [23]). *Assume that $f(x)$ has a bounded first derivative and $\min\limits_{1\le x\le 1}f(x)\ge\mu>0$. If $\frac{\lambda}{n}\to 0$, then for $n\to\infty$*

$$\frac{\chi_n^2-\lambda}{\sqrt{2\lambda}}\xrightarrow{d} N(0,1).$$

Proof. Assume that $r(x)=\frac{1}{f(x)}$. On each interval $A_i=[a_i,b_i)$, $i=\overline{1,\lambda}$, we write

$$\begin{aligned}f(x)-Eg_n(x)&=f(a_i)+(x-a_i)f^{(1)}(\xi_i')-\\&\quad-\frac{1}{h}\int\limits_{\Delta_i}\big[f(a_i)+(x-a_i)f^{(1)}(\xi_i'')\big]dx=\\&=(x-a_i)f^{(1)}(\xi_i')-\frac{h}{2}f^{(1)}(\xi_i''),\quad \xi_i',\xi_i''\in\Delta_i.\end{aligned}$$

For brevity, we can write

$$|Eg_n(x)-f(x)|=O\Big(\frac{1}{\lambda}\Big),\quad x\in[0,1].$$

Therefore

$$\Big|\frac{\chi_n^2-\lambda}{\sqrt{2\lambda}}-\frac{\lambda^{1/2}(U_n^{(2)}-1)}{\sqrt{2}}\Big|\le c_3\,\frac{n}{\sqrt{\lambda}}\sum_{i=1}^{\lambda}\int\limits_{\Delta_i}\big(g_n(x)-Eg_n(x)\big)^2\cdot$$

$$\cdot\Big|\frac{1}{f(x)}-\frac{1}{Eg_n(x)}\Big|dx\le c_4\,\frac{1}{n\sqrt{\lambda}}\sum_{i=1}^{\lambda}(\nu_i-np_i)^2.$$

Hence we have

$$\Big|\frac{\chi_n^2-1}{\sqrt{2\lambda}}-\frac{\lambda^{1/2}(U_n^{(2)}-1)}{\sqrt{2}}\Big|\le c_5\,\frac{1}{\sqrt{\lambda}}\to 0. \qquad \square$$

6. R–1 – **Kernel estimator.** Let $X_i=(\overline{X}_i,Y_i)$, $\overline{X}_i=(X_i^{(1)},\dots,X_i^{(p)})$, $i=\overline{1,n}$, be a sequence of independent, identically distributed random variables with density $f(x,y)$, $x=(x_1,x_2,\dots,x_p)$. Let $f_0(x)$ be the martingale density of distribution of a random vector $\overline{X}_1$, and the regression function Y_1 on $\overline{X}_1$ be denoted by $m(x)$, i.e. $g(x)\equiv m(x)=E(Y_1|\overline{X}_1=x)$. Let, further, $\delta_{0\lambda}(x,X_i)=\lambda^pK(\lambda(x-\overline{X}_i))Y_i$ and $\delta_{1\lambda}(x,X_i)=\lambda^pK(\lambda(x-\overline{X}_i))$.

Then, as said in Subsection 2, (2.1) gives kernel estimators of the Nadaraya-Watson regression function

$$g_n(x)=\sum_{i=1}^{n}Y_iK(\lambda(x-\overline{X}_i))\Big/\sum_{i=1}^{n}K(\lambda(x-\overline{X}_i)).$$

Denote

$$\varphi_s(x)=\int y^sf(x,y)\,dy,\quad s\ge 0.$$

We introduce the following assumptions:

1. $K(x)\in H_2$ (see § 3).
2. $EY^4<\infty$.
3. The functions $f_0(x)\equiv\varphi_0(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ are bounded and have bounded partial derivatives of first order, while $\varphi_4(x)$ is only bounded.
4. The weight function $r(x)$ satisfies the conditions of Subsection 3, and, moreover, it is integrable. The functions $m^k(x)r(x)$, $k=1,2$, are bounded and integrable.

In the considered case, conditions 1°–4° of Theorem 2.1 are fulfilled for $\alpha=p$ and $\beta=1$. Indeed,

$$1^\circ.\ E\Big(\int\delta_{0\lambda}^2(x,X_1)r(x)\,dx\Big)^\gamma=E\Big(\int\lambda^{2p}K^2(\lambda(x-\overline{X}_1)Y_1^2r(x)\,dx\Big)^\gamma=$$

$$=\lambda^{\gamma p}\int y^{2\gamma}\Big(\lambda^p\int K^2(\lambda(x-u))r(x)\,dx\Big)^\gamma f(u,y)\,du\,dy\le$$

$$\le c_1EY^{2\gamma}\Big(\int K^2(u)\,du\Big)^\gamma\lambda^{\gamma p},\quad \gamma=1,2,$$

and also

$$E\Big(\delta^2_{1\lambda}(x,X_1)r(x)\,dx\Big)^{\gamma}\le c_2\lambda^{\gamma p}.$$

2°. Let us show that

$$E\big(\delta^s_{ij}(X_1,X_2,\lambda)\big)\le c_3\lambda^{(s-1)p},\quad i=0,1,\quad s=2,4. \tag{6.1}$$

a) Let $i=0$, $j=0$. Then

$$\begin{aligned}E\big(\delta^s_{00}(X_1,X_2,\lambda)\big)&=E\Big(\int\delta_{0\lambda}(x,X_1)\delta_{0\lambda}(x,X_2)r(x)\Big)^s=\\&=E\Big(Y_1^sY_2^s\lambda^{2sp}\Big(\int K(\lambda(x-\overline{X}_1)K(\lambda(x-\overline{X}_2))r(x)\,dx\Big)^s\le\\&\le c_4\lambda^{sp}EY_1^sY_2^sK_1^s(\lambda(\overline{X}_2-\overline{X}_1))\le\\&\le c_5\lambda^{(s-1)p}\int K_1^s(t)\,dt,\end{aligned}$$

where

$$K_1(t)=\int|K(u-t)|\,|K(u)|\,du.$$

b) Let $i=1$, $j=1$. Then

$$\begin{aligned}&E\big(\delta^s_{11}(X_1,X_2,\lambda)\big)=\\&\quad=\lambda^{sp}E\Big(\lambda^p\int K(\lambda(x-\overline{X}_1))K(\lambda(x-\overline{X}_2))m^2(x)r(x)\,dx\Big)^s\le\\&\quad\le c_6\lambda^{sp}EK_1^s(\lambda(\overline{X}_2-\overline{X}_1))\le c_7\lambda^{(s-1)p}\int K_1^s(t)\,dt.\end{aligned}$$

The other cases of (6.1) are shown similarly and so we omit them.

Let us now establish the validity of condition 3°. Since $\alpha_\lambda(u,v)=\delta_\lambda(u,v)-E\delta_\lambda(u,X_1)$ and $\delta_\lambda(u,v)=\lambda^pK(\lambda(u-v))[Y_1-m(u)]$, we have

$$\begin{aligned}\sigma_n^2&=2\iint\big(E\alpha_\lambda(u_1,X_1)\alpha_\lambda(u_2,X_1)\big)^2r(u_1)r(u_2)\,du_1\,du_2=\\&=2\iint\big(E\delta_\lambda(u_1,X_1)\delta_\lambda(u_2,X_1)-E\delta_\lambda(u_1,X_1)E\delta_\lambda(u_2,X_1)\big)^2\cdot\\&\qquad\cdot r(u_1)r(u_2)\,du_1\,du_2=\\&=2\iint\big(E\delta_\lambda(u_1,X_1)\delta_\lambda(u_2,X_1)\big)^2r(u_1)r(u_2)\,du_1\,du_2+R_n,\end{aligned}\tag{6.2}$$

where

$$\begin{aligned}R_n=2&\iint\big(E\delta_\lambda(u_1,X_1)E\delta_\lambda(u_2,X_1)\big)^2r(u_1)r(u_2)\,du_1\,du_2-\\&-4\iint E\delta_\lambda(u_1,X_1)\delta_\lambda(u_2,X_1)E\delta_\lambda(u_1,X_1)E\delta_\lambda(u_2,X_1)\cdot\\&\qquad\cdot r(u_1)r(u_2)\,du_1\,du_2.\end{aligned}$$

But, as simple calculations show, with our assumptions 1–4 we have

$$R_n = O(1). \tag{6.3}$$

Now we proceed to calculating the asymptotics of the principal term in the right-hand side of (6.2). It can be represented as follows:

$$2\iint \big(E\delta_\lambda(u_1, X_1)\delta_\lambda(u_2, X_1)\big)^2 r(u_1)r(u_2)\,du_1\,du_2 =$$
$$= 2\lambda^{4p}\iint \Big[\int \big(\varphi_2(x) - (m(u_1)+m(u_2))\varphi_1(x) + m(u_1)m(u_2)f_0(u_1)\big)\cdot$$
$$\cdot K(\lambda(u_1 - x))K(\lambda(u_2 - x))\,dx\Big]^2 r(u_1)r(u_2)\,du_1\,du_2.$$

After expanding $f_0(u_1 - \frac{t}{\lambda})$, $\varphi_1(u_1 - \frac{t}{\lambda})$ and $\varphi_2(u_1 - \frac{t}{\lambda})$ by the Taylor formula and applying (6.3), we have

$$\sigma_n^2 = 2\lambda^{2p}\iint F^2(u_1, u_2)K_0^2(\lambda(u_1 - u_2))r(u_1)r(u_2)\,du_1\,du_2 + $$
$$+ O(\lambda^{p-1}) + O(1), \tag{6.4}$$

where

$$F(u_1, u_2) = \varphi_2(u_1) - (m(u_2) + m(u_2))\varphi_1(u_1) + m(u_1)m(u_2)f_0(u_1),$$
$$K_0(u) = \int K(t)K(u - t)\,dt.$$

As is known [24],

$$\lambda^p \int K_0^2(\lambda(u_1 - u_2))F^2(u_1, u_2)r(u_1)\,du_1 \to F^2(u_2, u_2)r(u_2)\int K_0^2(t)\,dt$$

as $n \to \infty$. Hence, by the Lebesgue theorem on limit passage under the integral sign, from (6.4) follows that for $n \to \infty$

$$\overline{\lambda}^{-p}\sigma_n^2 \to \sigma^2 = 2\int F^2(u, u)r^2(u)\,du \int K_0^2(t)\,dt =$$
$$= 2\int D^2(Y|X = x)f_0^2(x)r^2(x)\,dx \int K_0^2(t)\,dt. \tag{6.5}$$

Therefore $\sigma_n^2 \to \infty$ as $n \to \infty$, i.e. the first part of condition 3° is satisfied. It remains to test the second part of condition 3°, i.e., to show that $L_n = o(\sigma_n^4)$.

As before, we have

$$L_n = E\Big[\iint \alpha_\lambda(u_1, X_1)\alpha_\lambda(u_2, X_2)E\alpha_\lambda(u_1, X_1)\alpha_\lambda(u_2, X_1)\cdot$$
$$\cdot r(u_1)r(u_2)\,du_1\,du_2\Big]^2 =$$

$$= \lambda^{8p} \iiiint \operatorname{cov}\big(K(\lambda(u_1 - \overline{X}_1))[Y_1 - m(u_1)], K(\lambda(v_1 - \overline{X}_1))[Y_1 - m(v_1)]\big) \cdot$$
$$\cdot \operatorname{cov}\big(K(\lambda(u_2 - \overline{X}_1))[Y_1 - m(u_2)], K(\lambda(v_2 - \overline{X}_1))[Y_1 - m(v_2)]\big) \cdot$$
$$\cdot \operatorname{cov}\big(K(\lambda(u_1 - \overline{X}_1))[Y_1 - m(u_1)], K(\lambda(u_2 - \overline{X}_1))[Y_1 - m(u_2)]\big) \cdot$$
$$\cdot \operatorname{cov}\big(K(\lambda(v_1 - \overline{X}_1))[Y_1 - m(v_1)], K(\lambda(v_2 - \overline{X}_1))[Y_1 - m(v_2)]\big) \cdot$$
$$\cdot r(u_1)r(u_2)r(v_1)r(v_2)\, du_1\, du_2\, dv_1\, dv_2. \tag{6.6}$$

It is easy to obtain

$$\big|\operatorname{cov}\big(K(\lambda(u - \overline{X}_1))[Y_1 - m(u)], K(\lambda(v - \overline{X}_1))[Y_1 - m(v)]\big)\big| \le$$
$$\le c_8[1 + M(u,v)]\,[K_1(\lambda(v-u)) + \lambda^{-p}]\lambda^{-p},$$

where

$$M(u,v) = |m(u)| + |m(v)| + |m(u)m(v)|.$$

Using this and (6.6), we have

$$L_n \le c_9 \lambda^{4p} \iiiint [1 + M(u_1, v_1)][K_1(\lambda(v_1 - u_1)) + \lambda^{-p}] \cdot$$
$$\cdot [1 + M(u_2, v_2)][K_1(\lambda(v_2 - u_2)) + \lambda^{-p}][1 + M(u_1, u_2)] \cdot$$
$$\cdot [K_1(\lambda(u_1 - u_2)) + \lambda^{-p}][1 + M(v_1, v_2)][K_1(\lambda(v_1 - v_2)) + \lambda^{-p}] \cdot$$
$$\cdot r(u_1)r(u_2)r(v_1)r(v_2)\, du_1\, du_2\, dv_1\, dv_2. \tag{6.7}$$

The integral in the right-hand side of (6.7) can be represented as a sum of integrals of the form

$$L_{1n} = \iiiint F(u_1, u_2, v_1, v_2) K_1(\lambda(v_1 - u_1)) K_1(\lambda(v_2 - u_2)) \cdot$$
$$\cdot K_1(\lambda(u_2 - u_1)) K_1(\lambda(v_2 - v_1)) r(u_1)r(u_2)r(v_1)r(v_2) \cdot$$
$$\cdot du_1\, du_2\, dv_1\, dv_2,$$

$$L_{2n} = \lambda^{-p} \iiiint F(u_1, u_2, v_1, v_2) K_1(\lambda(v_1 - u_1)) K_1(\lambda(u_2 - u_1)) \cdot$$
$$\cdot K_1(\lambda(v_2 - v_1)) r(u_1)r(u_2)r(v_1)r(v_2)\, du_1\, du_2\, dv_1\, dv_2,$$

$$L_{3n} = \lambda^{-2p} \iiiint F(u_1, u_2, v_1, v_2) K_1(\lambda(v_2 - v_1)) K_1(\lambda(v_1 - u_1)) \cdot$$
$$\cdot r(u_1)r(u_2)r(v_1)r(v_2)\, du_1\, du_2\, dv_1\, dv_2,$$

$$L_{4n} = \lambda^{-3p} \iiiint F(u_1, u_2, v_1, v_2) K_1(\lambda(v_2 - v_1)) r(u_1)r(u_2) \cdot$$
$$\cdot r(v_1)r(v_2)\, du_1\, du_2\, dv_1\, dv_2,$$

$$L_{5n} = \lambda^{-4p} \iiiint F(u_1, u_2, v_1, v_2) r(u_1) r(u_2) r(v_1) r(v_2) \cdot \\ \cdot du_1\, du_2\, dv_1\, dv_2,$$

where

$$F(u_1, u_2, v_1, v_2) = (1 + M(u_1, v_1))(1 + M(u_2, v_2)) \cdot \\ \cdot (1 + M(u_1, u_2))(1 + M(v_1, v_2)).$$

Let us consider the integral L_1. Simple calculations show that

$$\begin{aligned} L_{1n} \le c_{10} \lambda^{-p} & \iiint (1 + M(u_1, v_1))(1 + M(v_1, v_2))(c_{11} + M(u_1, u_2)) \cdot \\ & \cdot K_1(\lambda(v_1 - u_1)) K_1(\lambda(v_2 - v_1)) r(u_1) r(v_1) r(v_2)\, du_1\, dv_1\, dv_2 \le \\ & \le c_{12} \lambda^{-2p} \iint (1 + M(u_1, v_1))(c_{13} + M(u_1, v_1)) \cdot \\ & \cdot K_1(\lambda(v_1 - u_1)) r(u_1) r(v_1)\, du_1\, dv_1 \le \\ & \le c_{14} \lambda^{-3p}. \end{aligned} \tag{6.8}$$

As for the other integrals L_{2n}, L_{3n}, L_{4n}, L_{5n}, all of them, as one can easily check, are estimated similarly and we have

$$\begin{aligned} L_{2n} = O(\lambda^{-3p}), \quad L_{3n} = O(\lambda^{-3p}), \\ L_{4n} = O(\lambda^{-3p}), \quad L_{5n} = O(\lambda^{-4p}). \end{aligned} \tag{6.9}$$

Thus from (6.7), (6.8) and (6.9) we obtain

$$L_n \le c_{15} \lambda^p, \tag{6.10}$$

which and (6.5) imply that

$$\frac{L_n}{\sigma_n^4} \le c_{16} \lambda^{-p} \to 0. \tag{6.11}$$

Thus condition 3° is completely fulfilled. Further, with (6.5) taken into account, condition 4° takes the form

$$\frac{\lambda^p}{n} \to 0 \quad \text{as} \quad n \to \infty. \tag{6.12}$$

Combining (6.1), (6.5), (6.11) and (6.12), we come to the following basic theorem.

Theorem 6.1. *Let conditions* 1–4 *be fulfilled. If* $\frac{\lambda}{n} \to 0$ *as* $n \to \infty$*, then*

$$\lambda^{p/2} \Big(\frac{\overline{U}_n - \overline{\Delta}_n}{\sigma} \Big) \xrightarrow{d} N(0,1) \quad as \quad n \to \infty,$$

where

$$\overline{U}_n = \frac{n}{\lambda^p} \int \big(\widehat{g}_n(x) - E\widehat{g}_n(x) \big)^2 r(x)\, dx, \quad \overline{\Delta}_n = E\overline{U}_n,$$

$$\widehat{g}_n(x) = \frac{\lambda^p}{n}\sum_{i=1}^{n} K(\lambda(x-\overline{X}_i))[Y_i - m(x)],$$

$$\sigma^2 = 2\int D^2(Y|X=x) f_0^2(x) r^2(x)\,dx \int K_0^2(u)\,du.$$

Consider a random variable

$$W_n = \frac{n}{\lambda^p}\int \big(\varphi_n(x) - m(x) f_n(x)\big)^2 r(x)\,dx = \\ = \frac{n}{\lambda^p}\int \big(g_n(x) - m(x)\big)^2 f_n^2(x) r(x)\,dx,$$

where

$$\varphi_n(x) = \frac{\lambda^p}{n}\sum_{i=1}^{n} Y_i K(\lambda(x-\overline{X}_i)),$$

$$f_n(x) = \frac{\lambda^p}{n}\sum_{i=1}^{n} K(\lambda(x-\overline{X}_i)),$$

$$g_n(x) = \frac{\varphi_n(x)}{f_n(x)}\,,$$

which it is natural to consider as a mesure of closeness of the functions $g_n(x)$ and $m(x)$. It should be said that the functional W_n was the subject of investigations by many authors. These are some of them: E. Nadaraya [25], V. Konakov [10], Sh. Khashimov [26], P. Hall [27] and others. In these works sufficient conditions are given for establishing limit distribution of the functional W_n. Evidently, the first result in this direction was obtained in [25] and subsequently in [10].

Theorem 6.2 below provides, in our opinion, more natural sufficient conditions for solving this problem than in the works of the above-mentioned authors.

To obtain asymptotic distribution of the functional W_n, we have to show that by imposing some restrictions we can replace $E\widehat{g}_n(x)$ in $\overline{U}_n$ by a function identically equal to zero and Δ_n by some constant. For this, condition 3 should be replaced by assumption 3*: the functions $f_0 \equiv \varphi_0$, φ_1 and φ_2 have bounded partial derivatives up to second order, while φ_4 is bounded.

The following theorem is valid.

Theorem 6.2. *Let assumptions* 1, 2, 3* *and* 4 *be fulfilled. If* $\frac{\lambda}{n} \to 0$, $n\lambda^{-\frac{p}{2}-4} \to 0$ *and* $1 \le p \le 3$, *then*

$$\lambda^{p/2}\Big(\frac{W_n - \Delta}{\sigma}\Big) \xrightarrow{d} N(0,1),$$

where

$$\Delta = \int D(Y|\overline{X}=x) f_0(x) r(x)\,dx \int K^2(u)\,du,$$

$$\sigma^2 = \int D^2(Y|\overline{X}=x) f_0^2(x)^2 r(x)\,dx \int K_0^2(u)\,du.$$

Proof. In the first place we have to show that

$$\lambda^{p/2}(W_n - \overline{U}_n) = o_p(1). \tag{6.13}$$

We have

$$\lambda^{p/2}(W_n - \overline{U}_n) = 2n\lambda^{-p/2}\int \big(\eta_n(x) - E\eta_n(x)\big)E\eta_n(x) r(x)\,dx + \\ + n\lambda^{-p/2}\int \big(E\eta_n(x)\big)^2 r(x)\,dx = A_{1n} + A_{2n},$$

where

$$\eta_n(x) = \frac{\lambda^p}{n}\sum_{i=1}^n (Y_i - m(x))K(\lambda(x - \overline{X}_i)).$$

By virtue of conditions 1 and 3* we find that

$$|E\eta_n(x)| = \Big|\int K(u)\Big[\varphi_1\Big(x - \frac{u}{\lambda}\Big) - m(x) f_0\Big(x - \frac{u}{\lambda}\Big)\Big]du\Big| \le c_{17}\lambda^{-2},$$

and therefore

$$A_{2n} = n\lambda^{-p/2}\int \big(E\eta_n(x)\big)^2 r(x)\,dx = O\big(n\lambda^{-p/2-4}\big).$$

Further,

$$E|A_{1n}| = 2n\lambda^{-p/2}E\Big|\int \big(\eta_n(x) - E\eta_n(x)\big)E\eta_n(x) r(x)\,dx\Big| \le \\ \le 2n\lambda^{-p/2}\Big(E\iint \big(\eta_n(x) - E\eta_n(x)\big)\big(\eta_n(x_1) - E\eta_n(x_1)\big) \cdot \\ \cdot E\eta_n(x)E\eta_n(x_1) r(x) r(x_1)\,dx\,dx_1\Big)^{1/2} \le \\ \le 2n\lambda^{-p/2}\Big(\frac{\lambda^{2p}}{n}\iint f(u,y)\,du\,dy\Big[\int (y - m(x))K(\lambda(x-u)) \cdot \\ \cdot E\eta_n(x) r(x)\,dx\Big]^2\Big)^{1/2} \le c_{18}\sqrt{n}\,\lambda^{-\frac{p}{2}-2}.$$

Thus

$$\lambda^{p/2}E|W_n - \overline{U}_n| \le c_{19}\big(n\lambda^{-\frac{p}{2}-4} + \sqrt{n}\,\lambda^{-\frac{p}{2}-2}\big) = \\ = c_{19}\big(n\lambda^{-\frac{p}{2}-4}\big)^{1/2}\big[\big(n\lambda^{-\frac{p}{2}-4}\big)^{1/2} + \lambda^{-\frac{p}{4}}\big] \to 0.$$

Hence we obtain (6.13), which, in turn, leads to a conclusion that

$$\lambda^{p/2}\Big(\frac{W_n - \overline{\Delta}_n}{\sigma}\Big) \xrightarrow{d} N(0,1).$$

Further, to complete the proof of the theorem, we have to show that for $n \to \infty$

$$\lambda^{p/2}(E\overline{U}_n - \Delta) \to 0.$$

We have

$$\overline{\Delta}_n = E\overline{U}_n = \int \Big[\int \Big(\varphi_2\Big(x - \frac{u}{\lambda}\Big) - 2m(x)\varphi_1\Big(x - \frac{u}{\lambda}\Big) + \\ + m^2(x) f_0\Big(x - \frac{u}{\lambda}\Big) \Big) K^2(u)\,du \Big] r(x)\,dx - \\ - \lambda^{-p} \int \Big(\int \Big(\varphi_1\Big(x - \frac{u}{\lambda}\Big) - m(x) f_0\Big(x - \frac{u}{\lambda}\Big)\Big) K(u)\,du \Big)^2 r(x)\,dx.$$

Hence, by the Taylor formula, we obtain

$$\overline{\Delta}_n = \Delta + O(\lambda^{-2}) + O(\lambda^{-p-2}).$$

Thus

$$\lambda^{p/2}(\overline{\Delta}_n - \Delta) = O\big(\lambda^{\frac{p}{2}-2}\big) + O\big(\lambda^{-\frac{p}{2}-2}\big) \to 0. \qquad \square$$

Remark. By the proof of Theorem 6.2 it is not difficult to see that for $p = 1$ the condition on $\varphi_2(x)$ can be replaced by a less restrictive one, namely, by a requirement that $\varphi_2(x)$ have a bounded derivative of first order.

Let us now consider statistical problems connected with a regression curve.

By Theorem 6.2 we can construct a test for checking the hypothesis on a regression curve.

a) Let $V(x) = D(Y|\overline{X} = x) f_0(x)$ be known. It is required to check Hypothesis H_0 that $m(x) = m_0(x)$ ($m_0(x)$ is a given function). For this, we have to calculate W_n for $m(x) = m_0(x)$ and reject the hypothesis for $W_n \geq d_n(\alpha)$, where

$$d_n(\alpha) = \int V(x) r(x)\,dx \int K^2(x)\,dx + \lambda^{-p/2}\varepsilon_\alpha \sigma,$$

where ε_α is a quantile of level α of the distribution function of the standard normal law.

b) Let $V(x)$ be unknown. Then to check Hypothesis H_0 we should have an analogue of Theorem 6.2, where $V(x)$ in the normed constants is replaced by

$$V_n(x) = \varphi_n(x) - m_0^2(x) f_n(x),$$

where

$$\varphi_n(x) = \frac{\lambda^p}{n} \sum_{j=1}^{n} Y_j^2 K(\lambda(x - \overline{X}_j)),$$

$$f_n(x) = \frac{\lambda^p}{n} \sum_{j=1}^{n} K(\lambda(x - \overline{X}_j)).$$

Assume that

$$\Delta_n = \gamma_1 \int V_n(x) r(x)\,dx, \quad \sigma_n^2 = \gamma_0 \int V_n^2(x) r^2(x)\,dx,$$

where

$$\gamma_0 = \int K_0^2(u)\,du, \quad \gamma_1 = \int K^2(u)\,du.$$

It is easy to see that

$$EV_n(x) = V(x) + O(\lambda^{-2}),$$

where $O(\lambda_n^{-2})$ is uniform with respect to x.

Therefore

$$\Delta_n - \Delta = S_n^{(1)} + S_n^{(2)} + O(\lambda^{-2}), \tag{6.14}$$

where

$$S_n^{(1)} = \frac{1}{n}\sum_{j=1}^{n}(V_j^{(1)} - EV_j^{(1)}),$$

$$S_n^{(2)} = \frac{1}{n}\sum_{j=1}^{n}(V_j^{(2)} - EV_j^{(2)}),$$

$$V_j^{(1)} = Y_j^2 \lambda^p \int K(\lambda(x - \overline{X}_j)) r(x)\,dx,$$

$$V_j^{(2)} = \lambda^p \int K(\lambda(x - \overline{X}_j)) m_0^2(x) r(x)\,dx.$$

Assume that $E|Y_1|^{4+\delta} < \infty$, $\delta > 0$. In the first place, let us check that the conditions of the central limit theorem are fulfilled for the sums $\frac{S_n^{(1)}}{\sqrt{DS_n^{(1)}}}$. To do so, it is sifficient to state that for $n \to \infty$

$$nP\Big\{\Big|\frac{V_n - EV_n}{\sqrt{\mathcal{D}V_n}}\Big| \geq \varepsilon n^{1/2}\Big\} \to 0, \tag{6.15}$$

where

$$V_n = Y^2 \lambda^p \int K(\lambda(x - \overline{X})) r(x)\,dx,$$

while for (6.15), in turn, it is sufficient to indicate that

$$\frac{E|V_n - EV_n|^{2+\delta}}{n^{\delta/2}(\mathcal{D}V_n)^{1+\delta/2}} \to 0. \tag{6.16}$$

Condition (6.16) is fulfilled. Indeed,

$$E|V_n - EV_n|^{2+\delta} \leq c_{20} E|Y|^{4+2\delta}$$

and

$$\mathcal{D}V_n \sim d_1 = \int \varphi_4(u) r^2(u)\,du - \Big(\int \varphi_2(u) r(u)\,du\Big)^2 =$$

$$= E\big(Y^2 r(\overline{X}) - EY^2 r(\overline{X})\big)^2.$$

Therefore $\sqrt{n}\, S_n^{(1)}$ is asymptotically distributed normally with $N(0, d_1)$. In a similar manner one can prove that $\sqrt{n}\, S_n^{(2)}$ in the limit is also distributed normally with $N(0, d_2)$,

$$d_2 = E\big(m_0^2(\overline{X}) r(\overline{X}) - E m_0^2(\overline{X}) r(\overline{X})\big)^2.$$

From this and (6.14) we conclude that for $n \to \infty$

$$\lambda^{p/2}(\Delta_n - \Delta) \xrightarrow{P} 0. \tag{6.17}$$

Now let us prove that

$$E|\sigma_n^2 - \sigma^2| \to 0 \quad \text{as} \quad n \to \infty. \tag{6.18}$$

By the Cauchy-Schwartz inequality

$$E|\sigma_n^2 - \sigma^2| \le E^{1/2} \int [V_n(x) + V(x)]^2 r^2(x)\, dx \cdot$$
$$\cdot E^{1/2} \int [V_n(x) - V(x)]^2 r^2(x)\, dx. \tag{6.19}$$

Applying the Fubini theorem, we have

$$E \int V_n(x) V(x) r^2(x)\, dx = \int \lambda^p \int \big(\varphi_2(u) - m_0^2(x) f_0(u)\big) \cdot$$
$$\cdot K(\lambda(x-u)) V(x) r^2(x)\, dx\, du.$$

Hence, by the theorem from [24], we obtain

$$\lambda^p \int \big(\varphi_2(u) - m_0^2(x) f_0(u)\big) K(\lambda(x-u)) du \to V(x)$$

at the continuity points $\varphi_2(x)$ and $f_0(x)$ and by the Lebesgue theorem we now have

$$E \int V_n(x) V(x) r^2(x)\, dx \to \int V^2(x) r^2(x)\, dx. \tag{6.20}$$

Further, by the definition of $V_n(x)$ we have

$$E \int V_n^2(x) r^2(x)\, dx = \frac{\lambda^{2p}}{n} \int E(Y_1^2 - m_0^2(x))^2 K^2(\lambda(x - \overline{X}_1)) r^2(x)\, dx +$$
$$+ \frac{n-1}{n} \lambda^{2p} \int \big(E(Y_1^2 - m_0^2(x)) K(\lambda(x - \overline{X}_1))\big)^2 r^2(x)\, dx.$$

The first integral $\le c_{21} \frac{\lambda^p}{n}$, while the second converges, by virtue of the Lebesgue theorem to $\int V^2(x) r(x) dx$.

Thus

$$E \int V_n^2(x) r^2(x)\, dx \to \int V^2(x) r^2(x)\, dx. \tag{6.21}$$

Using (6.20) and (6.21) in (6.19), we complete the proof of (6.18).
Thus

$$\sigma_n^2 \xrightarrow{P} \sigma^2. \tag{6.22}$$

With (6.17) and (6.22) taken into account, Theorem 6.2 gives rise to

Theorem 6.3. *Let* $E|Y|^{4+\delta} < \infty$, $\delta > 0$. *In the conditions of Theorem* 6.2, *for* $n \to \infty$ *we have*

$$\lambda^{p/2}\sigma_n^{-1}(W_n - \Delta_n) \xrightarrow{d} N(0,1).$$

This theorem enables us to construct a test of level α for checking Hypothesis $H_0 : m(x) = m_0(x)$. For this we should reject Hypothesis H_0 if

$$W_n \geq \widetilde{d}_n(\alpha) = \Delta_n + \lambda^{-p/2}\varepsilon_\alpha\sigma_n.$$

Let us now consider a sequence of alternatives which are "close" to Hypothesis H_0 and have the form

$$H_n \ : \ \overline{m}_n(x) = m_0(x) + \gamma_n\eta(x) + o(\gamma_n),$$

where $\gamma_n \to 0$ in an appropriate manner and $o(\gamma_n)$ is uniform with respect to x. Such regression lines correspond to a sequence of densities close to $f(x,y)$

$$\overline{f}_n(x,y) = f(x,y) + \gamma_n\psi(x,y) + o(\gamma_n).$$

Here $\psi(x,y)$ is defined with respect to $\eta(x)$; we can take $\psi(x,y) = \eta(x) \times \int f(x,y)\,dy \cdot \eta_1(y)$, where $\int \eta_1(y)dy = 0$, $\int y\eta_1(y)dy = 1$.

Let $\overline{f}_{0n}(x)$, $\overline{\varphi}_{1n}(x)$ and $\overline{\varphi}_{2n}(x)$ denote respectively $f_0(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ calculated with respect to the density $\overline{f}_n(x,y)$.

Theorem 6.4. *Let* $\overline{f}_{0n}(x)$, $\overline{\varphi}_{1n}(x)$ *and* $\overline{\varphi}_{2n}(x)$ *satisfy the conditions of Theorem* 6.2. *If* $\gamma_n = n^{-\frac{1}{2}+\frac{p\delta}{4}}$ *and* $\lambda = n^\delta$, $\frac{2}{p+8} < \delta < \frac{2}{3p}$, $1 \leq p \leq 3$, *then for* $n \to \infty$

$$P_{H_n}(W_n \geq d_n(\alpha)) \to 1 - \Phi\Big(\varepsilon_\alpha - \frac{1}{\sigma}\int \eta^2(x)f_0^2(x)r(x)\,dx\Big).$$

Proof. We have

$$\begin{aligned} W_n = {}& \frac{n}{\lambda^p}\int \big(\varphi_n(x) - \overline{m}_n(x)f_n(x)\big)^2 r(x)\,dx + \\ & + 2\frac{n}{\lambda^p}\int \big(\varphi_n(x) - \overline{m}_n(x)f_n(x)\big)\big(\overline{m}_n(x) - m_0(x)\big)f_n(x)r(x)\,dx + \\ & + \frac{n}{\lambda^p}\int \big(\overline{m}_n(x) - m_0(x)\big)^2 f_n^2(x)r(x)\,dx = W_n^{(1)} + W_n^{(2)} + W_n^{(3)}. \end{aligned}$$

By Theorem 6.2, the random variable $\lambda^{p/2}(W_k^{(1)}-\Delta)$ is distributed in the limit normally with $N(0,\sigma)$. Further, with our choice of λ and γ_n, it follows that

$$\lambda^{p/2}W_n^{(3)}=\int \eta^2(x)f_0(x)r(x)\,dx+o_p(1).$$

Further, since $E(\lambda^p(Y_1-\overline{m}_n(x))K(\lambda(x-\overline{X}_1))=O(\lambda^{-2})$, we have

$$\lambda^{p/2}W_n^{(2)}=n^{1/2}\lambda^{-p/2}n^{\delta p/4}\int\Big(\frac{\lambda^p}{n}\sum_{j=1}^n\alpha_j(x)\Big)f_n(x)\eta(x)r(x)\,dx+ \\ +O_p\big(n^{\frac{2-(8+p)\delta}{4}}\big), \tag{6.23}$$

where

$$\alpha_j(x)=\eta_j(x)-E\eta_j(x),\quad \eta_j(x)=(Y_j-\overline{m}_n(x))K(\lambda(x-X_j)).$$

It is easy to calculate that

$$E\Big(\frac{\lambda^p}{n}\sum_{j=1}^n\alpha_j(x)\Big)^2\le c_{22}\,\frac{\lambda^p}{n}\,,\quad E\big(f_n(x)-Ef_n(x)\big)^2\le c_{23}\,\frac{\lambda^p}{n}.$$

Now (6.23) implies

$$\lambda^{p/2}W_n^{(2)}=n^{1/2}\lambda^{-p/2}n^{\delta p/4}\int\Big[\frac{\lambda^p}{n}\sum_{j=1}^n\alpha_j(x)\Big]Ef_n(x)\eta(x)r(x)\,dx+ \\ +O_p\big(n^{\frac{2-(8+p)\delta}{4}}\big)+O_p\big(n^{\frac{3p\delta-2}{4}}\big). \tag{6.24}$$

Further, a simple calculation of the dispersion of the integral in (6.4) shows that $\lambda^{p/2}W_n^{(2)}=o_p(1)$. Thus the theorem is proved. □

Remark. In the conditions of Theorem 6.4 one can prove similarly to the above that

$$P_{H_n}\{W_n\ge\widetilde{d}_n(\alpha)\}=1-\Phi\Big(\varepsilon_\alpha-\frac{1}{\sigma}\int\eta^2(x)f_0^2(x)r(x)\,dx\Big).$$

Since the integral $\int\eta^2(x)f_0^2(x)r(x)\,dx$ is equal to zero if and only if $\eta(x)=0$, it follows that the test for checking Hypothesis $H_0: m(x)=m_0(x)$ against the alternatives H_n is asymptotically strictly unbiased.

7. R–2 – **Regressogram.** Let A_i and $I_i(x)$, $0 \le x \le 1$, $i = \overline{1, \lambda}$, be defined as in D–3. Let $\delta_{0\lambda}(x, X_i) = \lambda \sum\limits_{j=1}^{\lambda} I_j(x) I_j(\overline{X}_i) \cdot Y_i$, $\delta_{1\lambda}(x, X_i) = \lambda \sum\limits_{j=1}^{\lambda} I_j(x) I_j(\overline{X}_i)$, where $X_i = (\overline{X}_i, Y_i)$. Then, as noted in § 2, from (2.1) we obtain the regressogram

$$g_n(x) = \frac{1}{N_j} \sum_{(\overline{X}_i \in A_j)} Y_i, \quad x \in A_j, \quad j = \overline{1, \lambda}, \tag{7.1}$$

where N_j is the number of $\overline{X}$'s occurring in A_j.

Let, further, $f_0(x) \equiv \varphi_0(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ be defined like in § 6, and $g(x) \equiv m(x) = E(Y | \overline{X} = x)$.

Assume that $f_0(x)$, $\varphi_1(x)$ and $\varphi_2(x)$, $x \in [0, 1]$, have a bounded derivative of first order. The functions $m^k(x) r(x)$, $k = 0, 1, 2$, and $\varphi_4(x)$ are bounded, and $EY^4 < \infty$.

In this case, conditions 1°–4° of Theorem 2.1 are fulfilled for $\alpha = 1$ and $\beta = 1$.

Indeed,

$$1^\circ.\ E\Big(\int \delta_{0\lambda}^2(x, X_1) r(x)\, dx\Big)^\gamma =$$

$$= \lambda^{2\gamma} E\bigg(Y_1^{2\gamma} \Big(\sum_{j=1}^{\lambda} I_j(\overline{X}_j) \int I_j(x) r(x)\, dx\Big)^\gamma\bigg) \le$$

$$\le c_1 \lambda^\gamma E Y_1^{2\gamma}, \quad \gamma = 1, 2,$$

and also

$$E\Big(\int \delta_{1\lambda}^2(x, X_1) r(x)\, dx\Big)^\gamma \le c_2 \lambda^\gamma.$$

2°. Let us show that

$$E\big(\delta_{ij}^s(X_1, X_2, \lambda)\big) \le c_3 \lambda^{s-1}, \quad s = 2, 4, \quad i, j = 0, 1.$$

a) Let $i = 0$, $j = 1$. Then

$$E\big(\delta_{01}^s(X_1, X_2, \lambda)\big) = E\Big(\int \delta_{0\lambda}(x, X_1) \delta_{1\lambda}(x, X_2) m(x) r(x)\, dx\Big)^s =$$

$$= \lambda^{2s} E\bigg(Y_1 \sum_{j=1}^{\lambda} \int I_j(x) m(x) r(x)\, dx\, I_j(\overline{X}_1) I_j(\overline{X}_2)\bigg)^s \le$$

$$\le c_4 \lambda^s E\Big(Y_1^s \sum I_j(\overline{X}_1) I_j(\overline{X}_2)\Big)^s =$$

$$= c_4 \lambda^s E\bigg(Y_1^s \sum_{j=1}^{\lambda} I_j(\overline{X}_1) I_j(\overline{X}_2)\bigg) \le$$

$$\le c_5 \lambda^{s-1} E Y_1^s.$$

b) Let $i=0$, $j=0$. Then

$$E\big(\delta_{00}^s(X_1,X_2,\lambda)\big)=E\Big(\int\delta_{0\lambda}(x,X_1)\delta_{0\lambda}(x,X_2)r(x)\,dx\Big)^s=$$
$$=\lambda^{2s}E\bigg(Y_1^sY_2^s\bigg(\int\sum_{j=1}^{\lambda}I_j(x)I_j(\overline{X}_1)I_j(\overline{X}_2)\bigg)^s\bigg)\le$$
$$\le c_6\lambda^sE\bigg(Y_1^sY_2^s\bigg(\sum_{j=1}^{\lambda}I_j(\overline{X}_1)I_j(\overline{X}_2)\bigg)^s\bigg)=$$
$$=c_6\lambda^s\sum_{j=1}^{\lambda}\Big(\int y^sI_j(x)f(x,y)\,dx\,dy\Big)^2=$$
$$=c_6\lambda^s\sum_{j=1}^{\lambda}\Big(\int I_j(x)\varphi_s(x)\,dx\Big)^2\le c_7\lambda^{s-1}.$$

c) Let $i=1$, $j=1$. Then

$$E\big(\delta_{11}^s(X_1,X_2,\lambda)\big)=E\Big(\int\delta_{1\lambda}(x,X_1)\delta_{1\lambda}(x,X_2)m^2(x)r(x)\,dx\Big)^s\le$$
$$\le c_8\lambda^sE\sum_{j=1}^{\lambda}I_j(\overline{X}_1)I_j(\overline{X}_2)\le c_9\lambda^{s-1}.$$

Now we proceed to proving the validity of condition 3°. Since $\delta_\lambda(x,X_i)=\delta_{1\lambda}(x,X_i)[Y_i-m(x)]$, we have

$$\sigma_n^2=2\iint\big(I_{1\lambda}(u_1,u_2)-I_{2\lambda}(u_1,u_2)\big)^2r(u_1)r(u_2)\,du_1\,du_2,$$

where

$$I_{1\lambda}(u_1,u_2)=E(Y_1-m(u_1))(Y_1-m(u_2))\delta_{1\lambda}(u_1,X_1)\delta_{1\lambda}(u_2,X_1),$$
$$I_{2\lambda}(u_1,u_2)=E(Y_1-m(u_1))\delta_{1\lambda}(u_1,X_1)E(Y_1-m(u_2))\delta_{1\lambda}(u_2,X_1).$$

It is not difficult to see that

$$I_{1\lambda}(u_1,u_2)=\lambda^2\sum_{j=1}^{\lambda}I_j(u_1)I_j(u_2)\bigg[\int_{\Delta_j}\big(\varphi_2(x)-m(u_2)\varphi_1(x)-$$
$$-m(u_1)\varphi_1(x)+m(u_1)m(u_2)f_0(x)\big)\,dx\bigg]$$

and

$$I_{2\lambda}(u_1,u_2)=\lambda^2\sum_{j=1}^{\lambda}I_j(u_1)\bigg[\int_{\Delta_j}\big(\varphi_1(x)-m(x_1)f_0(x)\big)\,dx\bigg]\cdot$$

$$\cdot \sum_{k=1}^{\lambda} I_k(u_2)\Big[\int\limits_{\Delta_k} \big(\varphi_1(x) - m(x_2)f_0(x)\big)\,dx\Big].$$

Hence, after some simple transformations, we obtain

$$2\iint I_{1\lambda}^2(u_1,u_2)r(u_1)r(u_2)\,du_1\,du_2 =$$
$$= 2\lambda^2\sum_{k=1}^{\lambda}\int\limits_{\Delta_k} r(u_2)\,du_2 \int\limits_{\Delta_k} F^2(u_1,u_2)r(u_1)\,du_1 + O(1),$$

where

$$F(u_1,u_2) = \varphi_2(u_1) - m(u_2)\varphi_1(u_1) - m(u_1)\varphi_1(u_1) + m(u_1)m(u_2)f_0(u_1).$$

By the Taylor formula we have

$$\int\limits_{\Delta_k} F^2(u_1,u_2)r(u_1)\,du_1 = \frac{1}{\lambda}\,F^2(u_2,u_2)r(u_2) + O\Big(\frac{1}{\lambda^\alpha}\Big).$$

Therefore

$$2\iint I_{1\lambda}^2(u_1,u_2)r(u_1)r(u_2)\,du_1\,du_2 =$$
$$= 2\lambda\int F^2(u_2,u_2)r^2(u_2)\,du_2 + O(1) =$$
$$= 2\lambda\int \mathcal{D}^2(Y|X=x)f_0^2(x)r^2(x)\,dx + O(1).$$

Further, in quite a simple manner it is shown that

$$\iint I_{1\lambda}(u_1,u_2)I_{2\lambda}(u_1,u_2)r(u_1)r(u_2)\,du_1\,du_2 = O(1),$$
$$\int I_{2\lambda}^2(u_1,u_2)r(u_1)r(u_2)\,du_1\,du_2 = O(1).$$

Hence

$$\lambda^{-1}\sigma_n^2 \to \sigma^2 = 2\int \mathcal{D}^2(Y|X=x)f_0^2(x)r^2(x)\,dx. \tag{7.2}$$

Therefore $\sigma_n^2 \to \infty$ for $n \to \infty$. It remains to check the second part of condition 3°, i.e., to show that $L_n = 0(\sigma_n^4)$.

Since $\delta_{1\lambda}(x,y)$ is majorized by the kernel $\lambda K(\lambda(x-y))$, where $K(x)$ is the indicator of the interval $[-1,1]$, i.e.,

$$\delta_{1\lambda}(x,y)|y-m(x)| \le \lambda K(\lambda(x-y))|y-m(x)|,$$

we establish, similarly to relation (6.10), that

$$L_n \le c_{10}\lambda.$$

Hence, by virtue of (7.1), it follows that

$$\frac{L_n}{\sigma_n^4} \le c_{11}\lambda^{-1},$$

and therefore condition 3° is fulfilled. Further, with (7.2) taken into account, condition 4° takes the form $\frac{\lambda}{n} \to 0$ as $n \to \infty$.

The following theorem is thus valid.

Theorem 7.1. *Let the above-given conditions be fulfilled for $f_0(x)$, $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_4(x)$ and $m(x)$. If, in addition to this, $EY^4 < \infty$ and $\frac{\lambda}{n} \to 0$ as $n \to \infty$, then*

$$\lambda^{1/2}\sigma^{-1}(\overline{U}_n - \overline{\Delta}_n) \xrightarrow{d} N(0,1),$$

where

$$\overline{U}_n = \frac{n}{\lambda}\int \big(\widehat{g}_n(x) - E\widehat{g}_n(x)\big)^2 r(x)\,dx, \quad \overline{\Delta}_n = E\overline{U}_n,$$

$$\widehat{g}_n(x) = \frac{1}{n}\sum_{i=1}^n \delta_{1\lambda}(x, \overline{X}_i)[Y_i - m(x)].$$

Assume that

$$W_n = \frac{n}{\lambda}\int \widehat{g}_n^2(x) r(x)\,dx = \frac{n}{\lambda}\int \big(g_n(x) - m(x)\big)^2 \widehat{f}_n^2(x) r(x)\,dx,$$

where $g_n(x)$ is estimator (7.1) and $\widehat{f}_n(x)$ is a histogram.

Theorem 7.2. *Let the conditions of Theorem* 7.1 *be fulfilled. Furthermore, if $n\lambda^{-5/2} \to 0$ as $n \to \infty$, then*

$$\lambda^{1/2}\sigma^{-1}(W_n - \Delta) \xrightarrow{d} N(0,1),$$

where

$$\Delta = \int \mathcal{D}(Y|X=x) f_0(x) r(x)\,dx,$$

$$\sigma^2 = 2\int \mathcal{D}(Y|X=x) f_0^2(x) r^2(x)\,dx.$$

Proof. We have

$$\lambda^{1/2}(W_n - \overline{U}_n) = 2n\lambda^{-1/2}\int \big(\widehat{g}_n(x) - E\widehat{g}_n(x)\big) E\widehat{g}_n(x) r(x)\,dx + \\ + n\lambda^{-1/2}\int \big(E\widehat{g}_n(x)\big)^2 r(x)\,dx = \\ = A_{1n} + A_{2n}.$$

It is easy to see that

$$|E\widehat{g}_n(x)| = \Big|\lambda\sum_{j=1}^{\lambda} I_j(x)\Big[\int\limits_{\Delta_j}\varphi_1(x)\,dt - m(x)\lambda\sum_{j=1}^{\lambda} I_j(x)\int\limits_{\Delta_j} f_0(t)\,dt\Big]\Big| \le c_{12}\lambda^{-1},$$

and therefore

$$A_{2n} = n\lambda^{-1/2}\int \big(E\widehat{g}_n(x)\big)^2 r(x)\,dx = O(n\lambda^{-5/2}). \tag{7.3}$$

Further,

$$\begin{aligned}E|A_{1n}| &= 2n\lambda^{-1/2}E\Big|\int\big(\widehat{g}_n(x)-E\widehat{g}_n(x)\big)E\widehat{g}_n(x)r(x)\,dx\Big| \le\\ &\le 2n\lambda^{-1/2}\Big\{\frac{1}{n}E\Big(\int\delta_{1\lambda}(x,X_1)[Y_1-m(x)]E\widehat{g}_n(x)r(x)\,dx\Big)^2\Big\}^{1/2} \le\\ &\le c_{13}\sqrt{n}\lambda^{-3/2}. \end{aligned}\tag{7.4}$$

By (7.3) and (7.4) we have

$$\begin{aligned}\lambda^{1/2}E|W_n-\overline{U}_n| &\le c_{14}\big(\sqrt{n}\,\lambda^{-3/2}+n\lambda^{-5/2}\big) =\\ &= c_{14}(n\lambda^{-5/2})^{1/2}\big((n\lambda^{-5/2})^{1/2}+\lambda^{-1/4}\big)\to 0.\end{aligned}$$

From this and Theorem 7.1 it follows that

$$\lambda^{1/2}\Big(\frac{W_n-\overline{\Delta}_n}{\sigma}\Big)\xrightarrow{d} N(0,1).$$

To complete the proof of the theorem, it remains to show that

$$\lambda^{1/2}(E\overline{U}_n-\Delta)\to 0.$$

We have

$$\overline{\Delta}_n = E\overline{U}_n =$$
$$= \frac{1}{\lambda}\int\big[E\delta^2_{1\lambda}(x,X_1)(Y_1-m(x))^2-(E\delta_{1\lambda}(x,X_1)(Y_1+m(x)))^2\big]r(x)\,dx.$$

Hence, by the Taylor formula, we obtain

$$\begin{aligned}\overline{\Delta}_n &= \frac{1}{\lambda}\int\Big\{\lambda\big[\varphi_2(x)-2m(x)\varphi_1(x)+m^2(x)f_0(x)\big]+\\ &\quad + m(x)O(1)+m^2(x)O(1)+O\Big(\frac{1}{\lambda^2}\Big)\Big\}r(x)\,dx =\\ &= \int\big[\varphi_2(x)-2m(x)\varphi_1(x)+m^2(x)f_0(x)\big]r(x)\,dx+O\Big(\frac{1}{\lambda}\Big)+O\Big(\frac{1}{\lambda^3}\Big) =\\ &= \int\mathcal{D}(Y|X=x)f_0(x)r(x)\,dx+O\Big(\frac{1}{\lambda}\Big).\end{aligned}$$

Thus

$$\lambda^{1/2}(\overline{\Delta}_n-\Delta)=O\Big(\frac{1}{\sqrt{\lambda}}\Big)\to 0. \qquad \square$$

Let $r(x)\equiv 1$, $x\in[0,1]$. By (7.1), W_n can also be written as

$$W_n=\frac{\lambda}{n}\sum_{k=1}^{\lambda}\int\limits_{A_k}\Big[\sum_{(\overline{X}_i\in A_k)}(Y_i-N_k m(x)\Big]^2 dx,$$

where N_k is the number of $\overline{X}$ occurring in A_K.

Let, further, t_k, $k=\overline{1,\lambda}$ be the middle of the intervals A_k, $k=\overline{1,\lambda}$, and assume that

$$W_n^*=\frac{1}{n}\sum_{k=1}^{\lambda}\Big(\sum_{(\overline{X}_i\in A_k)}Y_i-N_k m(t_k)\Big)^2.$$

Theorem 7.3. *Let the conditions of Theorem* 7.2 *be fulfilled and, in addition to this,* $m(x)$, $x\in[0,1]$, *have a bounded derivative of first order. If* $\frac{n}{\lambda^{3/2}}\to 0$ *as* $n\to\infty$, *then*

$$\lambda^{1/2}\Big(\frac{W_n^*-\Delta}{\sigma}\Big)\xrightarrow{d} N(0,1).$$

Proof. It is sufficient to show that

$$\sqrt{\lambda}(W_n-W_n^*)\xrightarrow{P} 0.$$

We have

$$W_n-W_n^*=\frac{2}{n}\sum_{k=1}^{\lambda}N_k\sum_{(\overline{X}_i\in A_k)}Y_i\Big(m(t_k)-\lambda\int\limits_{A_k}m(x)\,dx\Big)+$$
$$+\frac{1}{n}\Big(\lambda\sum_{k=1}^{\lambda}N_k^2\int\limits_{A_k}m^2(x)\,dx-\sum_{k=1}^{\lambda}N_k^2 m^2(t_k)\Big)=$$
$$=\Sigma_1+\Sigma_2.$$

It is easy to see that

$$|\Sigma_1|\le c_{15}\,\frac{1}{n\lambda}\sum_{k=1}^{\lambda}N_k\sum_{(\overline{X}_i\in A_k)}|Y_i|$$

and

$$E|\Sigma_1|\le c_{15}\frac{1}{n\lambda}\sum_{k=1}^{\lambda}E\Big(E\Big(N_k\sum_{(\overline{X}_i\in A_k)}|Y_i|\big|N_k\Big)\Big)=$$
$$=c_{15}\frac{1}{n\lambda}\sum_{k=1}^{\lambda}\Big(\sum_{m=0}^{n}E\Big(N_k\sum_{(\overline{X}_i\in A_k)}|Y_i|\big|N_k=m\Big)P\{N_k=m\}\Big)=$$

$$
\begin{aligned}
&= c_{15}\frac{1}{n\lambda}\sum_{k=1}^{\lambda}\Big(\sum_{m=0}^{n} m^2 E|Y_1|P\{N_k=m\}\Big) = \\
&= c_{15}\frac{1}{n\lambda}E|Y_1|\sum_{k=1}^{\lambda}\sum_{m=0}^{n} m^2 P\{N_k=m\} = \\
&= c_{15}E|Y_1|\frac{1}{n\lambda}\sum_{k=1}^{\lambda}\big[np_k(1-p_k)+(np_k)^2\big] \le \\
&\le c_{16}\sum_{k=1}^{\lambda}\Big(n\,\frac{1}{\lambda}+\frac{n^2}{\lambda^2}\Big)\frac{1}{n\lambda} \le c_{17}\Big(\frac{1}{\lambda}+\frac{n}{\lambda^2}\Big), \qquad (7.5)
\end{aligned}
$$

where $p_k = P\{\overline{X}_1 \in A_k\}$.

Let us now consider Σ_2. We have

$$
\begin{aligned}
E|\Sigma_2| &\le \frac{1}{n}\sum_{k=1}^{\lambda} EN_k^2\Big|\lambda\int\limits_{A_k} m^2(x)\,dx - m^2(t_k)\Big| \le \\
&\le c_{17}\,\frac{1}{n\lambda}\sum_{k=1}^{\lambda} EN_k^2 = c_{17}\,\frac{1}{n\lambda}\sum_{k=1}^{\lambda}\big(np_k(1-p_k)+(np_k)\big)^2 \le \\
&\le c_{18}\Big(\frac{1}{\lambda}+\frac{n}{\lambda^{\alpha}}\Big).
\end{aligned}
$$

Therefore

$$
\sqrt{\lambda}\,E|W_n - W_n^*| \le c_{18}\Big(\frac{1}{\sqrt{\lambda}}+\frac{n}{\lambda^{3/2}}\Big) \to 0. \qquad \square
$$

Remark. Limit distribution of a maximal deviation of the regressogram was obtained in [21].

8. R–3 – **Kernel estimator of the regression function connected with the convolution problem.** Let $X = Y + Z$, Y and Z be independent random variables, Z having normal distribution $N(0,\sigma^2)$, where σ is known. By observation data on X it is required to estimate the regression function Y on X, i.e., to estimate $g_0(x) = E(Y|X=x)$. It is easy to see that

$$
g_0(x) = \sigma^2\,\frac{f^{(1)}(x)}{f(x)} + x,
$$

where

$$
f(x) = \frac{1}{\sqrt{2\pi}\sigma}\int \exp\Big(\frac{-(x-u)^2}{2\sigma^2}\Big)p(u)\,du
$$

and $p(x)$ is the distribution density of the random value Y. Denote $g(x) = g_0(x) - x$. Let, further, $\delta_{0\lambda}(x,X_i) = \lambda^2\sigma^2K^{(1)}(\lambda(x-X_i)$, $\delta_{1\lambda}(x,X_i) =$

$\lambda K((x-X_i)\lambda)$. Then from (2.1) we obtain the estimator for $g(x)$ [21, 28]:

$$g_n(x)=\frac{\lambda\sigma^2\sum\limits_{i=1}^n K^{(1)}(\lambda(x-X_i))}{\sum\limits_{i=1}^n K(\lambda(x-X_i))}.$$

We introduce the assumptions:

1. $K(x)\in H_2$ (see § 2) and has an integrable bounded derivative of first order, and $xK^{(1)}(x)\in L_1(-\infty,\infty)$.
2. A weight function $r(x)\geq 0$ is bounded and integrable.
3. $\max\limits_x |g^i(x)r(x)|<\infty$, $i=1,2$, and $g^2(x)r(x)\in L_1(-\infty,\infty)$.

In the cosidered case conditions 1°–4° of Theorem 2.1 are fulfilled for $\alpha=3$ and $\beta=\frac{1}{3}$.

Indeed,

$$1^\circ.\ \ E\Big(\int \delta^2_{0\lambda}(x,X_1)r(x)\,dx\Big)^\gamma=E\Big(\int \lambda^4\big[K^{(1)}(\lambda(x-X_1))\big]^2 r(x)\,dx\Big)^\gamma\leq \leq c_1\lambda^{3\gamma},\quad \gamma=1,2,$$

and also

$$E\Big(\int \delta^2_{1\lambda}(x,X_1)r(x)\,dx\Big)^\gamma\leq c_2\lambda^{3\gamma}.$$

2°. Let us establish that

$$E\big(\delta^s_{ij}(X_1,X_2,\lambda)\big)\leq c_3\lambda^{3s-1},\quad i,j=0,1,\quad s=2,4.$$

a) Let $i=0$, $j=0$. Then

$$E\big(\delta^s_{00}(X_1,X_2,\lambda)\big)=\sigma^{4s}\lambda^{3s}\int\Big(\lambda\int K^{(1)}((x-u)\lambda)K^{(1)}(\lambda(x-v))\cdot \cdot r(x)\,dx\Big)^s f(u)f(v)\,du\,dv\leq \leq c_4\lambda^{3s-1}\int \overline{K}^s(u)\,du,$$

where

$$\overline{K}(x)=\int |K^{(1)}(x-t)|\,|K^{(1)}(t)|\,dt.$$

b) Let $i=0$, $j=1$. Then

$$E\big(\delta^s_{01}(X_1,X_2,\lambda)\big)=\sigma^{2s}\lambda^{2s}\int\Big[\lambda\int K^{(1)}(\lambda(x-u))K(\lambda(x-v))\cdot \cdot g(x)r(x)\,dx\Big]^s f(u)f(v)\,du\,dv\leq \leq c_5\lambda^{2s-1}\int \overline{\overline{K}}^s(u)\,du\leq c_5\lambda^{3s-1}\int \overline{\overline{K}}^s(u)\,du,$$

where

$$\overline{\overline{K}}(x) = \int |K^{(1)}(x-t)| K(t)\, dt.$$

c) Let $i = 1$, $j = 1$. Then

$$E\big(\delta_{11}^s(X_1, X_2, \lambda)\big) = \int \Big[\int \lambda^2 K(\lambda(x-u)) K(\lambda(x-v)) \cdot \\ \cdot g^2(x) r(x)\, dx \Big]^s f(u) f(v)\, du\, dv \le \\ \le c_6 \lambda^{3s-1} \int K_0^s(u)\, du,$$

where

$$K_0(x) = \int K(x-t) K(t)\, dt.$$

Let us now show that condition 3° is valid. Indeed, since $\alpha_\lambda(u,v) = \delta_\lambda(u,v) - E\delta_\lambda(u, X_1)$, we have

$$\sigma_n^2 = \iint \big(\operatorname{cov}(\delta_\lambda(u_1, X_1)\delta_\lambda(u_2, X_1))\big)^2 r(u_1) r(u_2)\, du_1\, du_2 = \\ = 2 \iint \big(E\delta_\lambda(u_1, X_1)\delta_\lambda(u_2, X_1)\big)^2 r(u_1) r(u_2)\, du_1\, du_2 + R_n, \quad (8.1)$$

where

$$R_n = 2 \iint \big(E\delta_\lambda(u_1, X_1) E\delta_\lambda(u_2, X_1)\big)^2 r(u_1) r(u_2)\, du_1\, du_2 - \\ - 4 \iint E\delta_\lambda(u_1, X_1)\delta_\lambda(u_2, X_1) E\delta_\lambda(u_1, X_1) \cdot \\ \cdot E\delta_\lambda(u_2, X_1) r(u_1) r(u_2)\, du_1\, du_2, \\ \delta_\lambda(x, X_i) = \delta_{0\lambda}(x, X_i) - \delta_{1\lambda}(x, X_1) g(x).$$

But, as simple calculations show, with our assumptions 1–3 we have

$$R_n = O(\lambda^4).$$

It is obvious that

$$E\delta_\lambda(u_1, X_1)\delta_\lambda(u_2, X_1) = E\delta_{0\lambda}(u_1, X_1)\delta_{0\lambda}(u_2, X_1) - \\ - E\delta_{0\lambda}(u_1, X_1)\delta_{1\lambda}(u_2, X_1) g(u_2) - \\ - E\delta_{0\lambda}(u_2, X_1)\delta_{1\lambda}(u_1, X_1) g(u_1) + \\ + E\delta_{1\lambda}(u_1, X_1)\delta_{1\lambda}(u_2, X_1) g(u_1) g(u_2) = \\ = A_{1\lambda} + A_{2\lambda} + A_{3\lambda} + A_{4\lambda}.$$

By the definition of $\delta_{0\lambda}(x,y)$ and $\delta_{1\lambda}(x,y)$ and assumptions 1–3 we easily find

$$A_1 = \sigma^4 \lambda^3 f(u_1) K_1((u_2 - u_1)\lambda) + O(\lambda^2),$$

$$A_2 = g(u_2)O(\lambda^2), \quad A_3 = g(u_1)O(\lambda^2),$$
$$A_4 = g(u_1)g(u_2)O(\lambda),$$

where

$$K_1(x) = \int K^{(1)}(x-u)K^{(1)}(u)\,du.$$

Therefore

$$E\delta_\lambda(u_1,X_1)\delta_\lambda(u_2,X_1) = \sigma^4\lambda^3 f(u_1)K_1((u_2-u_1)\lambda) +$$
$$+(g(u_1)+g(u_2))O(\lambda^2) + g(u_1)g(u_2)O(\lambda) + O(\lambda^2).$$

This and (8.1) imply

$$\sigma_n^2 = 2\sigma^8\lambda^6 \iint f^2(u_1)K_1^2(\lambda(u_2-u_1))r(u_1)r(u_2)\,du_1\,du_2 + O(\lambda^4).$$

Therefore

$$\lambda^{-5}\sigma_n^2 \to \sigma_0^2 = 2\sigma^8 \int f^2(u)r^2(u)\,du \int K_1^2(u)\,du. \tag{8.2}$$

Thus $\sigma_n^2 \to \infty$ for $n \to \infty$. It remains to check the second part of condition 3°, i.e.,

$$L_n = o(\sigma_n^4).$$

It is not difficult to check that the following inequality is valid:

$$L_n = E\Big[\iint \alpha_\lambda(u_1,X_1)\alpha_\lambda(u_2,X_2)E\alpha_\lambda(u_1,X_1)\alpha_\lambda(u_2,X_1)\,\cdot$$
$$\cdot\, r(u_1)r(u_2)\,du_1\,du_2\Big]^2 \le$$
$$\le c_7\lambda^{12} \iiiint R_\lambda(u_1,v_1)R_\lambda(u_2,v_2)R_\lambda(u_1,u_2)R_\lambda(v_1,v_2)\,\cdot$$
$$\cdot\, r(u_1)r(u_2)r(v_1)r(v_2)\,du_1\,du_2\,dv_1\,dv_2, \tag{8.3}$$

where

$$R_\lambda(x,y) = \overline{K}(\lambda(x-y)) + \big(|g(x)|+|g(y)|\big)O\Big(\frac{1}{\lambda}\Big) +$$
$$+ |g(x)g(y)|O(\lambda^{-2}) + O\Big(\frac{1}{\lambda}\Big),$$
$$\overline{K}(u) = \int |K^{(1)}(u-t)|\,|K^{(1)}(t)|\,dt.$$

(8.3) contains integrals of the form

$$L_\lambda^{(1)} = \iiiint \overline{K}(\lambda(v_1-u_1))\overline{K}(\lambda(v_2-u_2))\overline{K}(\lambda(u_2-u_1))\overline{K}(\lambda(v_2-v_1))\,\cdot$$
$$\cdot\, r(u_1)r(u_2)r(v_1)r(v_2)\,du_1\,du_2\,dv_1\,dv_2,$$

$$L_\lambda^{(2)} = \iiiint \overline{K}(\lambda(v_1-u_1))\overline{K}(\lambda(u_2-u_1))\overline{K}(\lambda(v_2-v_1))\cdot \\ \cdot\big[(|g(u_2)|+|g(v_2)|)O(\lambda^{-1})+|g(u_2)g(v_2)|O(\lambda^{-2})+O(\lambda^{-1})\big]\cdot \\ \cdot r(u_1)r(u_2)r(v_1)r(v_2)\,du_1\,du_2\,dv_1\,dv_2,$$

$$L_\lambda^{(3)} = \iiiint \overline{K}(\lambda(v_1-u_1))\overline{K}(\lambda(v_2-v_1))\big[(|g(u_2)|+|g(v_2)|)O(\lambda^{-1})+ \\ +|g(u_2)g(v_2)|O(\lambda^{-2})+O(\lambda^{-1})\big]\big[(|g(u_1)|+|g(u_2)|)O(\lambda^{-1})+ \\ +|g(u_1)g(u_2)|O(\lambda^{-2})+O(\lambda^{-1})\big]r(u_1)r(u_2)\cdot \\ \cdot r(v_1)r(v_2)\,du_1\,du_2\,dv_1\,dv_2,$$

$$L_\lambda^{(4)} = \iiiint \overline{K}(\lambda(v_2-v_1))\big[(|g(u_1)|+|g(v_1)|)O(\lambda^{-1})+ \\ +|g(u_1)g(v_1)|O(\lambda^{-2})+O(\lambda^{-1})\big]\big[(|g(u_2)|+|g(v_2)|)O(\lambda^{-1})+ \\ +|g(u_2)g(v_2)|O(\lambda^{-2})+O(\lambda^{-1})\big]\big[(|g(u_1)|+|g(u_2)|)O(\lambda^{-1})+ \\ +|g(u_1)|\,|g(u_2)|O(\lambda^{-2})+O(\lambda^{-1})\big]\cdot \\ \cdot r(u_1)r(u_2)r(v_1)r(v_2)\,du_1\,du_2\,dv_1\,dv_2,$$

$$L_\lambda^{(5)} = \iiiint \big[(|g(u_1)|+|g(v_1)|)O(\lambda^{-1})+|g(u_1)g(v_1)|O(\lambda^{-2})+ \\ +O(\lambda^{-1})\big]\big[(|g(u_2)|+|g(v_2)|)O(\lambda^{-1})+|g(u_2)g(v_2)|O(\lambda^{-2})+ \\ +O(\lambda^{-1})\big]\big[(|g(u_1)|+|g(u_2)|)O(\lambda^{-1})+|g(u_1)g(u_2)|O(\lambda^{-2})+ \\ +O(\lambda^{-1})\big]\big[(|g(v_1)|+|g(v_2)|)O(\lambda^{-1})+|g(v_1)g(v_2)|O(\lambda^{-2})+ \\ +O(\lambda^{-1})\big]r(u_1)r(u_2)r(v_1)r(v_2)\,du_1\,du_2\,dv_1\,dv_2.$$

By assumptions 1–3 and simple calculations we have

$$L_\lambda^{(1)} = O(\lambda^{-3}),\quad L_\lambda^{(2)} = O(\lambda^{-4}),\quad L_\lambda^{(3)} = O(\lambda^{-4}), \\ L_\lambda^{(4)} = O(\lambda^{-4}),\quad L_\lambda^{(5)} = O(\lambda^{-4}).$$

Therefore

$$L_n \le c_8\lambda^9.$$

But

$$\frac{L_n}{\sigma_n^4} \le \frac{c_8\lambda^9}{(\lambda^{-5}\sigma_n^2)^2\lambda^{10}} = c_9\lambda^{-1} \to 0.$$

Thus condition 3° is completely fulfilled. Further, with condition (8.2) taken into account, condition 4° takes the form

$$\frac{\lambda}{n} \to 0 \quad \text{pri} \quad n\to\infty.$$

The fulfilment of conditions 1°–4° makes it possible to formulate the following theorem.

Theorem 8.1. *Let assumptions* 1–3 *be fulfilled. If* $\frac{\lambda}{n} \to 0$ *for* $n \to \infty$*, then*

$$\lambda^{1/2}\Big(\frac{\overline{U}_n - \Delta_n}{\sigma_0}\Big) \xrightarrow{d} N(0,1),$$

where

$$\overline{U}_n = \frac{n}{\lambda^3}\int \big(\widehat{g}_n(x) - E\widehat{g}_n(x)\big)^2 r(x)\,dx, \quad \Delta_n = E\overline{U}_n,$$

$$\widehat{g}_n(x) = \frac{1}{n}\sum_{j=1}^{n}\big[\sigma^2\lambda^2 K^{(1)}(\lambda(x-X_j)) - \lambda K(\lambda(x-X_j))g(x)\big].$$

Denote

$$U_n^* = \frac{n}{\lambda^3}\int \widehat{g}_n^2(x)r(x)\,dx = \frac{n}{\lambda^3}\int \big(g_n(x)-g(x)\big)^2 f_n^2(x)r(x)\,dx,$$

where

$$f_n(x) = \frac{\lambda}{n}\sum_{i=1}^{n}K(\lambda(x-X_i)).$$

Theorem 8.2. *Let all conditions of Theorem* 8.1 *be fulfilled. In additition to this,* $K(x) = K(-x)$ *and* $K^{(1)}(x) \to 0$ *as* $|x| \to \infty$*. If* $\frac{\sqrt{n}}{\lambda^3} \to \infty$ *as* $n \to \infty$*, then*

$$\lambda^{1/2}\Big(\frac{U_n^* - \Delta}{\sigma_0}\Big) \xrightarrow{d} N(0,1),$$

where

$$\Delta = \sigma^4\int f(x)r(x)\,dx\int [K^{(1)}(x)]^2 dx,$$

$$\sigma_0^2 = 2\sigma^8\int f^2(x)r^2(x)\,dx\int K_1^2(x)\,dx.$$

Proof. Using the equalities (see (5.3), [21])

$$E\lambda^2 K^{(1)}((x-X_1)\lambda) = \lambda\int K(\lambda(x-t))f^{(1)}(t)\,dt,$$

it can be easily established that

$$E\widehat{g}_n(x) = \sigma^2 f^{(1)}(x) - g(x)f(x) + O(\lambda^2) = O(\lambda^{-2}),$$

and therefore

$$\frac{n}{\lambda^3}\int \big(E\widehat{g}_n(x)\big)^2 r(x)\,dx \le c_{10}\,\frac{n}{\lambda^7},$$

$$\frac{n}{\lambda^3}E\Big|\int \widehat{g}_n(x)E\widehat{g}_n(x)r(x)\,dx\Big| \le c_{11}\,\frac{\sqrt{n}}{\lambda^{7/2}}.$$

Hence

$$\lambda^{1/2}E|U_n^* - \overline{U}_n| \le c_{12}\Big(\frac{\sqrt{n}}{\lambda^3} + \frac{n}{\lambda^{13/2}}\Big) \to 0.$$

This and Theorem 8.1 imply

$$\lambda^{1/2}\Big(\frac{U_n^* - \Delta}{\sigma_0}\Big) \xrightarrow{d} N(0,1).$$

To complete the proof of the theorem, it remains to show that

$$\lambda^{1/2}(\Delta_n - \Delta) \to 0.$$

We have

$$\Delta_n = E\overline{U}_n = \frac{1}{\lambda^3}\int E\delta_\lambda^2(x, X_1)r(x)\,dx - \frac{1}{\lambda^3}\int \big(E\delta_\lambda(x, X_1)\big)^2 r(x)\,dx = \\ = E_\lambda^{(1)} + E_\lambda^{(2)},$$

where

$$\delta_\lambda(x, X_1) = \sigma^2\lambda^2 K^{(1)}((x - X_1)\lambda) - \lambda K(\lambda(x - X_i))g(x).$$

Now it is not difficult to see that

$$E_\lambda^{(1)} = \sigma^2\int f(x)r(x)\,dx\int [K^{(1)}(x)]^2 dx + O\Big(\frac{1}{\lambda}\Big),$$

$$E_\lambda^{(2)} = O\Big(\frac{1}{\lambda}\Big).$$

Therefore

$$\lambda^{1/2}(\Delta_n - \Delta) = O\Big(\frac{1}{\sqrt{\lambda}}\Big). \quad \square$$

9. H–1 – **Kernel estimator.** Let X_i, $i = 1, 2, \dots$, be a sequence of independent, identically distributed random variables with values in R_1. Let their common distribution function $F(x)$ be absolutely continuous, and $f(x)$ be the corresponding distribution density. As said above, the hazard function $h(x)$ is defined by the equality

$$g(x) \equiv h(x) = \frac{f(x)}{1 - F(x)}\,.$$

Let $\delta_{0\lambda}(x, X_i) = \lambda K(\lambda(x - X_i))$, $\delta_{1\lambda}(x, X_i) = 1 - \mathcal{K}(\lambda(x - X_i))$, where $\mathcal{K}(u) = \int\limits_{-\infty}^{u} K(t)dt$.

Then (2.1) gives the class of Watson–Leadbetter estimators for $h(x)$:

$$g_n(x) = \frac{\frac{\lambda}{n}\sum\limits_{i=1}^{n}\mathcal{K}(\lambda(x - X_i))}{1 - \frac{1}{n}\sum\limits_{i=1}^{n}\mathcal{K}(\lambda(x - X_i))}\,.$$

We make the following assumptions:

1. $K(x) \in H_2$ (see § 3).
2. $f(x)$ is bounded and has bounded derivatives up to second order.
3. The weight function $r(x)$ is bounded and integrable.
4. $\max\limits_x h(x)r(x) < \infty$ and $h^2(x)r(x) \in L_1(-\infty,\infty)$.

Conditions 1°–4° of Theorem 2.1 are fulfilled for $\alpha = \beta = 1$.

Indeed, first we note that it the check of conditions 1° and 2° is easy and hence it is left out. It is likewise easy to establish the validity of condition 3°.

Since

$$\begin{aligned}\alpha_\lambda(x, X_i) = \lambda K((x - X_i)\lambda) - (1 - \mathcal{K}(\lambda(x - X_i)))h(x) - \\ - E\big[\lambda K(\lambda(x - X_i)) - (1 - \mathcal{K}(\lambda(x - X_i)))h(x)\big],\end{aligned}$$

simple calculations show that

$$\begin{aligned}\sigma_n^2 = 2 \iint \Big[\lambda f(u_1)K_0(\lambda(u_2 - u_1)) - \big(f(u_1)h(u_1) + f(u_2)h(u_2)\big) \cdot \\ \cdot \int K(v)(1 - \mathcal{K}(\lambda(u_2 - u_1) - v))dv + \\ + O(1)h(u_1)h(u_2) + O(1)\Big]^2 r(u_1)r(u_2)\, du_1\, du_2 + O(1).\end{aligned}$$

Hence we obtain

$$\lambda^{-1}\sigma_n^2 = 2 \iint \lambda K_0^2(\lambda(u_2 - u_1))f^2(u_1)r(u_1)r(u_2)\, du_1\, du_2 + O\Big(\frac{1}{\lambda}\Big).$$

Therefore, by virtue of the Lebesgue theorem on majorized convergence, we have

$$\lambda^{-1}\sigma_n^2 \to \sigma^2 = 2 \int f^2(x)r^2(x)\, dx \int K_0^2(t)\, dt, \quad K_0 = K * K. \tag{9.1}$$

Thus $\sigma_n^2 \to \infty$. Now let us check the second part of condition 3°.

Denote

$$H_\lambda(x, X_i) = K(\lambda(x - X_i)) - \frac{1}{\lambda}\big(1 - \mathcal{K}(\lambda(x - X_i))h(x)\big).$$

Then

$$\begin{aligned}L_n = E\Big[\iint \alpha_\lambda(u_1, X_1)\alpha_\lambda(u_2, X_2)E\alpha_\lambda(u_1, X_1)\alpha_\lambda(u_2, X_2)r(u_1)r(u_2)du_1du_2\Big]^2 = \\ = \lambda^8 \iiiint \operatorname{cov}\big(H_\lambda(u_1, X_1), H_\lambda(v_1, X_1)\big) \operatorname{cov}\big(H_\lambda(u_2, X_2), \\ H_\lambda(v_2, H_2)\big) \operatorname{cov}\big(H_\lambda(u_1, X_1), H_\lambda(u_2, X_1)\big) \cdot \\ \cdot \operatorname{cov}\big(H_\lambda(v_1, X_1), H_\lambda(v_2, X_1)\big) r(u_1)r(u_2) \cdot r(v_1)r(v_2)\, du_1\, du_2\, dv_1\, dv_2.\end{aligned}$$

Since

$$\int |H_\lambda(u,t)|\,|H_\lambda(v,t)|f(t)\,dt \le \int K(\lambda(u-t))K(\lambda(v-t))f(t) + c_1\,\frac{1}{\lambda^2} \le$$
$$\le c_2\lambda^{-1}K_0(\lambda(u-v)) + c_1\,\frac{1}{\lambda^2}\,,$$

and

$$EH_\lambda(u,X_1) \le c_3\,\frac{1}{\lambda}\,(1+h(u)),$$

by simple calculations we obtain

$$L_n \le c_4\lambda^4 \iiiint \big[K_0(\lambda(u_1-v_1)) + \lambda^{-1}(1+h(u_1))\big]\cdot$$
$$\cdot\big[K_0(\lambda(u_2-v_2)) + \lambda^{-1}(1+h(v_2))\big]\cdot\big[K_0(\lambda(u_1-u_2)) + \lambda^{-1}(1+h(u_2))\big]\cdot$$
$$\cdot\big[K_0(\lambda(v_1-v_2)) + \lambda^{-1}(1+h(v_1))\big]\cdot r(u_1)r(u_2)r(v_1)r(v_2)\,du_1\,du_2\,dv_1\,dv_2 \le$$
$$\le c_4\lambda.$$

But

$$\frac{L_n}{\sigma_n^4} = c_4\,\frac{\lambda}{(\lambda^{-1}\sigma_n^2)^2\lambda^2} \le c_5\,\frac{1}{\lambda} \to 0.$$

Thus condition 3° is fulfilled. Further, with (9.1) taken into account, condition 4° takes the form

$$\frac{\lambda}{n} \to 0 \quad \text{as} \quad n\to\infty.$$

Combining all the statements obtained above, we come to the following theorem.

Theorem 9.1. *Let assumptions* 1–4 *be fulfilled. If* $\frac{\lambda}{n}\to 0$ *for* $n\to\infty$, *then*

$$\lambda^{1/2}\Big(\frac{\overline{U}_n-\Delta_n}{\sigma}\Big) \xrightarrow{d} N(0,1),$$

where

$$\overline{U}_n = \frac{n}{\lambda}\int \big(\widehat{g}_n(x) - E\widehat{g}_n(x)\big)^2 r(x)\,dx, \quad \Delta_n = E\overline{U}_n,$$
$$\widehat{g}_n(x) = \frac{1}{n}\sum_{i=1}^n \big[\lambda K(\lambda(x-X_i)) - (1-\mathcal{K}(\lambda(x-X_i)))h(x)\big].$$

Theorem 9.2. *Let the conditions of Theorem* 9.1 *be fulfilled and, in addition to this,* $K(x)=K(-x)$ *and* $f^{(2)}\in L_1(-\infty,\infty)$. *If* $n\lambda^{-9/2}\to 0$ *for* $n\to\infty$, *then*

$$\sqrt{\lambda}\,\Big(\frac{U_n^{(1)}-\Delta}{\sigma}\Big) \xrightarrow{d} N(0,1), \tag{9.1*}$$

where

$$U_n^{(1)} = \frac{n}{\lambda}\int \widehat{g}_n^2(x)r(x)\,dx = \frac{n}{\lambda}\int \big(g_n(x)-h(x)\big)^2(1-\overline{F}_n(x))^2 r(x)\,dx,$$

$$\Delta = \int f(x)r(x)\,dx \int K^2(x)\,dx,$$

$$\overline{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n}\mathcal{K}(\lambda(x-X_i)).$$

Proof. We have

$$E\widehat{g}_n(x) = E\lambda K(\lambda(x-X_1)) - h(x)(1-E\mathcal{K}((x-X_i)\lambda)) = C_\lambda^{(1)}(x)+C_\lambda^{(2)}(x),$$

and

$$C_n^{(1)}(x) = f(x) + O(\lambda^{-2});$$

also

$$E\mathcal{K}(\lambda(x-X_i)) = F(x) + E\int_{-\infty}^{x}\big[\lambda K((t-X_1)\lambda) - f(t)\big]\,dt.$$

But

$$\left|E\int_{-\infty}^{x}\big[\lambda K((t-X_1)\lambda)-f(t)\big]\,dt\right| \le \int \big|E\lambda K((t-X_1)\lambda)-f(t)\big|\,dt \le$$

$$\le \frac{1}{2\lambda^2}\int x^2K(x)\,dx\int |f^{(2)}(u)\,du|\,.$$

Hence

$$C_n^{(2)}(x) = -h(x)(1-F(x)) + O(\lambda^{-2}).$$

Therefore

$$E\widehat{g}_n(x) = f(x) - h(x)(1-F(x)) + O(\lambda^{-2}) = O(\lambda^{-2}).$$

Hence we find that

$$\lambda^{1/2}E|U_n^{(1)} - \overline{U}_n| \le c_6\sqrt{n}\,\lambda^{-5/2} + c_7 n\lambda^{-9/2} \to 0.$$

Therefore

$$\lambda^{1/2}(U_n^{(1)} - \overline{U}_n) \xrightarrow{P} 0. \tag{9.1}$$

It is obvious that

$$\Delta_n = E\overline{U}_n = \int f(x)r(x)\,dx\int K^2(u)\,du + O\Big(\frac{1}{\lambda}\Big).$$

This and (9.2) eventually give (9.1*). □

Let now $\overline{F}_n(x) \equiv F_n(x)$, i.e., $\mathcal{K}(x) \equiv I(x) = 1$ $(= 0)$ respectively for $x > 0$ $(x \leq 0)$. $F_n(x)$ is an ordinary empirical distribution function, i.e.,

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(x - X_i).$$

Then

$$g_n(x) = \frac{f_n(x)}{1 - F_n(x)}, \quad f_n(x) = \frac{\lambda}{n} \sum_{i=1}^{n} K(\lambda(x - X_i)).$$

By the proofs of Theorems 9.1 and 9.2 we obtain

$$\sqrt{\lambda}\Big(\frac{U_n^{(2)} - \Delta}{\sigma}\Big) \xrightarrow{d} N(0,1),$$

where

$$U_n^{(2)} = \frac{n}{\lambda} \int \big(f_n(x) - h(x)(1 - F_n(x))\big)^2 r(x)\,dx.$$

Let $-\infty \leq a_1 < a_2 < a_F = \inf\{x : F(x) = 1\}$ and $r_0(x)$ be a non-negative bounded function defined in the interval $(a_1, a_2]$. Assume that

$$r(x) = (1 - F(x))^{-2} r_0(x) I_{[a_1, a_2]}(x).$$

Then

$$U_n^{(2)} = \frac{n}{\lambda} \int_{a_1}^{a_2} \big(f_n(x) - h(x)(1 - F_n(x))\big)^2 (1 - F(x))^{-2} r_0(x)\,dx.$$

Denote

$$U_n^* = \frac{n}{\lambda} \int_{a_1}^{a_2} \big(g_n(x) - h(x)\big)^2 r_0(x)\,dx, \quad g_n(x) = \frac{f_n(x)}{1 - F_n(x)}.$$

Further, in the conditions of Theorem 9.2, we show that

$$\sqrt{\lambda}(U_n^{(2)} - U_n^*) \xrightarrow{P} 0.$$

We have

$$|U_n^{(2)} - U_n^*| \leq \frac{4}{(1 - F(a_2))^2} \sup_{x \in R_1} |F_n(x) - F(x)| \cdot \cdot \frac{n}{\lambda} \int_{a_1}^{a_2} \big(g_n(x) - h(x)\big)^2 r_0(x)\,dx. \tag{9.2}$$

In Rice and Rosenblatt [29] it is shown that

$$g_n(x) = \frac{f_n(x)}{1 - F(x)} \Big[1 + O_p\Big(\frac{1}{\sqrt{n}}\Big)\Big].$$

Here $O_p(\cdot)$ is uniform with respect to $x \in (a_1, a_2]$.

Therefore

$$\frac{n}{\lambda}\int_{a_1}^{a_2}\big(g_n(x)-h(x)\big)^2 r_0(x)\,dx \le$$

$$\le \frac{c_8}{(1-F(a_2))^2}\,\frac{n}{\lambda}\int\big(f_n(x)-f(x)\big)^2 dx + O\Big(\frac{1}{\lambda}\Big)\int f_n^2(x)\,dx =$$

$$= \frac{c_8}{(1-F(a_2))^2}\,\eta_{1n} + O_p\Big(\frac{1}{\lambda}\Big)\eta_{2n}, \tag{9.3}$$

where

$$\eta_{1n} = \frac{n}{\lambda}\int\big(f_n(x)-f(x)\big)^2 dx, \quad \eta_{2n} = \int f_n^2(x)\,dx.$$

Since $\eta_{2n} \xrightarrow{P} \int f^2(x)dx$ ([21], [30]) and $E\eta_{1n} \le c_9 + c_{10}n\lambda^{-5}$, i.e., η_{1n} is bounded uniformy with respect to n, and also $\sup\limits_{x\in R_1}|F_n(x)-F(x)| = O_p(\frac{1}{\sqrt{n}})$, from (9.3) and (9.4) we obtain

$$\sqrt{\lambda}(U_n^{(2)} - U_n^*) = O_p\Big(\sqrt{\frac{\lambda}{n}}\Big). \tag{9.4}$$

Thus, by virtue of (9.5), we have established the validity of the following proposition.

Theorem 9.3. *In the conditions of Theorem* 9.2

$$\lambda^{1/2}(U_n^* - \Delta_0)\sigma_0^{-1} \xrightarrow{d} N(0,1),$$

where

$$\Delta_0 = \int_{a_1}^{a_2} h_0(x)r_0(x)\,dx\int K^2(u)\,du,$$

$$\sigma_0^2 = 2\int_{a_1}^{a_2} h_0^2(x)r_0^2(x)\,dx\int K_0^2(u)\,du,$$

$$h_0(x) = f(x)(1-F(x))^{-2}.$$

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