Proceedings of A. Razmadze
Mathematical Institute,
126, 2001

SOLUTION OF A PROBLEM OF THE THEORY OF FILTRATION THROUGH A PLANE EARTH DAM (COFFER-DAM) WHEN WATER DEPTH IN A DOWNSTREAM CAN BE NEGLECTED

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Abstract. Analytic solution of the mixed two-dimensional problem of stationary filtration through a plane earth dam with slopes and partially unknown boundary is given in the paper. Water level in a downstream is equal to zero. A dam base is assumed to be impermeable and porous medium to be isotropic, homogeneous and undeformable. Liquid motion in a porous medium is subjected to the Darcy law. Analytic formulas are obtained for calculation of geometric and mechanical characteristics (parameters). Parametric equation for an unknown portion of the boundary (for depression curve) is derived. In solving the problem the use is widely made of the theory of Fuchs class differential equations and nonlinear Schwartz differential equation.

In the present work we consider two-dimensional problem of the theory of stationary filtration through a plane earth dam (coffer-dam) with slopes, when water depth in a downstream is equal to zero. A boundary of the domain of filtrating liquid consists of an unknown depression curve and known segments of straight lines. Porous medium is assumed to be homogeneous,

2000 *Mathematics Subject Classification.* 34A20, 34B15.
Key words and phrases. Filtration, analytic functions, conformal mapping, differential equation.

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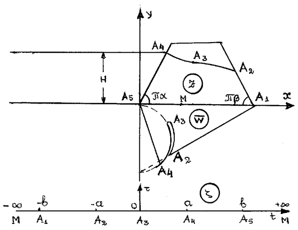

Fig. 1

The complex plane $z = x + iy$ coincides with the plane of liquid motion and we introduce complex planes $\omega(z) = \varphi(x, y) + i\psi(x, y)$, $w(z) = \omega'(z) = d\omega(z)/dz$, $\xi = t + i\tau$, $\overline{w(z)} = \overline{\omega'(z)}$, where $\omega(z)$ and $\omega'(z)$ are, respectively, the reduced complex potential and the complex velocity, $\varphi(x, y)$ is the potential of velocity, $\Psi(x, y)$ is the stream function, $\xi = t + i\tau$ is an auxiliary plane, $\overline{\omega'(z)}$ is the complex conjugate to $w(z) = \omega'(z)$ function. By $s(z)$, $s(\omega)$, $s(w)$ and $s(\overline{w})$ we denote respectively the domains of motion, complex potential, complex velocity and complex conjugate to $s(w)$ domain and by $l(z)$, $l(\omega)$, $l(w)$, $l(\overline{w})$ we denote boundaries corresponding to these domains.

Angular points and the point of inflection of the boundary $l(z)$, which are met when they move around the contour of the domain $s(z)$ in the positive direction are denoted by A_k, $k = \overline{1, 5}$; this notation will be used in the sequel for angular points of the boundaries $l(\omega)$, $l(w)$.

Boundary conditions along the contour $l(z)$ have the form: along the leaking interval $A_1A_2 : \varphi(x, y) + y = 0$, $y = -\operatorname{tg}(\pi\beta)(x - L_1)$ is the water depth in an upstream wail, H the length of the dam base, L_1 is the angle $\pi\beta$ at the point A_1; along an unknown depression curve $A_2A_3A_4 : \varphi(x, y) + y = 0$, $\psi(x, y) = Q$, where Q is liquid discharge for filtration; along the upstream wail $A_4A_5 : \varphi(x, y) = -H$, $y = \operatorname{tg}(\pi\alpha)x$, where $\pi\alpha$ is the angle at the point A_5; along the dam base (coffer-dam) $A_5A_1 : \Psi(x, y) = 0$, $y = 0$. It is proved that the domain $s(w)$ is a circular pentagon (Fig. 1-$\overline{w}$)–[1–15].

The half-plane $\operatorname{Im}(\xi) \geq 0$ is conformally mapped onto the domains $s(z)$, $s(\omega)$, $s(w)$. The mapping functions corresponding to these domains are denoted by $z(\xi) = z_2(\xi)$, $\omega(\xi) = z_1(\xi)$, $w(\xi) = z_1'(\xi)/z_2'(\xi)$, $dz_k(\xi)/d\xi = z_k'(\xi)$, $k = 1, 2$.

To the angular points A_k, $k = \overline{1,5}$ along the axis t there correspond the points $t = a_k$, $k = \overline{1,5}$, provided $-\infty < a_1 < a_2 < a_3 < a_4 < a_5 < +\infty$, where the point $t = \infty$ is mapped into the point M $(Fig.1)$.

To solve the filtration problem we first of all construct the point $w(\xi)$, since the domain $s(w)$ is nearly known, then by means of the function $w(\xi)$ we can construct functions $z(\xi)$ and $\omega(\xi)$ and find an equation for an unknown segment $A_2A_3A_4$ of the boundary $l(z)$, as well as other mechanical parameters of the problem.

By the functions $z(\xi)$ and $\omega(\xi)$ the domain $\mathrm{Im}(\xi) \geq 0$ is conformally mapped onto the domains $s(z)$ and $s(\omega)$, respectively. Consequently, in the domain $\mathrm{Im}(\xi) \geq 0$ we seek for a system of analytic functions $z_2(\xi) = x(t,\tau) + iy(t,\tau)$, $z_1(\xi) = \varphi(t,\tau) + i\Psi(t,\tau)$ which together with their derivatives $z_1'(\xi)$, $z_2'(\xi)$ satisfy the boundary conditions

$$z_k(t) = m_{k1}(t)\overline{z_1(t)} + m_{k2}(t)\overline{z_2(t)} + n_k(t), \ k = 1, 2, \ -\infty < t < +\infty, \qquad (1)$$

$$z_k'(t) = m_{k1}(t)\overline{z_1'(t)} + m_{k2}(t)\overline{z_2'(t)}, \ \ k = 1, 2, \ \ -\infty < t < +\infty, \qquad (2)$$

where m_{kj}, $n_k(t)$, $j, k = 1, 2$ are the given piecewise constant functions with discontinuity points $t = a_j$, $j = \overline{1,5}$; $z_1(t) = \varphi(t) + i\Psi(t)$, $z_2(t) = x(t) + iy(t)$, $\overline{z_1(t)} = \varphi(t) - i\Psi(t)$, $\overline{z_2(t)} = x(t) - iy(t)$, $z_k'(t) = dz_k(t)/dt$.

According to the Riemann theorem, from $t = a_k$, $k = \overline{1,5}$ we select arbitrarily and fix only three points. In our case the parameters a_k, $k = \overline{1,5}$ are fixed as follows: $a_1 = -b$, $a_2 = -a$, $a_3 = 0$, $a_4 = a$ and $a_5 = b$.

Consider an analytic vector $\Phi(\xi) = [\omega(\xi), z(\xi)]$ and its derivative $\Phi'(\xi) = [\omega'(\xi), z'(\xi)]$. By means of the vectors $\Phi(\xi)$ and $\Phi'(\xi)$ we can write boundary conditions (1), (2) as

$$\Phi^+(t) = g(t)\Phi^-(t) + f(t), \quad -\infty < t < +\infty; \qquad (3)$$

$$\Phi'^+(t) = g(t)\Phi^{'+}(t), \quad -\infty < t < +\infty, \qquad (3_1)$$

where $\Phi^{\pm}(t)$ and $\Phi^{'\pm}(t)$ are, respectively, limiting values of $\Phi(\xi)$ and $\Phi'(\xi)$ from the half-planes $\mathrm{Im}(\xi) > 0$ (for the sign $+$) and $\mathrm{Im}(\xi) < 0$ (for the sign $-$).

The matrix $g(t)$ and the vector $f(t) = [n_1(t), n_2(t)]$ are defined as follows:

$$g_\infty(t) = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}, \quad f_{-\infty}(t) = [0, 0], \quad -\infty < t < a_1;$$

$$g_1(t) = \begin{pmatrix} -1, & 2\sin(\pi\beta)\exp(-i\pi\beta) \\ 0, & \exp(-i2\pi\beta) \end{pmatrix},$$

$$f_1(t) = 2L_1 \sin\pi\beta e^{-i\pi\beta}[-1, i], \quad a_1 < t < a_2;$$

$$g_2(t) = \begin{pmatrix} 1, & 0 \\ -2i, & 1 \end{pmatrix}, \quad f_2(t) = 2Q[i, 1], \quad a_2 < t < a_4; \qquad (4)$$

$$g_4(t) = \begin{pmatrix} -1, & 0 \\ 0, & \exp(i2\pi\alpha) \end{pmatrix}, \quad f_4(t) = [-2H, 0], \quad a_4 < t < a_5;$$

$$g_{+\infty}(t) = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}, \quad f_{+\infty}(t) = [0, 0], \quad a_5 < t < +\infty.$$

Consider now characteristic equations for the points $t = a_j$, $j = \overline{1, 5}$:

$$\det \left| g_j^{-1}(a_j + 0) g_{j-1}(a_j - 0) - \lambda E \right| = 0, \tag{5}$$

where λ is parameter, $g_j(a_j + 0)$, $g_{j-1}(a_j - 0)$ are limiting values of the matrix $g(t)$ at the point $t = a_j$, from the right and from the left, respectively; $g_j^{-1}(a_j + 0)$ is the matrix inverse to $g_j(a_j + 0)$, $g_0(t) = g_{-\infty}(t)$.

If we denote characteristic roots of equation (5) for the point $t = a_j$ by λ_{kj}, $k = 1, 2$; $j = \overline{1, 5}$ and then consider the numbers $\alpha_{kj} = (2\pi i)^{-1} \ln \lambda_{kj}$, $k = 1, 2$, $j = \overline{1, 5}$, defined to within integral summands, we will obtain characteristic exponents. The above-mentioned integral summands are selected in such a way that $|\alpha_{1j} - \alpha_{2j}| = \nu_j$, where $\pi\nu_j$, $j = \overline{1, 5}$ are interior angles at vertices of the domain $S(w)$. For the considered by us circular polygon they have the form $\nu_1 = 1/2 - \beta$, $\nu_2 = 1/2 - \beta$, $\nu_3 = 2$, $\nu_4 = 1/2 - \alpha_u \nu_5 = 1/2 - \alpha$.

Characteristic exponents for the functions $\omega'(\xi)$, $z'(\xi)$ have at the angular points $A_j[\alpha_{1j}; \alpha_{2j}]$, $j = \overline{1, 5}$ and at the non-angular point $\xi = \infty$ the form $A_1[-1/2; \beta - 1]$, $A_2[1/2 - \beta, 0]$, $A_3[2; 0]$, $A_4[-1/2; -\alpha]$, $A_5[-1/2; \alpha - 1]$, $A_\infty[\alpha_{1\infty}; \alpha_{2\infty}]$. These exponents must satisfy the Fuchs condition

$$\sum_{k=1}^{5} [\alpha_{1k} + \alpha_{2k}] + \alpha_{1\infty} + \alpha_{2\infty} = 4, \quad \alpha_{1\infty} - \alpha_{2\infty} = 1. \tag{6}$$

It follows from condition (6) that $\alpha_{1\infty} = 3$, $\alpha_{2\infty} = 2$.

By means of the above exponents we make up the Fuchs class differential equation

$$u''(\xi) + \mathcal{P}(\xi) u'(\xi) + Q(\xi) u(\xi) = 0, \tag{7}$$

where

$$\mathcal{P}(\xi) = \sum_{j=1}^{5} [1 - \alpha_{1j} - \alpha_{2j}] (\xi - a_j)^{-1}, \tag{8}$$

$$Q(\xi) = \sum_{j=1}^{5} \left[\alpha_{1j} \alpha_{2j} (\xi - a_j)^{-2} + C_j (\xi - a_j)^{-1} \right]. \tag{9}$$

To an unknown segment A_2 A_3 A_4 of the boundary $\ell(z)$ of $S(z)$ with the point of inflection A_3 on the boundary $\ell(w)$ of the domain $S(w)$ there corresponds a circular cut with an interior angle 2π at the vertex A_3.

Fundamental investigations and methods of constructing analytic solutions of the problems of filtration $z(\xi)$, $\omega(\xi)$, $w(\xi)$ when a number of angular points on the contour $\ell(w)$ of the domain $S(w)$ does not exceed three points, or if they are more than three, but they are (with the exclusion of three points) removable singular points, belong to P. Polubarinova-Kochina. At present it is quite possible to extend the methods of P. Polubarinova-Kochina to the cases, when a number of angular points of the boundary $\ell(w)$ turns out in a general case to be more than three (see [6] and [15]).

Among unknown functions $z(\xi)$, $\omega(\xi)$, $w(\xi)$ the function $w(\xi)$ must satisfy a linear differential Schwartz equation

$$\{w,\xi\} \equiv w'''(\xi)/w'(\xi) - 1,5\big[w''(\xi)/w'(\xi)\big]^2 = R(\xi), \tag{10}$$

where

$$R(\xi) = -\mathcal{P}'(\xi) - \frac{1}{2}\mathcal{P}^2(\xi) + 2Q(\xi). \tag{11}$$

Represent the function $R(\xi)$ in the vicinity of $\xi = \infty$ in the form of a series

$$R(\xi) = \sum_{k=1}^{\infty} B_k \xi^{-k}. \tag{12}$$

If the point $\xi = \infty$ is the image of the non-angular point of the boundary of $S(w)$, then coefficients B_k, $k = 1, 2, 3$ must satisfy the conditions

$$B_1 = 0, \quad B_2 = 0, \quad B_3 = 0. \tag{13}$$

But if we assume that the point $\xi = \infty$ is the image of one of the vertices of the circular polygon, then the first equality $B_1 = 0$ (13) remains valid and the second coefficient $B_2 \neq 0$ takes the form

$$B_2 = (1 - \nu_j^2)/2, \tag{14}$$

where $\pi\nu_j$ is the angle at the vertex which is mapped into $t = \infty$, and the third of conditions (13) is eliminated.

Below we will derive conditions (13) on the basis of equation (7).

To construct linearly independent solutions of equation (7) we have first to construct its linearly independent local solutions in the vicinity of all singular points $t = a_k$, $k = \overline{1,5}$, $t = \infty$ and then connect these solutions uniquely.

As is known, for the points $t = a_j$, $j = \overline{1,5}$ to be regular singular points, for (7) it is necessary and sufficient that equalities (8) and (9) and the condition

$$B_1 = \sum_{k=1}^{5} C_k = 0 \tag{15}$$

to take place.

Equation (7) in the vicinity of singular points $\xi = a_j$ can be written as

$$(\xi - a_j)^2 u''(\xi) + (\xi - a_j) p_j(\xi) u'(\xi) + q_j(\xi) u(\xi) = 0, \tag{16}$$

where

$$p_j(\xi) = p_{0j} + (\xi - a_j) \sum_{k=1,\, k\neq j}^{5} (1 - \alpha_{1k} - \alpha_{2k})(\xi - a_j)^{-1}, \tag{17}$$

$$q_j(\xi) = \alpha_{1j}\alpha_{2j} + C_j(\xi - a_j) +$$
$$+(\xi-a_j)^2 \sum_{k=1,\, k\neq j} \big[\alpha_{1k}\alpha_{2k}(\xi - a_k)^{-2} + C_k(\xi-a_k)^{-1}\big]. \tag{18}$$

We can represent the functions $p_j(t)$, $q_j(t)$ in the vicinity of $\xi = a_j$ in the form

$$p_j(\xi) = \sum_{n=0}^{\infty} p_{nj}(\xi - a_j)^n, \quad p_{_{0j}} = 1 - \alpha_{1j} - \alpha_{2j}, \tag{19}$$

$$p_{nj} = (-1)^{n-1} \sum_{k=1,k\neq j}^{5} [1 - \alpha_{1k} - \alpha_{2k}](a_j - a_k)^{-n},$$

$$q_j(\xi) = \alpha_{1j}\alpha_{2j} + C_j(\xi - a_j) + \sum_{n=2}^{\infty} q_{nj}(\xi - a_j)^n, \tag{20}$$

$$q_{nj} = (-1)^{n-2} \sum_{k=1,k\neq j}^{5} [\alpha_{1k}\alpha_{2k}(n-1) + C_k(a_j - a_k)](a_j - a_k)^{-n}, \tag{21}$$

$$n = 2, 3, \dots .$$

$$q_{0j} = \alpha_{1j}\alpha_{2j}, \quad q_{1j} = C_j. \tag{22}$$

Note that the general solution (10) can be written by means of linearly independent solutions (7) as follows:

$$w(\xi) = [pw_1(\xi) + q]/[rw_1(\xi) + s], \tag{23}$$

where p, q, r and s are constant integrations of equation (10); they are complex numbers in the general case and $ps - rq \neq 0$, while $w_1(\xi) = u_1(\xi)/u_2(\xi)$, where $u_1(\xi)$ $u_2(\xi)$ are linearly independent particular solutions of equation (7).

Next we pass to the construction of local solutions first for the points $t = a_j$, $j = \overline{1,5}$ and then for the point $t = \infty$.

Local solutions of equation (16) for the points $t = a_j$ will be sought in the form

$$u_j(\xi) = (t - a_j)^{\alpha_j} \widetilde{u}_j(\xi), \quad \widetilde{u}_j(\xi) = 1 + \sum_{n=1}^{\infty} \gamma_{nj}(\xi - a_j)^n. \tag{24}$$

For determination of coefficients γ_{nj}, $n = \overline{1,\infty}$, $j = \overline{1,5}$ we have the following recursion formulas:

$$f_0(\alpha_j) = \alpha_j(\alpha_j - 1) + p_{_{0j}}\alpha_j + q_{0j} = 0, \tag{25}$$

$$\gamma_{1j} f_0(\alpha_j + 1) + f_1(\alpha_j) = 0 \tag{26}$$

$$\gamma_{2j} f_0(\alpha_j + 2) + \gamma_{1j} f_1(\alpha_j + 1) + f_2(\alpha_j) = 0 \tag{27}$$

. . . .

$$\gamma_{1j} f_0(\alpha_j + n) + \gamma_{(n-1)j} f_1(\alpha_j + n - 1) + \gamma_{(n-2)j} f_2(\alpha_j + n - 2) + \cdots + \gamma_{1j} f_{(n-1)}(\alpha_j + 1) + f_n(\alpha_j) = 0, \tag{28}$$

$$f_n(\alpha_j) = \alpha_j p_{n_j} + q_{nj}. \tag{29}$$

Determining equation (25) has two roots: α_{1j} and α_{2j}. Since for the points $t = a_k$, $k = 1, 2, 4, 5$ the difference $\alpha_{1j} - \alpha_{2j}$ is not an integral number, using formulas (26)–(28) we construct linearly independent local solutions

$$\begin{aligned} u_{kj}(\xi) &= (\xi - a_j)^{\alpha_{kj}} \widetilde{u}_{kj}(\xi), \\ \widetilde{u}_{kj}(\xi) &= 1 + \sum_{n=1}^{\infty} \gamma_{nj}^{k} (t - a_j)^n, \quad k = 1, 2. \end{aligned} \tag{30}$$

For the point $t = a_3$ we can construct the first solution α_{13} by using formulas (26)–(28), and to construct the second solution for α_{23}, we will act as follows. Because of the fact that $f_0(\alpha_{13}) = f_0(\alpha_{23} + 2) = 0$, equality (27) is not fulfilled. To fulfil (27), it is necessary and sufficient to require the condition [9–15]:

$$\gamma_{13} f_1(\alpha_{23} + 1) + f_2(\alpha_{23}) = 0, \tag{31}$$

which with regard for (26) can be written as

$$q_{13}(q_{13} + p_{13}) + q_{23} = 0, \tag{32}$$

where

$$q_{13} = C_3, \quad q_{23} = \sum_{k=1, k\neq 3}^{5} \big[\alpha_{1k}\alpha_{2k} + C_k(a_3 - a_k)\big](a_3 - a_k)^{-2}, \tag{33}$$

$$p_{13} = \sum_{k=1, k\neq 3}^{5} \big[1 - \alpha_{1k} - \alpha_{2k}\big](a_3 - a_k)^{-1}. \tag{34}$$

To construct at the point $\xi = a_3$ the second solution $u_{23}(\xi)$ we have first to find the coefficient γ_{23}^2, and then after having defined it we can find all γ_{n3}^2, $n = 1, 3, 4, \ldots$ by formulas (26) and (28). For the unique definition of $\gamma_{23}62$ we will act as follows. For $\alpha_j \neq 0$, from (27) we determine γ_{23} and then by means of the de L'Hospital method we reveal the indeterminacy. We have

$$\gamma_{23}^2 = -\frac{1}{2}\big[p_{_{13}}(p_{_{13}} + 2q_{13}) + p_{_{23}}\big]. \tag{35}$$

Let us now pass to the construction of local solutions in the vicinity of $\xi=\infty$.

Coefficients $\mathcal{P}(\xi)$ and $Q(\xi)$ of differential equation (7) in the vicinity of $\xi=\infty$ can be represented as

$$\mathcal{P}(\xi)=\xi^{-1}\sum_{n=0}^{\infty}p_{n\infty}\xi^{-n},\quad Q(\xi)=\xi^{-2}\sum_{n=0}^{\infty}q_{n\infty}\xi^{-n}, \tag{36}$$

$$p_{n\infty}=\sum_{k=1}^{5}(1-\alpha_{1k}-\alpha_{2k})a_k^n,\quad p_{0\infty}=\sum_{k=1}^{5}(1-\alpha_{1k}-\alpha_{2k}), \tag{37}$$

$$q_{n\infty}=\sum_{k=1}^{5}\big[\alpha_{1k}\alpha_{2k}(n+1)a_k^n+C_ka_k^{n+1}\big], \tag{38}$$

$$q_{0\infty}=\sum_{k=1}^{5}\big[\alpha_{1k}\alpha_{2k}+C_ka_k\big], \tag{39}$$

$$q_{1\infty}=\sum_{k=1}^{5}\big[\alpha_{1k}\alpha_{2k}2a_k+C_ka_k^2\big]. \tag{40}$$

We seek for local solutions in the vicinity of the point $\xi=\infty$ in the form

$$u_{\infty}(\xi)=\xi^{-\alpha_\infty}+\sum_{n=1}^{\infty}\gamma_{n\infty}\xi^{-\alpha_\infty+n}. \tag{41}$$

For determining γ_{nj}, $n=\overline{1,\infty}$ we have the following formulas:

$$f_\infty(\alpha_\infty)=\alpha_\infty(\alpha_\infty+1)-p_{0\infty}\alpha_\infty+q_{0\infty}=0, \tag{42}$$

$$\gamma_{1\infty}f_{0\infty}(\alpha_\infty+1)-p_{1\infty}\alpha_\infty+q_{1\infty}=0, \tag{43}$$

$$\gamma_{2\infty}f_{0\infty}(\alpha_\infty+2)+\gamma_{1\infty}f_{1\infty}(\alpha_\infty+1)+p_{2\infty}\alpha_\infty+q_{2\infty}=0, \tag{44}$$

. . . .

$$\gamma_{n\infty}f_{0\infty}+\gamma_{(n-1)\infty}f_{1\infty}(\alpha_\infty+n-1)+\gamma_{(n-2)\infty}f_{2\infty}(\alpha_\infty+n-2)+\cdots$$
$$+\gamma_{1\infty}f_{(n-1)\infty}(\alpha_\infty+1)-p_{n\infty}\alpha_\infty+q_{n\infty}=0, \tag{45}$$

. . . .

where

$$f_{k\infty}=q_{k\infty}-(\alpha_\infty+k)p_{k\infty}.$$

Determining equation (42) must have two roots $\alpha_{1\infty}=3$ and $\alpha_{2\infty}=2$, therefore its free term $q_{0\infty}$ must satisfy the condition

$$q_{0\infty}\equiv\sum_{k=1}^{5}\big[\alpha_{1k}\alpha_{2k}+C_ka_k\big]=6. \tag{46}$$

Condition (46) can also be written as

$$B_2=q_{0\infty}-6=0.$$

Since $\alpha_{1\infty} - \alpha_{2\infty} = 1$, by formulas (43)–(45) we can construct only one solution $u_{1\infty}(\xi)$ corresponding to the root $\alpha_{1\infty} = 3$, and to construct the second solution $u_{2\infty}(\xi)$ we have to act in the following way. Equation (43) is not fulfilled because the coefficient for $\gamma_{1\infty}$ vanishes, i.e., $f_{0\infty}(\alpha_{2\infty}+1) = f_{0\infty}(3) = 0$. For equality (43) to take place, it is necessary and sufficient that the condition

$$q_{1\infty} - 2p_{1\infty} = 0. \tag{47}$$

to be fulfilled. This condition is in fact the third equation

$$B_3 = q_{1\infty} - 2p_{1\infty} = 0,$$

of equality (13).

Having found equation (47), we can determine the coefficient $\gamma_{1\infty}$. Towards this end, we solve equation (43) with respect to $\gamma_{1\infty}$ for $\alpha_\infty \neq 2$ and then, using the de L'Hospital method, reveal the indeterminacy. We have

$$\gamma^2_{1\infty} = p_{1\infty}. \tag{48}$$

Having determined $\gamma_{1\infty}$, by formulas (44) and (45) we can define $\gamma^2_{n\infty}$, $n = 2, 3, \ldots$. Consequently, we can construct local linearly independent solutions

$$u_{k\infty}(\xi) = \xi^{-\alpha_{k\infty}} + \sum_{n=1}^{\infty} \gamma^k_{n\infty} \xi^{-\alpha_{k\infty}-n}, \quad k = 1, 2. \tag{49}$$

As is shown above, linearly independent local solutions can be written in the form

$$u_{kj}(\xi) = (\xi - a_j)^{\alpha_{kj}} \widetilde{u}_{kj}(\xi), \quad j = \overline{1,5}, \quad k = 1, 2,$$

where $(\xi - a_j)^{\alpha_{kj}}$, $j = \overline{1,5}$, $k = 1, 2$, are many-valued functions. Of these functions we choose and fix one-valued branches:

$$\begin{gathered} (t - a_j)^{\alpha_{kj}} > 0, \quad t > a_j \\ \left[(\xi - a_j)^{\alpha_{kj}}\right]^{\pm} = e^{\pm \pi i k_{kj}} (a_j - \xi)^{\alpha_{kj}}, \quad t < a_j. \end{gathered} \tag{50}$$

Equation (7) can be written in the form of a system

$$u'(\xi) = u(\xi)\mathcal{P}_0(\xi), \tag{51}$$

where

$$u(\xi) = \begin{pmatrix} u_1(\xi), & u_1'(\xi) \\ u_2(\xi), & u_2'(\xi) \end{pmatrix}, \quad \mathcal{P}_0(\xi) = \begin{pmatrix} 0, & -Q(\xi) \\ 1, & -\mathcal{P}(\xi) \end{pmatrix}, \tag{52}$$

are the matrices of the second order.

Having constructed $u_{kj}(\xi)$, $j = \overline{1,5}$, $u_{k\infty}$, $k = 1, 2$, we can construct local matrices for the points $t = a_j$, $j = \overline{1,5}$, $t = \infty$:

$$\Theta_j(t) = \begin{pmatrix} u_{1j}(t), & u'_{1j}(t) \\ u_{2j}(t), & u'_{2j}(t) \end{pmatrix}, \quad t > a; \quad \Theta^{\pm}_j(t) = \theta^{\pm}_j \Theta^*_j(t), \quad t < a_j; \tag{53}$$

$$\Theta_\infty(t)=\begin{pmatrix} u_{1\infty}(t), & u'_{1\infty}(t) \\ u_{2\infty}(t), & u'_{2\infty}(t) \end{pmatrix}, \ \theta_j^\pm=\begin{pmatrix} \exp(\pm i\pi\alpha_{1j}), & 0 \\ 0, & \exp(\pm i\pi\alpha_{2j}) \end{pmatrix}. \quad (54)$$

Real matrices $Q_{j-1}(t)$ and $\Theta_j^*(t)$ are in fact local solutions of system (51) in the vicinity of points $t = a_{j-1}(t > a_{j-1})$ and $t = a_j(t < a_j)$, respectively. Assume that the matrices $\Theta_{j-1}(t)$, and $\Theta_j^*(t)$ converge simultaneously in any part of the interval $a_{j-1} < t < a_j$. Then the matrices $\Theta_{j-1}(t)$ and $\Theta_j^*(t)$ are connected by the equality, or more precisely, by the identity

$$\Theta_j^*(t) = T_{j-1}\Theta_{j-1}(t). \quad (55)$$

From (55) we can define uniquely the matrix T_{j-1} if we fix the value $t = t_1$, where $t = t_1$, both matrices $\Theta_j^*(t_1)$ and $\Theta_{j-1}(t_1)$ converge. Indeed, let us consider two cases. (1) Domains of convergence of the matrices $\Theta_j^*(t)$ and $\Theta_{j-1}(t)$ intersect, i.e., they have general part, where $\Theta_j^*(t)$ and $\Theta_{j-1}(t)$ converge. Then from equality (55) we can define T_{j-1}. (2) The matrices $\Theta_j^*(t)$ and $\Theta_{j-1}(t)$ have no general part of the domain of convergence with respect to t. In this case, at a regular point $t = e_i = (a_{j-1} + a_j)/2$ we construct the matrix $\Theta e_j(t)$. Then one can always pass from the matrix $\Theta_j^*(t)$ to the matrix $\Theta e_j(t)$ by the formula

$$\Theta_j^*(t) = T_{e_j}\Theta e_j(t), \quad (56)$$

and then from the matrix $\Theta e_j(t)$ to the matrix $\Theta e_{j-1}(t)$ by the formula

$$\Theta_{ej}(t) = T_{j-1}^*\Theta_{j-1}(t). \quad (57)$$

Matrices Te_j and T_{j-1}^* can be defined from (56) and (57). The matrix $\Theta e_j(t)$ can be eliminated if there takes place case (1). Thus we get

$$\Theta_j^*(t) = Te_jT_{j-1}^*\Theta_{j-1}(t). \quad (58)$$

From (56) and (58) follows

$$T_{j-1} = Te_jT_{j-1}^*. \quad (59)$$

On the basis of the above-said we can consider that the matrix $\Theta_5(t)$ can be analytically continued along the entire real t-axis [9–15].

Pass now to the definition of the functions $\omega(\xi)$ and $z(\xi)$. Bearing this in mind, we consider the matrices

$$\chi_1^\pm(t) = T\Theta_5^\pm(t), \quad t > a_5, \quad (60)$$

where the matrix T is defined as

$$T = \begin{pmatrix} p, & q \\ r, & s \end{pmatrix}, \quad \det T = 1, \quad (61)$$

p, q, r, S are, as is said above, constant integrations of equation (10).

Matrices $\chi_1^{\pm}(t)$ defined by formula (60) is the general solution of system (51), and they must satisfy the boundary condition (3_1),

$$\chi^+(t) = g_5(t)\chi^-(t), \quad g_5(t) = E, \quad a_5 < t < +\infty, \tag{62}$$

where E is the unit matrix.

From condition (62) it follows that T is a real matrix. Next we define $\chi^{\pm}(t)$ for $t < a_5$. We have $\chi^{\pm}(t) = T\theta_5^{\pm}\Theta_5^*(t)$. Substituting these matrices in the boundary condition (3_1) we get

$$T\theta_5^+\Theta_5^*(t) = g_4(t)T\theta_5^-\Theta_5^*(t), \quad t < a_5. \tag{63}$$

From (63) follows the matrix equation

$$T\theta_5^+ = g_4(t)T_5\theta_5^-, \tag{64}$$

which results in

$$q = 0, \quad r = 0. \tag{65}$$

Matrices $\chi^{\pm}(t)$ for $t > a_4$ are defined as follows: $\chi^{\pm}(t) = T\theta_5^{\pm}T_4\Theta_4^{\pm}(t)$. Since $\Theta_4^+(t) = \Theta_4^-(t)$ for $t > a_4$, from the boundary condition we again arrive at (64). Passing from the matrix $\Theta_4(t)$ to the matrix $\Theta_4^*(t)$, after substitution in the boundary condition (3_1) we obtain the matrix equation

$$T\theta_5^+T_4\theta_4^+ = g_4T\theta_5^-T_4\theta_4^-. \tag{66}$$

From equation (66) it follows that

$$q_4 = 0, \tag{67}$$

$$pp_4 - sr_4\cos(\pi\alpha) = 0. \tag{68}$$

Matrices $\chi^{\pm}(t)$ for $t > a_3$ are defined in the following manner: $\chi^{\pm}(t) = T\theta_5^{\pm}T_4\theta_4^{\pm}T_3\Theta_3^{\pm}(t)$. Since $\Theta_3^+(t) = \Theta_3^-(t)$ both for $t > a_3$ and $t < a_3$, $\chi^{\pm}(t)$ satisfy condition (66). Now from the matrix $\Theta_3(t)$ we pass to the matrix $\Theta_3(t) = T_2\Theta_2(t)$ and then to the matrix $\Theta_2^*(t)$. Then matrices $\chi^{\pm}(t)$ are defined as follows: $\chi^{\pm}(t) = T\theta_5^{\pm}T_4\theta_4^{\pm}T_{32}\theta_2^{\pm}\Theta_2^*(t)$, where $T_{32} = T_3T_2$. Substituting these values of the matrices $\chi^{\pm}(t)$ in condition (3_1), we obtain the matrix equation

$$T\theta_5^+T_4\theta_4^+T_{32}\theta_2^+ = g_2T\theta_5^-T_4\theta_4^-T_{32}\theta_2^-. \tag{69}$$

From equation (69) follows a system of equations

$$r_4p_{32}\sin\pi\alpha + s_4r_{32} = 0, \tag{70}$$

$$r_4q_{32}\cos\pi(\alpha+\beta) - s_4s_{32}\sin(\pi\beta) = 0. \tag{71}$$

Matrices $\chi^{\pm}(t)$ for $t > a_1$ can be written as

$$\chi^{\pm}(t) = T\theta_5^{\pm}T_4\theta_4^{\pm}T_{32}\theta_2^{\pm}T_1\Theta_1^{\pm}(t), \tag{72}$$

and for $t < a_1$ as

$$\chi^{\pm}(t) = T\theta_5^{\pm}T_4\theta_4^{\pm}T_{32}\theta_2^{\pm}T_1\theta_1^{\pm}\Theta_1^{\pm}(t). \tag{73}$$

Substituting (73) in the boundary condition (3_1), we obtain the following matrix equality

$$T\theta_5^{+}T_4\theta_4^{+}T_{32}\theta_2^{+}T_1\theta_1^{+} = T\theta_5^{-}T_4\theta_4^{-}T_{32}\theta_2^{-}T_1\theta_1^{-}. \tag{74}$$

Equality (74) results in a system of equations

$$p_1p_{32}\sin\pi\beta + r_1q_{32} = 0, \tag{75}$$

$$q_1p_{32} + s_1q_{32}\sin\pi\beta = 0. \tag{76}$$

Systems of equations (70), (71) and (75), (76) are homogeneous with respect to (r_4, s_4) and (p_{32}, q_{32}), respectively. For their compatibility the corresponding conditions

$$p_{32}s_{32}/[r_{32}q_{32}] = -\cos[\pi(\alpha+\beta)]/[\sin(\pi\alpha)\sin\pi\beta], \tag{77}$$

$$p_1s_1/[r_1q_1] = 1/\sin^2\pi\beta \tag{78}$$

must be fulfilled.

Equalities (77) and (78) are connected with the double or unharmonic relations of four points of one and the same circle [8].

Matrices $\chi^{\pm}(t)$ in the vicinity of the point $t = \infty$ are defined as follows:

$$\chi^{\pm}(t) = T\theta_5^{\pm}T_4\theta_4^{\pm}T_{32}\theta_2^{\pm}T_1\theta_1^{\pm}T_{-\infty}\Theta_\infty^{\pm}(t), \quad t > -\infty, \tag{79}$$

$$\chi^{\pm}(t) = T\theta_5(t) = TT_{+\infty}\Theta_\infty^{\pm}(t), \quad t > a_5,$$

$$\Theta_5^{\pm}(t) = T_{+\infty}\Theta_\infty^{\pm}(t), \quad t > a_5, \quad \Theta_\infty^{+}(t) = \Theta_\infty^{-}(t). \tag{80}$$

The above defined matrices must satisfy the conditions

$$T\theta_5^{+}T_4\theta_4^{+}T_{32}\theta_2^{+}T_1\theta_1^{+}T_{-\infty}\Theta_\infty^{+}(t) =$$

$$= T\theta_5^{-}T_4\theta_4^{-}T_{32}\theta_2^{-}T_1\theta_1^{-}T_{-\infty}\Theta_\infty^{-}(t), \quad -\infty < t < a_1, \tag{81}$$

$$T\theta_5^{+}T_4\theta_4^{+}T_{32}\theta_2^{+}T_1\theta_1^{+}T_{-\infty}\Theta_\infty^{+}(t) = TT_{+\infty}\Theta_\infty^{+}(t), \quad t < +\infty, \tag{82}$$

$$T\theta_5^{-}T_4\theta_4^{-}T_{32}\theta_2^{-}T_1\theta_1^{-}T_{-\infty}\Theta_\infty^{-}(t) = TT_{+\infty}\Theta_\infty^{-}(t), \quad t < +\infty. \tag{83}$$

Consequently, for the singular points $t = a_j$, $j = 1,2,4,5$, we have obtained systems of real homogeneous with respect to the parameters p, q, r, s, p_j, q_j, r_j, s_j, $j = 1,2,4,5$ equations (65), (67); (70), (71); (75), (76). The conditions of compatibility of systems (70), (71) and (75), (76) have respectively the form (77), (78). For the point $t = a_3$ we have obtained only one equation (32).

From each system (70), (71) and (75), (76) we take only one equation and, in addition, system (77) and (78). To these equations we have to add system (15), (46), (47). From system (15), (32), (46), (47) we can define C_1, C_2, C_4, C_5 depending on C_3, a, b and then substitute their values in the rest equations. Here we recall that the parameters p_j, q_j, r_j, s_j, $j = \overline{1,5}$ depend implicitly on the parameters C_j, $j=\overline{1,5}$. Having omitted the parameters C_1, C_2, C_4, C_5, a number of the remaining equations will be equal to six and a number of the unknown parameters will be equal to four: a, b, C_3, p and s, since the parameters p,s are interconnected: $ps=1$. A number of equation is as a rule always more than a number of unknowns, even in the case of a linear polygon, what is quite explainable from the geometrical viewpoint [6–15].

We have to determine the functions $z(\xi)$, $\omega(\xi)$, $w(\xi)$ and for this aim to determine linearly independent solutions of (7), $u_1(\xi)$, $u_2(\xi)$ defined along the entire t-axis ($-\infty<t<+\infty$). Bearing this in mind, we write out all values $\chi^+(t)$ and have

$$\begin{aligned}
&\chi^+(t)=\begin{pmatrix}p, & 0\\ 0, & s\end{pmatrix}\begin{pmatrix}u_{15}(t), & u'_{15}(t)\\ u_{25}(t), & u'_{25}(t)\end{pmatrix}, \quad t>a_5;\\
&\chi^+(t)=\begin{pmatrix}-ip, & 0\\ 0, & -se^{i\pi\alpha}\end{pmatrix}\begin{pmatrix}u^*_{15}(t), & {u'_{15}}^*(t)\\ u^*_{25}(t), & {u'_{25}}^*(t)\end{pmatrix}, \quad t<a_5;\\
&\chi^+(t)=\begin{pmatrix}-ipp_4, & 0\\ -sr_4e^{i\pi\alpha}, & -ss_4e^{i\pi\alpha}\end{pmatrix}\begin{pmatrix}u_{14}(t), & u'_{14}(t)\\ u_{24}(t), & u'_{24}(t)\end{pmatrix}, \quad t>a_4;\\
&\chi^+(t)=\begin{pmatrix}-pp_4, & 0\\ isr_4e^{i\pi\alpha}, & -ss_4\end{pmatrix}\begin{pmatrix}u^*_{14}(t), & u'^*_{14}(t)\\ u^*_{24}(t), & u'_{24}(t)\end{pmatrix}, \quad t<a_4; \qquad (84)\\
&\chi^+(t)=\begin{pmatrix}-pp_4p_3, & -pp_4q_3\\ isr_4p_3e^{i\pi\alpha}-ss_4r_3, & isr_4q_3e^{i\pi\alpha}-ss_4s_3\end{pmatrix}\times\\
&\qquad\times\begin{pmatrix}u_{13}(t), & u'_{13}(t)\\ u_{23}(t), & u'_{23}(t)\end{pmatrix}, \quad t>a_3;\\
&\chi^+(t)=(-pp_4)\begin{pmatrix}p_{32}, & q_{32}\\ -ip_{32}, & e^{-i\pi\beta}q_{32}/\sin(\pi\beta),\end{pmatrix}\times\\
&\qquad\times\begin{pmatrix}u_{12}(t), & u'_{12}(t)\\ u_{22}(t), & u'_{22}(t)\end{pmatrix}, \quad t>a_2;\\
&\chi^+(t)=(-pp_4e^{-i\pi\beta})\begin{pmatrix}ip_{32}, & q_{32}e^{i\pi\beta}\\ p_{32}, & q_{32}/\sin(\pi\beta)\end{pmatrix}\times\\
&\qquad\times\begin{pmatrix}u^*_{12}(t), & {u'_{12}}^*(t)\\ u^*_{22}(t), & u'_{22}*(t)\end{pmatrix}, \quad t>a_2;\\
&\chi^+(t)=(-pp_4e^{-i\pi\beta})\cos\pi\beta\begin{pmatrix}ip_{32}p_1e^{i\pi\beta}, & q_{32}s_1\\ 0, & q_{32}s_1\operatorname{ctg}(\pi\beta)\end{pmatrix}\times\\
&\qquad\times\begin{pmatrix}u_{11}(t), & u'_{11}(t)\\ u_{21}(t), & u'_{21}(t)\end{pmatrix}, \quad t>a_1;\\
&\chi^+(t)=(-pp_4\cos\pi\beta e^{-i\pi\beta})\begin{pmatrix}p_{32}p_1, & -q_{32}s_1\\ 0, & -q_{32}s_1\operatorname{ctg}(\pi\beta)\end{pmatrix}\times\\
&\qquad\times\begin{pmatrix}u^*_{11}(t), & {u'_{11}}^*(t)\\ u^*_{21}(t), & {u'_{21}}^*(t)\end{pmatrix}, \quad t<a_1;
\end{aligned}$$

$$\chi^+(t) = \big(-pp_4\cos\pi\beta e^{-i\pi\beta}\big)\times$$
$$\times\begin{pmatrix} p_{32}p_1p_{-\infty}-q_{32}s_1r_{-\infty}, & p_{32}p_1q_{-\infty}-q_{32}s_1s_{-\infty}\\ -q_{32}s_1\operatorname{ctg}(\pi\beta)r_{-\infty}, & -q_{32}s_1\operatorname{ctg}(\pi\beta)\end{pmatrix}\times$$
$$\times\begin{pmatrix} u_{1\infty}(t), & u'_{1\infty}(t)\\ u_{2\infty}(t), & u'_{2\infty}(t)\end{pmatrix},\quad t>-\infty;$$
$$\chi^+(t) = \begin{pmatrix} p, & 0\\ 0, & s\end{pmatrix}\begin{pmatrix} p_{+\infty}, & q_{+\infty}\\ r_{+\infty}, & s_{+\infty}\end{pmatrix}\begin{pmatrix} u_{1\infty}(t), & u'_{1\infty}(t)\\ u_{2\infty}(t), & u'_{2\infty}(t)\end{pmatrix}\quad t>+\infty.$$

From (84) we define $u_1(t)$ and $u_2(t)$ along the t-axis ($-\infty<t<+\infty$) and have:

$$\begin{aligned}
u_1^+(t) =& pu_{15}^+,\quad t>a_5;\\
u_2^+(t) =& su_{25}^+,\quad t>a_5;\\
u_1^+(t) =& -ipu_{15}^*(t),\quad t<a_5;\\
u_2^+(t) =& -se^{i\pi\alpha}u_{25}^*(t),\quad t<a_5;\\
u_1^+(t) =& -ipp_4u_{14}(t),\quad t>a_4;\\
u_2^+(t) =& -sr_4e^{i\pi\alpha}u_{14}(t)-ss_4e^{i\pi\alpha}u_{24}(t),\quad t>a_4; \qquad (85)\\
u_1^+(t) =& -pp_4u_{14}^*(t),\quad t<a_4;\\
u_2^+(t) =& isr_4e^{i\pi\alpha}u_{14}^*(t)-ss_4u_{24}^*(t),\quad t<a_4;\\
u_1^+(t) =& -pp_4p_3u_{13}(t)-pp_4q_3u_{23}(t),\quad t>a_3;\\
u_2^+(t) =& [isr_4p_3e^{i\pi\alpha}-ss_4r_3]u_{13}(t)+\\
&+[isr_4q_3e^{i\pi\alpha}-ss_4s_3]u_{23}(t)\quad t>a_3;\\
u_1^+(t) =& -pp_4p_{32}u_{12}(t)-pp_4q_{32}u_{22}(t),\quad t>a_2;\\
u_2^+(t) =& ipp_4p_{32}u_{12}(t)-pp_4q_{32}e^{-i\pi\beta}/[\sin(\pi\beta)]u_{22}(t),\quad t>a_2;\\
u_1^+(t) =& -ipp_4e^{-i\pi\beta}p_{32}u_{12}^*(t)-pp_4q_{32}u_{22}^*(t),\quad t<a_2;\\
u_2^+(t) =& -ipp_4e^{-i\pi\beta}p_{32}u_{12}^*(t)-pp_4q_{32}e^{-i\pi\beta}/[\sin(\pi\beta)]u_{22}^*(t),\ \ t<a_2;\\
u_1^+(t) =& -ipp_4p_{32}p_1\cos(\pi\beta)u_{11}(t)-\\
&-pp_4q_{32}s_1e^{-i\pi\beta}\cos(\pi\beta)u_{21}(t),\quad t>a_1;\\
u_2^+(t) =& -ipp_4e^{-i\pi\beta}q_{32}/[\sin(\pi\beta)]u_{21}(t),\quad t>a_1;\\
u_1^+(t) =& -ipp_4\cos(\pi\beta)e^{-i\pi\beta}\big[p_{32}p_1u_{11}^*(t)-q_{32}s_1u_{21}^*(t)],\quad t<a_1;\\
u_2^+(t) =& pp_4\cos^2(\pi\beta)e^{-i\pi\beta}q_{32}s_1/[\sin(\pi\beta)]u_{21}^*(t),\quad t<a_1;\\
u_1^+(t) =& \big[-pp_4\cos(\pi\beta)e^{-i\pi\beta}\big]\big[(p_{32}p_1p_{-\infty}-q_{32}s_1r_{-\infty}\big]u_{1\infty}(t)+\\
&+\big[(p_{32}p_1q_{-\infty}-q_{32}s_1s_{-\infty}\big]u_{2\infty}(t),\quad t>a_{-\infty};\\
u_2^+(t) =& \big[-pp_4\cos(\pi\beta)e^{-i\pi\beta}\big]\big[-q_{32}s_1\operatorname{ctg}(\pi\beta)(r_{-\infty}u_{1\infty}(t)+s_{-\infty}u_{2\infty}(t)\big],\\
&t>-\infty;
\end{aligned}$$

$$u_1^+(t) = p\big[p_{+\infty}u_{1\infty}(t) + q_{+\infty}u_{2\infty}(t)\big], \quad t < +\infty;$$
$$u_2^+(t) = s\big[r_{+\infty}u_{1\infty} + s_+ u_{2\infty}(t)\big], \quad t < +\infty.$$

Having defined $u_1^+(t)$, $u_2^+(t)$, we determine the functions $z(\xi)$, $\omega(\xi)$, $w(\xi)$. But before we proceed to determining these functions, it is necessary to make the following important remark dealt with the above-said.

The series $u_{kj}(\xi)$, $k = 1, 2$, $j = \overline{1, 5}$, are the integral functions of parameters C_k [9–11], but with respect to ξ they converge very slowly and this is not advantageous for calculations. For example, coefficients γ_{nj}^k, $k = 1, 2$; $n = \overline{1, \infty}$, $j = \overline{1, 5}$ increase considerably as n grow and the corresponding multipliers $(\xi - a_j)^n$, on the contrary, decrease. Electronic computers fail to multiply γ_{nj}^k by $(t - a_j)^n$ despite the fact that these series converge near $t = a_j$. To eliminate this disadvantages it is proposed to replace these series by rapidly and uniformly convergent functional series. Bearing this in mind, it suffices to write these series in somewhat different form. Let us consider recurrence formulas (26)–(28) and formulas (19)–(21) for P_{jn}, q_{jn}. Instead of (24) we consider series

$$u_{kj}(t) = (t - a_j)^{\alpha_{kj}}\widetilde{u}_{kj}(t - a_j), \quad \widetilde{u}_{kj}(1 - a_j) = 1 + \\ + \sum_{n=1}^{\infty} \gamma_{nj}^k (t - a_j), \quad j = \overline{1, 5}, \quad k = 1, 2, \tag{86}$$

where γ_{nj}^k are defined in terms of $f_{nj}(\alpha_j)$, where

$$f_{nj}\big[(t - a),\, \alpha_{kj}\big] = \alpha_{kj}P_{nj}(t - a_j) + q_{nj}(t - a_j), \tag{87}$$

$$P_{nj}(t - a_j) = (-1)^{n-1}\sum_{k=1}^{5}\big[1 - \alpha_{1k} - \alpha_{2k}\big]\Big(\frac{t - a_j}{a_j - a_k}\Big)^n, \tag{88}$$

$$q_{nj}(t - a_j) = (-1)^{n-2}\sum_{k=1, k\neq j}^{5}\Big[\alpha_{1k}\alpha_{2k}(n-1) + C_k(a_j - a_k)\Big]\Big(\frac{t - a_j}{a_j - a_k}\Big)^n, \tag{89}$$

$$|t - a_j| < M_{in}\big\{|a_j - a_{j-1}|,\, |a_j - a_{j+1}|\big\}, \tag{90}$$
$$\Big|\frac{t - a_j}{a_j - a_k}\Big| < 1, \quad k \neq j.$$

It is seen from (88) and (89) that functional series (86) converge rapidly compared to series (30). Therefore in all the above formulas we replace series (30) by (86).

The function $w^+(t)$ along the real axis is defined by the equality

$$w^+(t) = u_1^+(t)/u_2^+(t), \quad -\infty < t < +\infty, \tag{91}$$

where u_k^+, $k = 1, 2$, are defined by formulas (85).

Note that in our earlier works similar problems were solved by means of constructing the canonical matrix for the homogeneous problem and then by its help we solved the inhomogeneous problem (3). But in our present work we will find a solution of the inhomogeneous problem (3) in a more effective and simple manner.

Matrices $\chi^{\pm}(t)$ defined by formulas (84) (matrix $\chi^{-}(t)$ is defined by the formula $\chi^{-}(t) = \overline{\chi^{+}(t)}$) satisfy boundary condition (3_1) because there take place equalities (63)–(83). And this means that columns of matrices $\chi^{+}(t)$ defined by formulas (63)–(69) satisfy boundary condition (3_1).

To obtain a solution of $w^{+}(t) = \omega'^{+}(t)/z'^{+}(t)$, we have to take elements of the first column of the matrix $\chi^{+}(t)$, since the function $w^{+}(t) = u_1^{+}(t)/u_2^{+}(t)$ provides the general solution (10), and the function $u_1'^{+}/u_2'^{+}(t)$ composed of elements of the second column does not satisfy equation (10).

Components of the vector $\Phi'(\xi)$ along the t-axis are defined as follows:

$$\omega'^{+}(t) = u_1^{+}(t), \quad z'^{+}(t) = u_2^{+}(t). \tag{92}$$

We write equalities (92) as

$$d\omega^{+}(t) = u_1^{+}(t)\,dt, \quad -\infty < t < +\infty; \tag{93}$$

$$dz^{+}(t) = u_2^{+}(t)\,dt, \quad -\infty < t < +\infty. \tag{94}$$

Integrating the above equalities in the intervals $(-\infty, t)$, (a_j, t), $j = \overline{1,5}$, we get

$$\omega^{+}(t) = \int_{-\infty}^{t} u_1^{+}(t)\,dt + \omega^{+}(-\infty), \tag{95}$$

$$z^{+}(t) = \int_{-\infty}^{t} u_2^{+}(t)\,dt + z^{+}(-\infty), \tag{96}$$

$$\omega^{+}(t) = \int_{a_j}^{t} u_1^{+}(t)\,dt + \omega^{+}(a_j + 0), \tag{97}$$

$$z^{+}(t) = \int_{a_j}^{t} u_2^{+}(t)\,dt + z^{+}(a_j + 0), \tag{98}$$

where $\omega^{+}(a_j+0)$, $z^{+}(a_j+0)$ are the limiting values of the above-mentioned functions at the points $t = a_j$, $j = \overline{1,5}$ from the right.

It is obvious that the functions $\omega^{+}(t)$, $z^{+}(t)$, defined by formulas (95)–(98) will satisfy boundary condition (3), since the vector $f(t) = [n_1(t), n_2(t)]$ is piecewise constant.

In formulas (95)–(98) we can separate the real parts from the imaginary ones and obtain the expression for the functions $\varphi(t)$, $\psi(t)$, $x(t)$ and $y(t)$.

Moreover, if in formulas (95)–(96) we take $t = a_1$ and in formulas (97)–(98) $t = a_{j+1}$, then we will find that

$$\omega^+(a_1 - 0) = \int\limits_{-\infty}^{a_1} u_1^+(t)\,dt + \omega^+(-\infty), \tag{99}$$

$$z^+(a_1 - 0) = \int\limits_{-\infty}^{a_1} u_2^+(t)\,dt + z^+(-\infty), \tag{100}$$

$$\omega^+(a_{j+1} - 0) = \int\limits_{a_j}^{a_{j+1}} u_1^+(t)\,dt + \omega^+(a_j + 0), \tag{101}$$

$$z^+(a_{j+1} - 0) = \int\limits_{a_j}^{a_{j+1}} u_2^+(t)\,dt + z^+(a_j + 0), \tag{102}$$

$$\omega^+(+\infty) = \int\limits_{a_5}^{+\infty} u_1^+(t)\,dt + \omega^+(a_5 + 0), \tag{103}$$

$$z^+(+\infty) = \int\limits_{a_5}^{+\infty} u_2^+(t)\,dt + z^+(a_5 + 0), \tag{104}$$

where $\omega^+(a_{j+1} - 0)$ and $z^+(a_{j+1} - 0)$ are limiting values of the above-mentioned functions at the singular points $t = a_j$ from the left, while at the point $t = +\infty$ we have

$$\omega^+(-\infty) = \omega^+(+\infty), \quad z^+(-\infty) = z^+(+\infty).$$

We rewrite a system of equations (15), (32), (46) and (47) in the form

$$\begin{aligned} & C_1 + C_2 + C_4 + C_5 = e_1, \\ & \frac{C_1}{a_3 - a_1} + \frac{C_2}{a_3 - a_2} + \frac{C_4}{a_3 - a_4} + \frac{C_5}{a_3 - a_5} = e_2, \\ & C_1 a_1 + C_2 a_2 + C_4 a_4 + e_5 a_5 = e_3, \\ & C_1 a_1^2 + C_2 a_2^2 + C_4 a_4^2 + C_5 a_5^2 = e_4, \end{aligned} \tag{105}$$

where

$$e_1 = -\,C_3, \quad e_3 = 6 - C_3 a_3 - \sum_{k=1}^{5} \alpha_{1k}\alpha_{2k},$$

$$e_2 = -\,C_3\Big[\sum_{k=1,k\neq 3}^{5} (1 - \alpha_{1k} - \alpha_{2k})(a_3 - a_k)^{-1} + C_3\Big] -$$

$$-\sum_{k=1,k\neq 3}^{5}\alpha_{1k}\alpha_{2k}(a_3-a_k)^{-2}, \tag{106}$$
$$e_4 = 2\sum_{k=1}^{5}(1-\alpha_{1k}-\alpha_{2k})a_4-\sum_{k=1}^{5}\alpha_{1k}\alpha_{2k}2a_k.$$

With respect to C_1, C_2, C_4, C_5 the solution of system (105) can be written as

$$C_k = F_k(C_3,a,b), \quad k=1,2,4,5, \tag{107}$$

where $F_k(C_3,a,b)$, $k=1,2,4,5$ can be defined from system (105) by using Cramer's formula.

Let us now pass to the determination of matrix elements when T_j passes from one local $\Theta_j^*(t)$ matrix to the other adjacent $\Theta_{j-1}(t)$ matrix.

We have the following matrix equalities:

$$\begin{gathered}\Theta_5^*(t)=T_4\Theta_4(t), \ \ \Theta_4^*(t)=T_3\Theta_3(t), \ \ \Theta_3(t)=T_2\Theta_2(t),\\ \Theta_2^*(t)=T_1\Theta_1(t), \ \ \Theta_1^*(t)=T_{-\infty}\Theta_\infty(t), \ \ \Theta_5(t)=T_{+\infty}(t).\end{gathered} \tag{108}$$

For each of equalities (108) along the t-axis we fix the points, where the both matrices appearing in the equalities $t=t_k$, $k=4,3,2,1$; $t=t_{-\infty}$, $t=t_{+\infty}$ respectively, converge

$$\begin{aligned} T_4&=\Theta_5^*(t_4)\Theta_4^{-1}(t_4), \quad T_3=\Theta_4^*(t_3)\Theta_3^{-1}(t_3),\\ T_2&=\Theta_3(t_2)\Theta_2^{-1}(t_2), \quad T_1=\Theta_2^*(t_1)\Theta_1^{-1}(t_1),\\ T_{-\infty}&=\Theta_1^*(t_{-\infty})\Theta_\infty^{-1}(t_{-\infty}), \quad T_{+\infty}=\Theta_5(t_{+\infty})\Theta_\infty^{-1}(t_{+\infty}),\end{aligned} \tag{109}$$

where $\Theta_4^{-1}(t_4)$, $\Theta_3^{-1}(t_3)$, $\Theta_2^{-1}(t_2)$, $\Theta_1^{-1}(t_1)$, $\Theta_\infty^{-1}(t_{-\infty})$, $\Theta_\infty^{-1}(t_{+\infty})$, are matrices, inverse respectively to matrices $\Theta_4(t_4)$, $\Theta_3(t_3)$, $\Theta_2(t_2)$, $\Theta_1(t_1)$, $\Theta_\infty(t_{-\infty})$, $\Theta_\infty(t_{+\infty})$.

Determine elements of the above-given matrices T_j and then substitute in them the values C_1, C_2, C_4, C_5 defined by formulas (107).

Before we proceed to the solution of system (67), (68), (70), (71), (75), (76), (77), (78), from equation (68) we define p/s, if p_4 and r_4 are known.

To solve a system of higher transcendent equations (67), (68), (70) and (71), (75), (76) it is necessary to establish the intervals of variation of the parameters C_3,a,b. Parameter C_3,a,b varies in the interval $0<a<b$.

Suppose that the domain $s(w)$ in the process of its deformation turns into a linear polygon. Then the domain $\mathrm{Im}(\xi)>0$ is mapped conformally onto the linear polygon by means of the Christoffel-Schwartz formula. The analytic function expressed by that formula must satisfy equation (10). From this identity for the parameters C_j, $j=\overline{1,5}$, we obtain the following limiting values:

$$\begin{aligned}
C_1^* &= \Big(\frac{1}{2}+\beta\Big)\Big[-\Big(\frac{1}{2}+\beta\Big)(b-a)^{-1}+b^{-1}-\\
&\quad -\Big(\frac{1}{2}+\alpha\Big)(3b+a)(a+b)^{-1}(2b)^{-1}\Big],\\
C_2^* &= \Big(\frac{1}{2}+\beta\Big)\Big[\Big(\frac{1}{2}+\beta\Big)(b-a)^{-1}+a^{-1}-\\
&\quad -\Big(\frac{1}{2}+\alpha\Big)(3a+b)\big[2a(a+b)^{-1}\big],\\
C_3^* &= -\Big\{\Big(\frac{1}{2}+\beta\Big)(2a+b)\big[a(a+b)\big]^{-1}-\Big(\frac{1}{2}+\alpha\Big)(a+b)\Big\},\\
C_4^* &= \Big(\frac{1}{2}+\alpha\Big)\Big\{\Big(\frac{1}{2}+\beta\Big)(3a+b)\big[2a(a+b)\big]^{-1}-\\
&\quad -a^{-1}-\Big(\frac{1}{2}+\alpha\Big)(b-a)^{-1}\Big\},\\
C_5^* &= \Big(\frac{1}{2}+\alpha\Big)\Big\{\Big(\frac{1}{2}+\beta\Big)(a+3b)\big[2b(a+b)\big]^{-1}-\\
&\quad -b^{-1}+\Big(\frac{1}{2}+\alpha\Big)(b-a)^{-1}\Big\}.
\end{aligned} \tag{110}$$

Limiting values of the parameters C_k, $k=\overline{1,5}$, C_k^*, $k=\overline{1,5}$ must satisfy the condition

$$\sum_{k=1}^{5} C_k^* = 0. \tag{111}$$

From condition (111) it follows that

$$\lambda_{1,2} = \big[\alpha-\beta \pm \sqrt{(\alpha-\beta)^2+2xy}/2y\big], \tag{112}$$

where

$$x=(3/2+2\alpha+\beta),\quad y=1/2+\beta,\quad \lambda=b/a. \tag{113}$$

Consider the other particular case. Assume that the domain $s(w)$ upon deformation turns into a circle. Then the domain $\mathrm{Im}(\xi)>0$ is mapped conformally onto a circle by means of the linear-fractional transformation. Then for the parameters C_j, $j=\overline{1,5}$ we obtain the other particular case,

$$C_j^{**}=0,\quad j=\overline{1,5}. \tag{114}$$

It remains to solve system (67), (68), (70), (71), (75) and (76) with respect to the parameters C_3, a and b. From that system we take system (67), (70), (75), since the conditions of compatibility of systems (70), (71) and (75), (76) have the form (77) and (78). For the sake of brevity we write system (67), (70) and (75) in the form

$$M_k(C_3,a,b)=0,\quad k=\overline{1,3}. \tag{115}$$

System (115) can be written pairwise,

$$\begin{cases} M_1(C_3, \quad a, b) = 0, \\ M_2(C_3, \quad a, b) = 0; \end{cases} \tag{116}$$

$$\begin{cases} M_1(C_3, \quad a, b) = 0, \\ M_3(C_3, \quad a, b) = 0. \end{cases} \tag{117}$$

Each pair of these equalities describes a curve in the space. These curves, according to the Riemann theory must intersect at the same point. When solving systems (116) and (117), by means of the projection method we will project these curves onto two mutually perpendicular planes. Consider the coordinate system a, b, C_3 and fix within admissible limits the parameter a. Then systems (116) and (117) will be solved for fixed a. An algorithm for the solution of the two equations with two unknowns will be given below.

Consider a system of equations

$$\begin{cases} N_1(C_3, \quad b) = 0, \\ N_2(C_3, \quad b) = 0, \end{cases} \tag{118}$$

where system (118) coincides with one of systems (116) and (117). We fix the values b and denote them by b^j. Varying the parameter C_3, we construct diagrams of dependencies $[C_3,\, N_k(C_3, b^j)]$, $k = 1, 2$. If these diagrams intersect the C_3-axis, then we will be able to find zeros of the function $N_k(C_3,\, b^j)$ for fixed b^j. Find zeros for the other b^j as well. Recall that we rely on the well-known Riemann's theorem that the problem has the unique solution. Suppose all zeros of the functions are already found. Denote them by $[b^{k_i}\, C_3^{k_i}]$. On the basis of those zeros we construct diagrams of dependencies $[b^{k_i}\, C_3^{k_i}]$, $k = 1, 2, \dots$. As a result we obtain two diagrams which must intersect. The coordinates of the point of intersection provide us with the solution of system (118). Increasing a number of different values b^j, we can get a solution of that system within any accuracy. But system (118) can also be solved by another well-known methods of approximation. Suppose we have found triples of real numbers (a^k, b^k, C_3^k), where $k = 1, 2$, indicates that the triple for $k = 1$ is a solution of the first system of (116) and for $k = 2$ is that of the second system. Varying a we can get triples of numbers for every system.

Further we construct diagrams in the two mutually intersecting planes (C_3, a) and (C_3, b) having a set of triples $a^k = \varphi_1(C_3^k)$, $b^k = \varphi_2(C_3^k)$, each. Then the curves in each of those planes must intersect, if the initial system has a unique solution. According to the Riemann theorem, we can conclude that the system has the unique solution. With the help of the diagram, the points of intersection of curves will enable one to define the initial solution of the system. Having defined the parameters a, b, c_3, from formulas (97) and (98) we can determine both the equation of a depression curve and the parameter Q connected with the liquid discharge for filtration.

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(Received 1.02.2001)

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