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APPLICATION OF THE METHODS OF ANALYTIC FUNCTIONS TO THE SOLUTION OF BOUNDARY VALUE PROBLEMS OF ELASTICITY FOR A NON-HOMOGENEOUS HALF-PLANE

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Abstract. In the present paper we consider boundary value problems for a non-homogeneous elastic half-plane whose Poisson coefficient is a continuous function of depth of special type and a shear modulus is constant.

1. Complex Representation of a General Solution of the Equation of the Non-Homogeneous Plane Theory of Elasticity

The basic equations of the plane theory of elasticity for a non-homogeneous isotropic body, when elastic characteristics are given in terms of arbitrary functions of the point coordinates, can in a general case be written as follows [1]:

$$\frac{\partial}{\partial \overline{z}}\Big[\frac{\mu}{\varkappa-1}\Big(\frac{\partial U}{\partial z}+\frac{\partial \overline{U}}{\partial \overline{z}}\Big)\Big]+\frac{\partial}{\partial z}\Big(\mu\frac{\partial U}{\partial \overline{z}}\Big)=0, \tag{1}$$

$$\begin{gathered} X_x+Y_y=\frac{4\mu}{\varkappa-1}\Big(\frac{\partial U}{\partial z}+\frac{\partial \overline{U}}{\partial \overline{z}}\Big)=4\frac{\partial^2 F}{\partial z\partial \overline{z}},\\ Y_y-X_x-2iX_y=-4\mu\frac{\partial U}{\partial \overline{z}}=4\frac{\partial^2 F}{\partial \overline{z}^2}, \end{gathered} \tag{2}$$

where $z=x+iy$, $\overline{z}=x-iy$ are complex variables; $\frac{\partial}{\partial z}=\frac{1}{2}(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y})$; $\frac{\partial}{\partial \overline{z}}=\frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})$; $U=u+iv$ is a complex displacement; X_x, Y_y, X_y are stress components; σ is the Poisson coefficient; μ is the shear modulus; F

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is the stress function; $\varkappa=\frac{3-\sigma}{1+\sigma}$ for the plane strained state and $\varkappa=3-4\sigma$ for the plane strain.

Let $\varkappa(x,y)$ be an arbitrary function of variables x and y in the region S occupied by a non-homogeneous elastic body and let $\mu(x,y)$. Then equation (1) takes the form

$$\frac{\partial}{\partial\overline{z}}\Big[\frac{1}{\varkappa-1}\Big(\frac{\partial U}{\partial z}+\frac{\partial\overline{U}}{\partial\overline{z}}\Big)+\frac{\partial U}{\partial z}\Big]=0,$$

whence

$$\frac{\varkappa}{\varkappa-1}\,\frac{\partial U}{\partial z}+\frac{1}{\varkappa-1}\,\frac{\partial\overline{U}}{\partial\overline{z}}=\phi(z). \tag{3}$$

Here $\phi(z)$ is an arbitrary analytic function in S.

Omitting the value $\frac{\partial\overline{U}}{\partial z}$ both from equation (3) and from its complex-conjugate equation, we obtain

$$2\mu\frac{\partial U}{\partial z}=\phi(z)-\frac{1}{\varkappa+1}\big(\phi(z)+\overline{\phi(z)}\big). \tag{4}$$

Let us consider non-homogeneity of the elastic properties of the body in terms of the n-th order trigonometric polynomial:

$$\frac{1}{\varkappa+1}=A_0+$$
$$+\sum_{k=1}^{n}\big(A_k\cos k\alpha y+B_k\sin k\alpha y\big)\quad(\mu=\text{const}), \tag{5}$$

where A_k $(k=0,1,2,\dots,n)$, α are real constants.

It is obvious that $\frac{1}{\varkappa+1}=\frac{1+\sigma}{4}$ for the plane strained state and $\frac{1}{\varkappa+1}=\frac{1}{4(1-\sigma)}$ for the plane deformation.

If we substitute expression (5) in (4), then we will easily get

$$2\mu\frac{\partial U}{\partial z}=(1-A_0)\phi(z)-A_0\overline{\phi(z)}-\big(\phi(z)+\overline{\phi(z)}\big)E(z,\overline{z}), \tag{6}$$

where

$$E(z,\overline{z})=\sum_{k=1}^{n}\big[A_k\cos k(\overline{a}_0z+a_0\overline{z})+B_k\sin k(\overline{a}_0z+a_0\overline{z})\big], \tag{7}$$

$a_0=\frac{\alpha}{2}i$.

Obviously, $(1-A_0)\phi(z)-A_0\overline{\phi(z)}-(\phi(z)+\overline{\phi(z)})E(z,\overline{z})$ is the analytic function of z and $\overline{z}$ in the region $(s,\overline{s})$, and the solution of equation (6) can be represented as [2]:

$$2\mu U = (1-A_0)\varphi(z) - A_0 z\overline{\phi(z)} - \overline{\psi(z)} + \frac{\overline{\phi(z)}}{a_0}\sum_{k=1}^{n}\frac{1}{k}\big[A_k \sin k(\overline{a}_0 z + a_0\overline{z}) - \\ - B_k \cos k(\overline{a}_0 z + a_0\overline{z})\big] - \int \phi(z)E(z,\overline{z})\,dz. \qquad (8)$$

Here $\psi(z)$ is an arbitrary analytic function of z, $\varphi(z) = \int \phi(z)dz$, and the integral is calculated as if the value $\overline{z}$ remains constant.

By virtue of (7) we have

$$\int \phi(z)E(z\overline{z})\,dz = \\ = \sum_{k=1}^{n}\int \phi(z)\big[A_k \cos k(\overline{a}_0 z + a_0\overline{z}) + B_k \sin k(\overline{a}_0 z + a_0\overline{z})\big]\,dz. \qquad (9)$$

It can be easily shown that (9) can be represented in the form

$$\int \phi(z)E(z\overline{z})\,dz = \sum_{k=1}^{n}\Big\{C_k^{*'}(z)\big[A_k \cos k(\overline{a}_0 z + a_0\overline{z}) + B_k \sin k(\overline{a}_0 z + a_0\overline{z})\big] - \\ - a_0 k G_k^*(z)\big[A_k \sin k(\overline{a}_0 z + a_0\overline{z}) - B_k \cos k(\overline{a}_0 z + a_0\overline{z})\big]\Big\}, \qquad (10)$$

where $G_k^*(z)$ $(k = 1,2,\dots,n)$ are analytic functions in S, satisfying the conditions

$$\phi(z) = G_1^{*''}(z) + a_0^2 G_1^*(z) = G_2^{*''}(z) + 4a_0^2 G_2^*(z) = \cdots = \\ = G_n^{*''}(z) + n^2 a_0^2 G_n^*(z). \qquad (11)$$

Indeed, differentiating both parts of equation (10) and comparing them with (9), we obtain (11).

Let us prove now the following

Lemma. *If $C_k(z)$ $(k = 1,2,\dots,n)$ are analytic functions in S, satisfying the conditions*

$$G_1(z) = G_1^*(z),\ \ G_k(z) = G_{k+1}^{''}(z) + (k+1)^2 a_0^2 G_{k+1}(z), \qquad (12)$$

then equation (11) *is valid and*

$$\phi(z) = \prod_{j=1}^{k}\Big(\frac{d^2}{dz^2} + j^2 a_0^2\Big)G_k(z)\ \ (k = 1,2,\dots,n), \qquad (13)$$

$$G_k^*(z) = \prod_{j=2}^{k}\Big[\frac{d^2}{dz^2} + (j-1)^2 a_0^2\Big]G_k(z)\ \ (k = 2,3,\dots,n). \qquad (14)$$

Proof. For $n = 1$ we have [3]:

$$\phi(z) = G_1^{*''}(z) + a_0^2 G_1^*(z). \tag{15}$$

By (12) and (15) we easily get

$$\begin{aligned}\phi(z) &= G_1^{*''}(z) + a_0^2 G_1^*(z) = G_1''(z) + a_0^2 G_1(z) = \\ &= \Big(\frac{d^2}{dz^2} + a_0^2\Big) G_1(z) = \Big(\frac{d^2}{dz^2} + a_0^2\Big)\Big(\frac{d^2}{dz^2} + 4a_0^2\Big) G_2(z) = \\ &= \Big(\frac{d^2}{dz^2} + a_0^2\Big)\Big(\frac{d^2}{dz^2} + 4a_0\Big)\Big(\frac{d^2}{dz^2} + 9a_0^2\Big) G_3(z) = \cdots = \\ &= \prod_{j=1}^{k} \Big(\frac{d^2}{dz^2} + j^2 a_0^2\Big) G_k(z).\end{aligned}$$

Next, substituting in (11) expression (13) instead of $\phi(z)$, we obtain

$$G_k^{*''}(z) + k^2 a_0^2 G_k^*(z) = \prod_{j=1}^{k} \Big(\frac{d^2}{dz^2} + j^2 a_0^2\Big) G_k(z),$$

whence

$$G_k^*(z) = \prod_{j=2}^{k} \Big[\frac{d^2}{dz^2} + (j-1)^2 a_0^2\Big] G_k(z).$$

Thus (13) and (14) are valid. □

It is not difficult to verify that conditions (11) are satisfied by virtue of (13) and (14), and hence the lemma is proved.

Introducing expressions (10) in formula (8), we arrive at

$$\begin{aligned}2\mu U &= (1 - A_0)\varphi(z) - A_0 z\overline{\phi(z)} - \overline{\psi(z)} + \\ &+ \frac{\overline{\phi(z)}}{a_0} \sum_{k=1}^{n} \frac{1}{k} \big[A_k \sin k(\overline{a}_0 z + a_0 \overline{z}) - B_k \cos k(\overline{a}_0 z + a_0 \overline{z})\big] - \\ &- \sum_{k=1}^{n} \Big\{ G_k^{*'}(z) \big[A_k \cos k(\overline{a}_0 z + a_0 \overline{z}) + B_k \sin k(\overline{a}_0 z + a_0 \overline{z})\big] - \\ &- a_0 k G_k^*(z) \big[A_k \sin k(\overline{a}_0 z + a_0 \overline{z}) - B_k \cos k(\overline{a}_0 z + a_0 \overline{z})\big] \Big\},\end{aligned} \tag{16}$$

where by $G_k^*(z)$ we mean expression (14).

Thus the following theorem is proved.

Theorem. *General solution of the equation of the plane theory of elasticity in displacements relating to the case of non-homogeneity of type* (5) *can in the absence of body forces be represented in the form of* (16).

To derive formulas for the stress components, we use formulas (2) and (16):

$$\frac{1}{2}(X_x+Y_y)=\Big\{A_0+\sum_{k=1}^{n}\big[A_k\cos k(\overline{a}_0z+a_0\overline{z})+$$
$$+B_k\sin k(\overline{a}_0z+a_0\overline{z})\big]\Big\}\big(\phi(z)+\overline{\phi(z)}\big), \tag{17}$$
$$\frac{1}{2}(Y_y-X_x-2iX_y)=A_0z\overline{\phi'(z)}+\overline{\Psi(z)}-$$
$$-\frac{\overline{\phi'(z)}}{a_0}\sum_{k=1}^{n}\frac{1}{k}\big[A_k\sin k(\overline{a}_0z+a_0\overline{z})-B_k\cos k(\overline{a}_0z+a_0\overline{z})\big]-$$
$$-\overline{\phi(z)}\sum_{k=1}^{n}\big[A_k\cos k(\overline{a}_0z+a_0\overline{z})+B_k\sin k(\overline{a}_0z+a_0\overline{z})\big]-$$
$$-a_0\sum_{k=1}^{n}\Big\{kG_k^{*'}(z)\big[A_k\sin k(\overline{a}_0z+a_0\overline{z})-B_k\cos k(\overline{a}_0z+a_0\overline{z})\big]+$$
$$+a_0k^2G_k^{*}(z)\big[A_k\cos k(\overline{a}_0z+a_0\overline{z})+B_k\sin k(\overline{a}_0z+a_0\overline{z})\big]\Big\}, \tag{18}$$

where $\psi'(z)=\Psi(z)$.

On the basis of formulas (2) and (16), we can easily calculate

$$f(x,y)=\frac{\partial F}{\partial x}+i\frac{\partial F}{\partial y}=A_0\varphi(z)+A_0z\overline{\phi(z)}+\overline{\psi(z)}-$$
$$-\frac{\overline{\phi(z)}}{a_0}\sum_{k=1}^{n}\frac{1}{k}\big[\sin k(\overline{a}_0z+a_0\overline{z})-B_k\cos k(\overline{a}_0z+a_0\overline{z})\big]+$$
$$+\sum_{k=1}^{n}\Big\{G_k^{*'}(z)\big[A_k\cos k(\overline{a}_0z+a_0\overline{z})+B_k\sin k(\overline{a}_0z+a_0\overline{z})\big]-$$
$$-a_0kG_k^{*}(z)\big[A_k\sin k(\overline{a}_0z+a_0\overline{z})-B_k\cos k(\overline{a}_0z+a_0\overline{z})\big]\Big\}. \tag{19}$$

2. Solution of the First and the Second Basic Problems for a Half-Plane

Let the region S, occupied by the body, consists of a lower non-homogeneous half-plane ($y<0$), bounded by the $0x$ axis. We denote it by S^- and the upper half-plane by S^+. The functions $\phi(z)$, $G_k(z)$ ($k=1,2,\dots,n$), $\Psi(z)$ will be assumed to be holomorphic in S^-.

Combining expressions (17) and (18), we get

$$Y_y-iX_y=A_0\phi(z)+A_0\overline{\phi(z)}+A_0z\overline{\phi'(z)}+\overline{\Psi(z)}+$$

$$
\begin{aligned}
&+\phi(z)\sum_{k=1}^{n}\big[A_k\cos k(\overline{a}_0z+a_0\overline{z})+B_k\sin k(\overline{a}_0z+a_0\overline{z})\big]-\\
&-\frac{\overline{\phi'(z)}}{a_0}\sum_{k=1}^{n}\frac{1}{k}\big[A_k\sin k(\overline{a}_0z+a_0\overline{z})-B_k\cos k(\overline{a}_0z+a_0\overline{z})\big]-\\
&-a_0\sum_{k=1}^{n}\Big\{kG_k^{*'}(z)\big[A_k\sin k(\overline{a}_0z+a_0\overline{z})-B_k\cos k(\overline{a}_0z+a_0\overline{z})\big]+\\
&+a_0k^2G_k^{*}(z)\big[A_k\cos k(\overline{a}_0z+a_0\overline{z})+B_k\sin k(\overline{a}_0z+a_0\overline{z})\big]\Big\}. \qquad (20)
\end{aligned}
$$

Now we write out one more formula obtained from (16) by differentiating with respect to x:

$$
\begin{aligned}
&2\mu\Big(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\Big)=(1-A_0)\phi(z)-A_0\overline{\phi(z)}-A_0z\overline{\phi'(z)}-\\
&-\overline{\Psi}(z)-\phi(z)\sum_{k=1}^{n}\big[A_k\cos k(\overline{a}_0z+a_0\overline{z})+B_k\sin k(\overline{a}_0z+a_0\overline{z})\big]+\\
&+\frac{\overline{\phi'(z)}}{a_0}\sum_{k=1}^{n}\frac{1}{k}\big[A_k\sin k(\overline{a}_0z+a_0\overline{z})-B_k\cos k(\overline{a}_0z+a_0\overline{z})\big]+\\
&+a_0\sum_{k=1}^{n}\Big\{kG_k^{*'}(z)\big[A_k\sin k(\overline{a}_0z+a_0\overline{z})-B_k\cos k(\overline{a}_0z+a_0\overline{z})\big]+\\
&+a_0k^2G_k^{*}(z)\big[A_k\cos k(\overline{a}_0z+a_0\overline{z})+B_k\sin k(\overline{a}_0z+a_0\overline{z})\big]\Big\}. \qquad (21)
\end{aligned}
$$

Suppose that X_x, Y_y, X_y tend to zero as z tends to infinity in any direction, remaining inside S^-, and the principal vector (x,y) of exterior forces applied to the segment $0x$ tends to the limit as its ends vanish. Applying formula (19) and that of [4], $X+iY=i[\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}]_{-\infty}^{\infty}$, the holomorphic functions $\phi(z)$ and $\Psi(z)$ for large $|z|$ can in complete analogy with the homogeneous half-plane ([4], §90) be represented as

$$
\begin{aligned}
\phi(z)&=-\frac{X+iY}{2\pi A_0}\,\frac{1}{z}+o\Big(\frac{1}{z}\Big),\\
\Psi(z)&=\frac{X-iY}{2\pi}\,\frac{1}{z}+o\Big(\frac{1}{z}\Big).
\end{aligned}
\qquad (22)
$$

To conditions (22) we add the conditions

$$
\begin{aligned}
G_k(z)&=-\frac{X+iY}{2\pi(k!)^2a_0^{2k}A_0}\,\frac{1}{z}+o\Big(\frac{1}{z}\Big),\\
G_k^{*}(z)&=-\frac{X+iY}{2\pi k^2a_0^2A_0}\,\frac{1}{z}+o\Big(\frac{1}{z}\Big).
\end{aligned}
\qquad (23)
$$

obtained from formulas (13) and (14) and also the conditions

$$\begin{aligned} \varphi(z) &= -\frac{X+iY}{2\pi A_0}\ln z + o(1) + \text{const}, \\ \psi(z) &= \frac{X-iY}{2\pi}\ln z + o(1)\,\text{const}, \end{aligned} \tag{24}$$

obtained from formulas (22) by integrating with respect to z.

a) **The First Basic Problem.** In this problem the exterior stresses are given: the pressure $P(t) = -Y_y^-$ and the tangential stress $T(t) = X_y^-$ are applied to the whole boundary $0x$ which will in the sequel be denoted by L. It will be assumed that $P(t)$ and $T(t)$ satisfy the Hölder condition on L, including the point at infinity, and vanish as $t \to \infty$. The signs $(-)$ and $(+)$ denote boundary values adopted by the function from the lower and from the upper half-plane.

According to (20), the boundary condition takes the form

$$\begin{aligned} &\Bigl(A_0 + \sum_{k=1}^{n} A_k\Bigr)\phi^-(t) + a_0\sum_{k=1}^{n} k\bigl(B_k G_k^{*'-}(t) - a_0 k A_k G_k^{*-}(t)\bigr) + \\ &+A_0\overline{\phi^-(t)} + A_0 t\overline{\phi'^-(t)} + \overline{\Psi^-(t)} + \frac{\overline{\phi'^-(t)}}{a_0}\sum_{k=1}^{n}\frac{1}{k}B_k = \\ &= -P(t) - iT(t) \quad \text{on} \quad L. \end{aligned} \tag{25}$$

Denote by $\Omega(z)$ a piecewise holomorphic function defined as

$$\Omega(z) = \\ = \begin{cases} \Bigl(A_0 + \sum\limits_{k=1}^{n}\Bigr)\phi(z) + a_0\sum\limits_{k=1}^{n} k\bigl(B_k G_k^{*'}(z) - a_0 k A_k G_k^*(z)\bigr) & \text{in } S^-, \\ -A_0\overline{\phi}(z) - A_0 z\overline{\phi'}(z) - \overline{\Psi}(z) - \frac{\overline{\Phi}'(z)}{a_0}\sum\limits_{k=1}^{n}\frac{1}{k}B_k & \text{in } S^+, \end{cases} \tag{26}$$

where $\overline{\gamma}(z)$ is the function obtained from $\gamma(z)$ as follows: $\overline{\gamma}(z) = \overline{\gamma(\overline{z})}.$ If $\gamma(z)-$ is holomorphic in S^-, then $\overline{\gamma}(z)$ is holomorphic in S^+, and $\overline{\gamma(t)^-} = \overline{\gamma}(t)^+ (\overline{\gamma(t)^+} = \overline{\gamma}(t)^-)$.

From formula (26) we have

$$\Psi(z) = -\overline{\Omega}(z) - A_0\phi(z) - A_0 z\phi'(z) + \frac{\phi'(z)}{a_0}\sum_{k=1}^{n}\frac{1}{k}B_k \quad \text{in} \quad S^-. \tag{27}$$

Substitution of expression (27) in (25) results in

$$\Omega^+(t) - \Omega^-(t) = P(t) + iT(t) \quad \text{on} \quad L. \tag{28}$$

The solution of the boundary value problem (28), vanishing at infinity, can be written in the form [5]

$$\Omega(z)=\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{P(t)+iT(t)}{t-z}\,dt \quad \text{in} \quad S^-,$$

where

$$\Omega(z)=\Big(A_0+\sum_{k=1}^{n}\Big)\phi(z)+a_0\sum_{k=1}^{n}k\big(B_kG_k^{*'}(z)-a_0kA_kG_k^*(z)\big).$$

Substituting in the later the expressions given by formulas (12)–(14), we obtain the ordinary differential equation of the second order with constant coefficients:

$$D_0G_n^{(2n)}(z)+D_1a_0G_n^{(2n-1)}(z)+D_2a_0^2G_n^{(2n-2)}(z)+\cdots+D_{2n-1}a_0^{2n-1}G_n'(z)+$$
$$+D_{2n}a_0^{2n}G_n(z)=\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{P(t)+iT(t)}{t-z}\,dt, \tag{29}$$

where

$$D_{2j}=\big[1^2\cdot2^2\cdots j^2+\cdots+(n-j+1)^2(n-j+2)^2\cdots n^2\big]\sum_{k=0}^{n}A_k-$$
$$-\big[1^2\cdot2^2\cdots(j-1)^2+\cdots+(n-j+2)^2(n-j+3)^2\cdots n^2\big]\sum_{k=1}^{n}k^2A_k+$$
$$+\big[1^2\cdot2^2\cdots(j-2)^2+\cdots+(n-j+3)^2(n-j+4)^2\cdots n^2\big]\sum_{k=1}^{n}k^4A_k+$$
$$+\cdots+(-1)^{j-1}(1^2+2^2+\cdots+n^2)\sum_{k=1}^{n}k^{2j-2}A_k+$$
$$+(-1)^j\sum_{k=1}^{n}k^{2j}A_k, \tag{30}$$

$$D_{2j-1}=\big[1^2\cdots(j-1)^2+\cdots+(n-j+2)^2(n-j+3)^2\cdots n^2\big]\sum_{k=1}^{n}kB_k-$$
$$-\big[1^2\cdot2^2\cdots(j-2)^2+\cdots+(n-j+3)^2(n-j+4)^2\cdots n^2\big]\sum_{k=1}^{n}k^3B_k+$$
$$+\cdots+(-1)^j(1^2+2^2+\cdots+n^2)\sum_{k=1}^{n}k^{2j-3}B_k+$$
$$+(-1)^{j+1}\sum_{k=1}^{n}k^{2j-1}B_k, \quad j=0,1,2,\dots,n. \tag{31}$$

For example, for $n=3$, from formulas (29)–(31) it follows that

$$D_0G_3^{(6)}(z)+D_1a_0G_3^{(5)}(z)+D_2a_0G_3^{(4)}(z)+D_3a_0^3G_3^{'''}(z)+$$
$$+D_4a_0^4G_3^{''}(z)+D_5a_0^5G_3^{'}(z)+D_6a_0^6G_3(z)=$$
$$=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{P(t)+iT(t)}{t-z}\,dt,$$

where

$$D_0=A_0+A_1+A_3,\quad D_1=B_1+2B_2+3B_3,\quad D_2=14A_0+13A_1+$$
$$+10A_2+5A_3,\quad D_3=13B_1+20B_2+15B_3,\quad D_4=49A_0+$$
$$+36A_1+9A_2+4A_3,\quad D_5=36B_1+18B_2+12B_3,\quad D_6=36A_0.$$

Having found the solution (29) vanishes at infinity, we can find the functions $\phi(z)$ and $\Psi(z)$ by using formulas (14) and (27).

b) **The Second Basic Problem** requires definition of elastic equilibrium of the body by the given displacements of its boundary points. Solution of this problem is in complete analogy with the previous one. Thus from condition (21) we have

$$\Big(1-A_0-\sum_{k=1}^{n}\Big)\phi^-(t)-a_0\sum_{k=1}^{n}k\big(B_kG_k^{*'-}(t)-a_0kA_kG_k^{*-}(t)\big)-A_0\overline{\Phi^-(t)}-$$
$$-A_0t\overline{\phi'^-(t)}-\overline{\Psi^-(t)}-\frac{\overline{\phi'^-(t)}}{a_0}\sum_{k=1}^{n}\frac{1}{k}B_k=2\mu\big(g_1'(t)+ig_2'(t)\big),\qquad(32)$$

where $g_1(t)$ and $g_2(t)$ are the given on L functions with their derivatives $g_1'(t)$ and $g_2'(t)$ which satisfy the Hölder condition, including the point at infinity, and vanish for $t=\infty$.

Denote

$$\Omega(z)=$$
$$=\begin{cases}\Big(A_0+\sum\limits_{k=1}^{n}-1\Big)\phi(z)+a_0\sum\limits_{k=1}^{n}k\big(B_kG_k^{*'}(z)-a_0kA_kG_k^{*}(z)\big)\ \text{ in }\ S^-,\\ -A_0\overline{\phi}(z)-A_0z\overline{\phi}(z)-\overline{\Psi}(z)-\frac{\overline{\phi}'(z)}{a_0}\sum\limits_{k=1}^{n}\frac{1}{k}B_k\ \text{ in }\ S^+.\end{cases}\qquad(33)$$

Obviously,

$$\Psi(z)=-\overline{\Omega}(z)-A_0\phi(z)-A_0z\phi'(z)+\frac{\phi'(z)}{a_0}\sum_{k=1}^{n}\frac{1}{k}B_k\ \text{ in }\ S^-.\qquad(34)$$

On the basis of (33) and (34), condition (32) takes the form

$$\Omega^+(t)-\Omega^-(t)=2\mu\big(g_1'(t)+ig_2'(t)\big)\ \text{ on }\ L,$$

whence

$$\Omega(z)=\frac{\mu}{\pi i}\int\limits_{-\infty}^{\infty}\frac{g_1(t)+ig_2(t)}{t-z}\,dt. \tag{35}$$

From formulas (33) and (35) follows

$$\Big(A_0-1+\sum_{k=1}^{n}A_k\Big)\phi(z)+a_0\sum_{k=1}^{n}k\big(B_kG_k^{*'}(z)-a_0kA_kG_k^{*}(z)\big)=$$
$$=\frac{\mu}{\pi i}\int\limits_{-\infty}^{\infty}\frac{g_1'(t)+ig_2'(t)}{t-z}\,dt \quad \text{in} \quad S^-.$$

Substituting in the latter formulas (12)–(14), we get

$$D_0^1G_n^{(2n)}(z)+D_1^1a_0G_n^{(2n-1)}(z)+D_2^1a_0^2G_n^{(2n-2)}(z)+\cdots+$$
$$+D_{2n-1}^1a_0^{2n-1}G_n'(z)+D_{2n}^1a_0^{2n}G_n(z)=\frac{\mu}{\pi i}\int\limits_{-\infty}^{\infty}\frac{g_1'(t)+ig_2'(t)}{t-z}\,dt.$$

The values D_{2j}^1, D_{2j-1}^1 $(j=0,1,2,\dots,n)$ can be obtained if in (30) and (31) A_0 will be replaced by A_0-1.

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