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ON THE CALCULATION OF RADII CONNECTED WITH SOME QUASI-CONFORMAL MAPPINGS OF MULTIPLY-CONNECTED DOMAINS

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ABSTRACT. By use of a method, allowing to find some quasi-conformal mappings of finite doubly amd multiply-connected domains to concrete canonical domains, the values of radii are calculated approximately and by their help the final forms of these canonical domains are found.

1. In [1] a method of solution of one mixed boundary problem connected with linear elliptic systems of first order differential equations is given. It is assumed that in finite triply connected domain G of complex plane bounded by simple closed piece-wise analytic contours $\Gamma_1, \Gamma_2, \Gamma_3$ (one of which, say, Γ_3 includes the rest) a uniformly elliptic system

$$\begin{aligned} -v_x &= du_x + cu_y + h, \\ v_y &= au_x + bu_y + g, \\ a > 0, \quad A &= ac - \left(\frac{d+b}{2}\right) \geq k_0 > 0, \end{aligned} \tag{1}$$

is given the coefficients of which have continuous partial derivatives in Holder sense in closed domain $\overline{G}$.

By the elimination of the function $v(x, y)$ from (1) we come to elliptic differential equation of second order with respect to $u(x, y)$:

$$\begin{aligned} M[u] = au_{xx} + (b+d)u_{xy} + cu_{yy} + (a_x + d_y)u_y + \\ +(b_x + c_y)u_y = -(g_x + h_y). \end{aligned} \tag{2}$$

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The second order equation $M_1[v] = 0$ with one unknown function $v(x, y)$ can be obtained similarly.

In the concrete case when $h = g = 0$, $b = d$, and $ac - b^2 = 1$ the corresponding system is called Beltrami system and the equations $M[u] = 0$, $M[v] = 0$ coinside.

In [1] the following mixed boundary problem is solved: to find a function in the class of twice continuously differentiable in G functions, the partial derivatives of which are continuous up to boundary, which satisfies to the equation (2) in G and to the following conditions

$$u|_{t\in\Gamma_i} = f(t) + a_i \quad (i = 1, 2, 3),$$
$$\int\limits_{\Gamma_i} [(du_x + cu_y + h)dx - (au_x + bu_y + g)dy] = 0 \ \ (i = 1, 2),$$

on the boundary $\Gamma = \bigcup\limits_1^3 \Gamma_i$. Here $f(t) \in C^1$ is a given function of points of the contour Γ, and a_1, a_2 a_3 are constants not given in advance and which must be defined in the process of solution of the problem. Only one of them is being fixed (say, $a_3 = r_0$).

If we introduce the complex function $w(z) = u + iv$ and the known differential operators $w_{\bar z}$, w_z the system (1) can be written in equivalent complex form

$$w_{\bar z} = q_1(z)w_z + q_2(z)\overline{w(z)} + c(z), \tag{3}$$

where the coefficients $q_1(z)$, $q_2(z)$, $c(z)$ are expressed explicitely [1] by the coefficients of the system (1) and $|q_1(z)| + |q_2(z)| \le q_0 < 1$ ($q_0 = const$) for all $z \in G$.

It is said that the complex function $w = w(z)$ realizes quasi-conformal mapping of the domain G to the domain G_1 corresponding to the given system of the elliptic equations if $w = w(z)$ maps homeomorphically G to G_1 and satisfies to this system.

Let us take an arbitrary point z_0 (let z_0=0)in the domain *int* Γ_1 and consider in G an equation of the form

$$w_{\bar z} = q_1(z)w_z + q_2(z)e^{i\alpha(z)}\overline{w_z}. \tag{4}$$

By direct check we get convinced that if in (4) $\alpha(z) = 2\arg z + 2\,\mathrm{Im}\, w(z)$, where $w(z)$ is the solution to the equation

$$w_{\bar z} = q_1 w_z + q_2\overline{w(z)} + \frac{q_1}{z} + \frac{q_2}{z} \tag{5}$$

in the domain G then the function

$$w = ze^{\omega(z)} \tag{6}$$

satisfies in G to the equation (4).

Under the accepted assumptions the problem [1] is being solved: to find a homeomorphic solution of the equation (4) $w = \varphi(z)$ such that $\varphi(G)$ is a circular ring $r_1 < |w| < R$ with a cut along the arc of the circle $|w| = r_2 (r_1 < r_2 < R)$, where among the indicated radii only the outer R is being fixed in advance. Such canonical domain is denoted by K.

Having written (5) as a system, assuming $\omega = U + iV$ and solving for it the concrete mixed boundary problem, we prove the following

Theorem. *The function*

$$w = \varphi(z) = ze^{\omega^*(z)}$$

where $\omega^*(z) = U(x, y)+iV(x, y)$ *satisfies to* (5)*, realizes the quasi-conformal mapping* (*corresponding to* (4)) *of triply connected domain* G *to the domain* K *if in* (4) *we mean that* $\alpha(z) = 2\arg z+2V(z)$*. At this the complete circles* $|w| = r_1$ *and* $|w| = R$ *are images of the contours* Γ_1 *and* Γ_2 *respectively.*

Here $U(x, y)$ *and* $V(x, y)$ *are solutions to the system*

$$-V_x = dU_x + cU_y + h_1,$$
$$V_y = aU_x + bu_y + g_1$$

corresponding to (5)*:*

$$\begin{gathered} U(x, y) = \ln\ r_1 \cdot u_1(x, y) + \ln r_2 \cdot u_2(x, y) + \\ + \ln R \cdot [1 - (u_1 + u_2)] - \ln |z|, \\ V(x, y) = \int\limits_{(x_0,y_0)}^{(x,y)} \Big(a\frac{\partial U}{\partial x} + b\frac{\partial U}{\partial y} + g_1\Big)\, dy - \\ -\Big(d\frac{\partial U}{\partial x} + c\frac{\partial U}{\partial y} + h_1\Big)\, dx + C, \end{gathered} \tag{7}$$

where $(x_0, y_0) \in G$ *is an arbitrary point, and* C *is a real constant.*

The real functions $h_1(x, y)$, $g_1(x, y)$ *are present because of the presence of free term* $\frac{q_1}{z} + \frac{q_2}{\bar z}$.

$u_1(x, y)$ *and* $u_2(x, y)$ *are solutions of the following boundary problems:*

$$\begin{matrix} M[u] = 0 & M[u] = 0 \\ u|_{\Gamma_1} = 1,\ u\,|_{\Gamma_2} = u|\Gamma_3 = 0; & u|_{\Gamma_2} = 1,\ u|_{\Gamma_1} = u|\Gamma_3 = 0 \end{matrix} \tag{8}$$

respectively.

The numbers r_1 *and* r_2 $(r_1 < r_2)$ *from* (7) *are defined by formulas*

$$\ln r_1 = \ln R - \frac{2\pi c_{22}}{\Delta}, \quad \ln r_2 = \ln R + \frac{2\pi c_{21}}{\Delta}, \tag{9}$$

where $\Delta = c_{11}c_{22} - c_{12}c_{21}$ *and*

$$c_{ij} = \int\limits_{\Gamma_i} \Big(d\frac{\partial u_j}{\partial x} + c\frac{\partial u_j}{\partial y}\Big)dx - \Big(a\frac{\partial u_j}{\partial x} + b\frac{\partial u_j}{\partial y}\Big)\,dx \tag{10}$$

$$(i,j) = 1,2), \quad (c_{ii} > 0,\ c_{ij} < 0,\ i \neq j,\ \Delta > 0).$$

In the case when the contour Γ is absent, i.e. the number of connectivity of Γ equals to two, for calculation of one radius r ($r_1 = r$) we come the equation

$$(\ln r - \ln R)c_{11} = -2\pi$$

and according to the Green's formula

$$c_{11} = I_G[u_1] = \iint\limits_G \Big[a\Big(\frac{\partial u_1}{\partial x}\Big)^2 + (b+d)\frac{\partial u_1}{\partial x}\frac{\partial u_1}{\partial y} + c\Big(\frac{\partial u_1}{\partial y}\Big)^2\Big]\,dxdy$$

we have

$$\ln r = \ln R - \frac{2\pi}{I_G[u_1]}. \tag{11}$$

2. In this paper one part of the algorythm (7)–(10) is realized numerically. Namely, $u_1(x,y)$, $u_2(x,y)$, $I_G[u_1]$, $I_G[u_2]$, c_{12}, c_{21} and the radii of the canonical domain $K(r_1,r_2;1)$ are found approximately. A square with tops at the points $(-1,-1)$, $(-1,1)$, $(1,1)$ and $(1,-1)$ was taken in the role of the initial domain. Two rectangles with sides paralel to axes and boundaries Γ_1, Γ_2 were removed from it.

In the Table 1 the results of numerical experiments for various cases of quasi-conformal mappings (including conformal case) are represented. In the experiments sizes of the rectangle holes were varied, namely, by $G(\Gamma_1,\Gamma_2)$ the initial square domain is denoted from which two rectangles with tops at the points $(-0.35,-0.35)$, $(-0.35,0.2)$, $(0.2,0.2)$, $(0.2,-0.35)$ (denoted by Γ_1) and $(0.25,-0.8)$, $(0.25,-0.6)$, $(0.7,-0.6)$ $(0.7,-0.8)$ (denoted by Γ_2) are removed. By $G(\Gamma_1,\Gamma_2^+)$, $G(\Gamma_1^+,\Gamma_2)$ and $G(\Gamma_1^+,\Gamma_2^+)$ are denoted the domains in which the second (Γ_2), the first (Γ_1) and both (Γ_1,Γ_2) holes are enlarged respectively comparing with $G(\Gamma_1,\Gamma_2)$. In particular, the tops of Γ_1^+ are at the points $(-0.4,-0.4)$, $(-0.4,0.25)$, $(0.25,0.25)$, $(0.25,-0.4)$ and the tops of the Γ_2^+ are at the points $(0.2,-0.85)$, $(0.2,-0.55)$, $(0.75,-0.55)$ and $(-0.75,-0.85)$.

The local variation method [2] was used for calculation of the functions $u_1(x,y)$, $u_2(x.y)$. The justifying and estimates of convergence rate in various metrics are given in [3]. The following estimate $\|u - u^{\delta h}\|_1 \leq M\delta^2$ ($M = const$) is valid among them where $\| * \|$ is the first norm of a function (see [4]), u and $u^{\delta h}$ are the exact and approximate solutions to the corresponding variational problem, δ is the step of the grid, h is the value of variation (under the condition $h = 0(\delta^4)$).

In the doubly connected case the modulus $M(a,b,c,d;G) = 1/\rho$ was found according to (11) (see the Table 2). The results of calculations (in conformal case) were compared with the results of numerical experiments represented in [5]. In Table 2 $G_a^{(1)}$ is a square frame: $G_a^{(1)} = \{\text{ext } Q_a\} \cap \{\text{int } Q_1\}$ where Q_a is a square with center at the origin and the length of the side $2a$ (the sides are paralel to axes).

Table 1

	$G(\Gamma_1, \Gamma_2)$	$G(\Gamma_1, \Gamma_2^+)$	$G(\Gamma_1^+, \Gamma_2^+)$	$G(\Gamma_1^+, \Gamma_2)$
	$a = 1.2,\ b = -4,\ c = 4,\ d = 1.$			
r_1	0.4083	0.4425	0.5147	0.4767
r_2	0.9112	0.9694	0.9278	0.8769
	$a = x^4 + y^4 + 0.5,\ b = d = \sqrt{x^4 + y^4},\ c = 1.$			
r_1	0.0513	0.0574	0.0885	0.0794
r_2	0.3831	0.4601	0.4148	0.3481
	$a = \frac{1}{4}e^{x^2+y^2},\ b = 0,\ c = e^{-(x^2+y^2)} + 1,\ d = \frac{1}{2}\sqrt{e^{x^2+y^2}}.$			
r_1	0.0813	0.0921	0.1403	0.1245
r_2	0.6879	0.6868	0.5836	0.5784
	$a = 1.2,\ b = d = \sqrt{0.32},\ c = 1.1$ (Beltrami system)			
r_1	0.1075	0.1126	0.1497	0.1438
r_2	0.4885	0.5260	0.4685	0.4418
	$a = 2 + 2(x^2 + y^2),\ b = d = \sqrt{x^2 + y^2},\ c = 0.5$ (Beltrami system)			
r_1	0.2093	0.2115	0.2764	0.2722
r_2	0.7586	0.7910	0.7600	0.7301
	$a = c = 1,\ b = d = 0.$ (conformal case)			
r_1	0.0875	0.0939	0.1231	0.1170
r_2	0.3486	0.3955	0.3558	0.3238

Table 2

	$G_{0.5}^{(1)}$	$G_{0.8}^{(1)}$
	$a = 1.2,\ \ b = -4,\ \ c = 4,\ \ d = 1$	
$\frac{1}{\varrho}$	1.2831	1.0739
	$a = x^4 + y^4 + 0.5\ b = d = \sqrt{x^4 + y^4},\ c = 1$	
$\frac{1}{\varrho}$	1.9690	1.1762
	$a = \frac{1}{4}e^{x^2+y^2},\ b = 0,\ c = e^{-(x^2+y^2)} + 1,\ d = \frac{1}{2}\sqrt{e^{x^2+y^2}}$	
$\frac{1}{\varrho}$	1.8613	1.1899
	$a = c = 1,\ b = 0$ (conformal case,appr.sol.)	
$\frac{1}{\varrho}$	1.8300	1.1993
	$a = c = 1,\ b = 0$ (conformal case, exact sol.)	
$\frac{1}{\varrho}$	1.8477	1.2015

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