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ON THE DIRICHLET PROBLEM IN SMIRNOV CLASSES OF HARMONIC FUNCTIONS

Z. MESHVELIANI AND V. PAATASHVILI

Abstract. The questions of the solvability of the Dirichlet problem are studied in Smirnov classes of harmonic functions, whose integral means are bounded along some family of curves which converge to the boundary.

Lately, a vast number of works are devoted to the investigation of boundary value problems in domains with non-smooth boundaries. In particular, the Dirichlet and Neumann problems have been studied for harmonic functions from different classes.

In [1] and [2] (see also [3]), the Dirichlet and Neumann problems are studied in Smirnov classes of harmonic functions.

In the present work we investigate the Dirichlet problem in classes of the same type.

Let Γ be a simple, closed, oriented, piecewise Ljapunov curve bounding a finite domain D. The curve Γ will be assumed to have one angular point A at which at has interior (with respect to D) angle of size $\delta\pi$, $0 \leq \delta \leq 2$. In what follows, a set of such curves will be assumed by $C_A^{1,\delta}$.

Let $z = z(w)$ be a function, mapping conformally a circle $U = \{w : |w| < 1\}$ onto D, and let $w = w(z)$ be an inverse function. Suppose $\Gamma_r = \{z : z = z(w),\ |w| = r,\ 0 < r < 1\}$, $\gamma = \{w : |w| = 1\}$.

Denote by $E_p(D)$, $p > 0$ a class of Smirnov analytic functions, i.e., a set of analytic in D functions ϕ for which

$$\sup_{0<r<1} \int_{\Gamma_2} |\phi(z)|^p |dz| < \infty.$$

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If the domain of D is the unit circle U, then the class $E_p(U)$ coincides with Hardy's class H_p. By analogy with the class $E_p(D)$, we define a class $\widetilde{e}_p(D)$, i.e, a set of harmonic in D functions u for which

$$\sup_{0<r<1}\int\limits_{\Gamma_2}|u(z)|^p|dz|<\infty.$$

Suppose

$$e_p(D)=\{u:u=\operatorname{Re}\phi,\ \ \phi\in E_p(D)\}.$$

If the domain D is the unit circle U, then the class $\widetilde{e}_p(D)$ coincides with the class h_p, and taking into account the M. Riess theorem (see [4], p. 431), we obtain

$$\widetilde{e}_p(U)=e_p(U)=h_p.$$

Classes $e_p(D)$ and $\widetilde{e}_p(D)$ differ in a general case, and $e_p(D)\subset\widetilde{e}_p(D)$ (see [5]).

We consider the Dirichlet problem which is formulated as follows: Find a function D defined in the domain $u(z)$ and satisfying the following conditions:

$$\begin{cases}\Delta u=0, \ \ u\in\widetilde{e}_p(D), \ \ p>1,\\ u^+(t)=f(t), \quad f\in L^p(\Gamma).\end{cases}\tag{1}$$

It is not difficult to state that if the boundary of a simply connected domain is a curve of the class $C_A^{1,\delta}$, $0\le\delta<p$ then any function of the class $\widetilde{e}_p(D)$ has angular boundary values for almost all $t\in\Gamma$.

Lemma. *Let D be a simply connected, finite domain bounded by the curve $\Gamma\in C_A^{1,\delta}$, $0\le\delta\le 2$. If $u\in\widetilde{e}_p(D)$, then the function $u^*(w)=u(z(w))$, where $z=z(w)$ is the conformal mapping of U onto D, is of the class h_β for some $\beta>0$.*

Proof. Since $\Gamma\in C_A^{1,\delta}$, for the derivative $z(w)$ of the conformal mapping $z=z(w)$, $z(a)=A$ the representation

$$z'(w)=(w-a)^{\delta-1}\cdot z_0(w),\tag{2}$$

is valid, where $z_0(w)$ is a continuous, different from zero analytic function on $\overline{U}$ for $0<\delta\le 2$, and for $\delta=0$, $[z_0(w)]^{\pm1}\in\bigcap\limits_{p>0}H_p$ (see [6], [7]).

We now have

$$\int\limits_{\gamma_r}|u^*(w)|^\beta|dw|=\int\limits_{\gamma_r}\Big|\frac{u(z(w))\cdot\sqrt[p]{z'(w)}}{\sqrt[p]{z'(w)}}\Big|^\beta|dw|\le$$

$$\le\bigg(\int\limits_{\gamma_r}\Big(\big|u(z(w))\sqrt[p]{z'(w)}\big|^\beta\Big)^{\frac{p}{\beta}}|dw|\bigg)^{\beta/p}\bigg(\int\limits_{\gamma_r}\frac{dw}{|z'(w)|^{p\beta/(p-\beta)}}\bigg)^{\frac{p-\beta}{p}}\tag{3}$$

Since $u(z) \in \widetilde{e}_p(D)$, the first multiplier in (3) is bounded with respect to r. If we choose β such that $\frac{p\beta}{p-\beta} < 1$, then the second multiplier will likewise be the same. $\square$

Theorem 1. *Let D be a simply connected, finite domain bounded by the curve $\Gamma \in C_A^{1,\delta}$, $0 \le \delta \le 2$. Then:*

(a) if $0 \le \delta < p$, then the problem

$$\begin{cases} \Delta u = 0, \quad u \in \widetilde{e}_p(D), \quad p > 1 \\ u^+(t) = 0 \end{cases} \tag{4}$$

has only a trivial solution;

(b) if $1 < p \le \delta$, then (4) has at least one non-zero solution, and such is the function

$$u_0(z) = \operatorname{Re} \frac{w(z) + w(A)}{w(z) - w(A)}. \tag{5}$$

Proof. Suppose $\delta = 0$. Then, if $u(z)$ is a solution of problem (4), we can easily show that $u^*(w) = u(z(w))$ is the function of the class h_q, where q is an arbitrary positive number satisfying the condition $q < p$. But in this case it is representable by the Poisson integral of its boundary values. Taking into account the fact that by our supposition $u^+(t) \equiv 0$, we get $u(z) \equiv 0$.

If $0 < \delta < p$, then $\sqrt[p]{z'} \in w_p(\Gamma)$, and in this case $\widetilde{e}_p(D) = e_p(D)$ (see [5]). The fulfilment of the equality, as is known (see [1]), provides for the homogeneous Dirichlet problem in the class $e_p(D)$ only a trivial solution.

Let $1 < p \le \delta$. Since $\Gamma \in C_A^{1,\delta}$, according to (2) we have $|z'| \sim |1-w|^{\delta-1}$. (We put $a = 1$). To prove the lemma, it suffices to establish that if $u^*(w) = u_0(z(w))$, then

$$I = \sup_{0<r<1} \int_0^{2\pi} |u^*(re^{i\theta})|^p \left(\sqrt{1 + r^2 - 2r\cos\theta}\right)^{\delta-1} d\theta = \sup_{0<r<1} I_r < \infty.$$

We have

$$I_r = \int_0^{2\pi} \left| \frac{1-r^2}{1+r^2-2r\cos\theta} \right|^{\frac{p}{2}} \left(1 + r^2 - 2r\cos\theta\right)^{(\delta-1)/2} \le$$

$$\le 2^{p-1} \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos\theta} \frac{d\theta}{(1+r^2-2r\cos\theta)^{(p-\delta)/2}}. \tag{6}$$

Since $p \le \delta$, therefore $(1 + r^2 - 2r\cos\theta)^{(p-\delta)/2} \le 2^{\delta-p}$, and from (6) follows

$$I_r \le 2^\delta \cdot \pi.$$

Consequently,

$$I \le 2^\delta \cdot \pi.$$

Thus, if $\Gamma \in C_A^{1,\delta}$, $1 < p \le \delta$ then problem (4) in the class $\widetilde{e}_p(D)$ has a non-trivial solution $u_0(z)$ which is expressed by formula (5). □

In particular, if $\Gamma \in C_A^{1,\delta}$, $1 < \delta \le 2$, then the homogeneous Dirichlet problem (4) in the class $\widetilde{e}_\delta(D)$ has a non-trivial solution, whereas the same problem in the class e_δ has only a trivial solution (see [1]).

Theorem 2. *Let D be a simply connected, finite domain bounded by the curve $\Gamma \in C_A^{1,\delta}$, $0 < \delta \le 2$. Then:*

(*a*) *for the condition $0 < \delta < p$ problem* (1) *is uniquely solvable and the solution is the function*

$$u_f(z) = \operatorname{Re}\Big[\frac{1}{2\pi i}\int\limits_{\Gamma} \frac{f(t)}{w(t)}\,\frac{w(t)+w(z)}{w(t)-w(z)}\cdot w'(t)\,dt\Big]. \tag{7}$$

(*b*) *if $1 < p < \delta$, then problem* (1) *is solvable, and a set of its solutions contains all functions of the type*

$$u(z) = u_f(z) + u_0(z),$$

where

$$u_f(z) = \operatorname{Re}\bigg[\frac{1}{2\pi i}\cdot\frac{1}{w(z)-w(A)}\times \times\int\limits_{\Gamma} \frac{f(t)(w(t)-w(A))}{w^2(t)}\cdot\frac{w^2(t)+w^2(z)}{w(t)-w(z)}w'(t)\,dt\bigg], \tag{8}$$

and $u_0(z)$ is given by equality (5)*;*

(*c*) *for $\delta = p$, problem* (1) *is solvable if for the function f besides the condition $f \in L^p(\Gamma)$ the following condition is also fulfilled: there is a function ψ, such that*

$$\psi \in L^\infty(\Gamma), \quad \int\limits_{\Gamma} \frac{|\psi(\zeta)|^{\delta'}}{|\zeta - A|}\,d\zeta < \infty, \tag{9}$$

where $\delta' = \frac{\delta-1}{\delta}$ and $\frac{f}{\psi} \in L^p(\Gamma)$.

When (9) *is fulfilled, a set of solutions of problem* (1) *contains all functions of the type*

$$u(z) = u_f(z) + u_0(z),$$

where

$$u_f\big(z(re^{i\theta})\big) = \frac{1}{2\pi}\int\limits_0^{2\pi} f\big(z(e^{it})\big)\frac{1-r^2}{1+r^2-2r\cos(t-\theta)}\,dt$$

(*it is assumed that $w(A) = 1$*), *and $u_0(z)$ is given by equality* (5).

In particular, for $\psi = \frac{1}{\ln|\zeta - A|}$, we obtain that problem (1) *is solvable if $f(t)\cdot\ln|t-A| \in L^p(\Gamma)$.*

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Authors' addresses:

A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, Aleksidze St., Tbilisi 380093
Georgia

Sukhumi branch of
I. Javakhishvili Tbilisi State University
12, Jikia St., Tbilisi 380086
Georgia