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THE NEUMANN PROBLEM IN DOMAINS WITH PIECEWISE SMOOTH BOUNDARIES IN WEIGHT CLASSES OF HARMONIC SMIRNOV TYPE FUNCTIONS

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Abstract. In the present paper we investigate the Neumann problem in domains with piecewise smooth boundaries in weight classes of harmonic functions Smirnov type. It is found that the solvability and solution of the problem depend on weight and on the boundary of the domain.

A great number of works appeared for the recent years are devoted to the investigation of boundary value problems in domains with non-smooth boundaries. The emphasis in these works is placed on the case of boundary with cusps. In particular, the Dirichlet and Neumann problems have been studied for harmonic functions from different classes.

In [1], [2] (see also [3]), the Neumann problem is investigated in classes $e'_p(D)$, $p > 1$ (see equality (3) below), when the boundary of the domain D is a simple piecewise smooth curve. It turns out that violation of the boundary smoothness affects essentially the solvability of the problem. The Dirichlet problem in [4] has been considered in the weight class $e_p(D,\rho)$, $\rho(z) = (z - A)^\beta$, $\beta \in \mathbb{R}$, $A \in \Gamma$ (see equality (2) below). For $-\frac{1}{p} < \beta < \frac{1}{p'}$, the questions of Noethericity of the Dirichlet and Neumann problems have been studied in [5] in classes which correspond in our notation to the classes $e'_p(D,\rho)$ and $e'_p(D,\rho)$ (see equalities (2) and (4) below).

In all the above-mentioned works the most significant step is the reduction of the problems under consideration to the investigation of the problem of linear conjugation, indicated by N. Muskhelishvili (see [6], §41).

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In the present paper we study the Neumann problem in the classes $e'_p(D,\rho)$, $p>1$ for $\rho(z)=(z-A)\beta$, $\beta\in\mathbb{R}$. Using N. Muskhelishvili's method, this problem is reduced to the problem of linear conjugation for analytic functions representable by the Cauchy type integral. In new assumptions we have managed to obtain a well visible picture of solvability of the Neumann problem: the dependence of the problem of the solvability and solution on the functions $\rho(z)$ and on the geometry of the boundary. In all the cases, when the solution does exist, it is constructed explicitly by using the Cauchy type integrals and functions mapping conformally a circle onto a given domain.

In Section 1^0 we give main definitions; in Section 2^0 the Neumann problem is reduced to the problem of linear conjugation (20). The functions $y(w)$ and $X(w)$ introduced by equalities (24) and (30) are studied in Sections 3^0 and 4^0. By means of these functions the problem is reduced to the Sokhotskii problem whose solution is given in Section 5^0. The problem of linear conjugation (20) is considered in Section 7^0, and in Section 7^0 we present in the form of the theorem the statements regarding the solvability of the Neumann problem which follow from Sections 3^0–6^0.

1^0. Let Γ be a simple, closed, oriented piecewise smooth curve bounding a finite domain D. The curve Γ will be assumed to have one angular point A at which it has the inner, with respect to D, angle of size ν,π, $0\le\nu\le 2$. In what follows, a set of such curves will be denoted by C^ν_A.

Let $z=z(w)$ be a function, mapping conformally a circle $U=\{w:|w|<1\}$ onto D, and $w=w(z)$ be an inverse function. Suppose $\gamma_r=\{z: z=z(w),\ |w|=r,\ 0<r<1\}$, $\gamma=\{w:|w|=11\}$.

Let $\rho(z)=(z-A)^\beta$, $\beta\in\mathbb{R}$, be an analytic function on a plane, cut along the smooth curve which joins the point $z=A$ with $z=\infty$ and does not intersect Γ.

Denote by $E_p(D,\rho)$, $p>0$ a set of analytic in D functions ϕ for which

$$\sup_{0<r<1}\int_{\Gamma_r}\big|\phi(z)\cdot\rho(z)\big|^p|dz|<\infty. \tag{1}$$

This class for $\beta=0$ coincides with the Smirnov class $E_p(D)$. If the domain D is the unit circle $U=\{w:|w|<1\}$, then the class $E_p(U)$ coincides with the Hardy class H_p.

Suppose

$$e_p(D,\rho)=\big\{u:u=R\phi,\ \phi\in E_p(D,\rho)\big\}, \tag{2}$$

$$E'_p(D)=\big\{\phi:\phi'\in E_p(D)\big\},$$

$$e'_p(D)=\big\{u:u=R\phi,\ \phi\in E'_p(D)\big\}, \tag{3}$$

$$E'_p(D,\rho)=\big\{\phi:\phi'\cdot\rho\in E_p(D)\big\},$$

$$e'_p(D,\rho)=\big\{u:u=R\phi,\ \phi'\cdot\rho\in E_p(D)\big\}. \tag{4}$$

Since the functions from the class $E_p(D)$ possess angular boundary values at almost all points of Γ (see [7], p. 483), for every function from the class $e'_p(D,\rho)$ there exist at almost all points of Γ the angular boundary values of the derivative directed to the inner normal.

By $\mathcal{K}^p_n(\Gamma)$, $n \in N$ we denote a class of functions ψ, analytic on the set $\mathbb{C}\backslash\Gamma$, for which the representation

$$\phi(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\varphi(\tau)}{\tau - z} d\tau + q_n(z),$$
$$\varphi \in L^p(\Gamma), \ \ p \geq 1, \ \ q_n(z) = \sum_{j=0}^{n} a_j z^j, \tag{5}$$

is valid. If n is a negative integer, then under $\mathcal{K}^p_n(\Gamma)$ we mean a class of functions representable by the Cauchy type integral which at infinity have zero of order not less than $(-n)$. Obviously, the class $\mathcal{K}^p_{-1}(\Gamma)$ coincides with a set of functions representable by the Cauchy type integral with density from $L^p(\Gamma)$, and in the sequel it will be denoted by $\mathcal{K}^p(\Gamma)$. That is,

$$\mathcal{K}^p(\Gamma) = \Big\{\phi : \exists\varphi, \ \varphi \in L^p(\Gamma), \ \ \phi(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\varphi(\tau)}{\tau - z}\, d\tau\Big\}. \tag{6}$$

Suppose

$$\widetilde{\mathcal{K}}^p(\Gamma) = \cup(\Gamma)\mathcal{K}^p_n(\Gamma). \tag{7}$$

Let $w(t)$ be a measurable, almost everywhere finite on Γ function. We will say that $w(t)$ belongs to the class $W_p(\Gamma)$, $p > 1$, if it differs from zero almost everywhere on Γ, and the operator

$$T : f \longmapsto Tf; \ \ (Tf)(\zeta) = \frac{w(\zeta)}{\pi i}\int\limits_{\Gamma} \frac{f(t)}{w(t)} \cdot \frac{dt}{t - \zeta}; \ \ \zeta \in \Gamma, \tag{8}$$

is continuous in the space $L^p(\Gamma)$. Note that if $w \in W^p$, then $w^p \in A^p$, where A^p is the Muckenhoupt class.

The function $X(z)$, $z \in \mathbb{C}\backslash\Gamma$ is said to be a quotient function for $G(\zeta)$ in the class $\widetilde{\mathcal{K}}^p(\Gamma)$, $p > 1$ if: 1) $X \in \widetilde{\mathcal{K}}^p(\Gamma)$, 2) $\frac{1}{X} \in \widetilde{\mathcal{K}}'^p(\Gamma)$, $p' = \frac{p}{p-1}$, 3) $X^+ \in W_p(\Gamma)$, $G(\zeta) = \frac{X^+(\zeta)}{X^-(\zeta)}$, $\zeta \in \Gamma$.

A class of functions, continuous on the curve Γ, will be denoted by $C(\Gamma)$.

Let $\Gamma \in C^\nu_A$ and $\alpha(t)$ be an angle made by a directed tangent at the point t and the $0x$-axis. Since $\Gamma \in C^\nu_A$, we can define $\alpha(t)$ in such a way that it is continuous on the set $\Gamma\backslash\{A\}$, and

$$\alpha(A-) - \alpha(A+) = \lim_{t\to A-} \alpha(t) - \lim_{t\to A+} \alpha(t) = \pi(1+\delta). \tag{9}$$

If $\overrightarrow{n}$ is an inner normal at the point t and $u \in e'_p(D,p)$, then for angular boundary values of the functions $\frac{\partial u}{\partial n}$ almost everywhere on Γ we have (see [6] and [3])

$$\Big(\frac{\partial u}{\partial n}\Big)^+(t) = \Big(\frac{\partial u}{\partial x}\Big)^+(t)\cdot\cos(n;x) + \Big(\frac{\partial u}{\partial y}\Big)^+(t)\cdot\cos(n;y) = \\ = \Big(\frac{\partial u}{\partial x}\Big)^+(t)\big(-\sin\alpha(t)\big) + \Big(\frac{\partial u}{\partial y}\Big)^+(t)\cdot\cos\alpha(t). \tag{10}$$

2^0. Let D be a simply connected bounded domain with boundary Γ belonging to C_A^ν. Consider the Neumann problem formulated as follows: define in D a function u, satisfying the conditions

$$\begin{cases} \Delta u = 0, \ \ u \in e'_p(D\rho), & p > 1, \\ \Big(\frac{\partial u}{\partial n}\Big)^+(t) = f(t), & f \in L^p(\Gamma,\rho). \end{cases} \tag{11}$$

For $f \equiv 0$ we have the homogeneous problem corresponding to (11):

$$\begin{cases} \Delta u = 0, & u \in e'_p(D\rho), \ \ p > 1, \\ \Big(\frac{\partial u}{\partial n}\Big)^+(t) & \equiv 0. \end{cases} \tag{11_0}$$

Equalities in (11) and (11_0) are understood almost everywhere. Upon investigation of problems (11) and (11_0) the use will be made of the result obtained in [10].

If $\Gamma \in C_A^\nu$, $0 \le \nu \le 2$ bounds the finite domain D, $z = z(w)$ is the function mapping conformally the unit circle U onto D, and $z(a) = A$, then

$$\begin{cases} z'(w) = (w-a)^{\delta-1}\cdot z_0(w), \\ z_0(w) = \exp\Big[\frac{1}{2\pi i}\int_\gamma \frac{\varphi_0(\tau)}{\tau - w}\,d\tau\Big], \quad \varphi_0 \in C(\Gamma), \end{cases} \tag{12}$$

$$\begin{cases} z(w) - z(a) = (w-a)^{\delta}\cdot z_1(w), \\ z_1(w) = \exp\Big[\frac{1}{2\pi i}\int_\gamma \frac{\varphi_1(\tau)}{\tau - w}\,d\tau\Big], \quad \varphi_1 \in C(\Gamma). \end{cases} \tag{13}$$

Since $e'_p(D,\rho)$, there exists the function $\phi \in E'_p(D,\rho)$ for which $u = \operatorname{Re}\phi$. Denote

$$\widetilde{\phi} = \phi'. \tag{14}$$

Taking into account the relation (3), we can easily see that $\widetilde{\phi}$ satisfies the conditions

$$\begin{cases} \widetilde{\phi} \in E_p(D,\rho), & p > 1, \\ \operatorname{Re}\big[i\widetilde{\phi}(t)e^{\alpha(t)i}\big] = f(t), & f \in L^p(\Gamma,\rho). \end{cases} \tag{15}$$

Following [6] (§41), we reduce problem (15) to the problem of linear conjugation. Towards this end we suppose

$$\Psi(w)=\widetilde{\phi}\big(z(w)\big)\cdot\sqrt[p]{z'(w)}\cdot\rho\big(z(w)\big). \tag{16}$$

This implies that

$$\begin{cases}\Psi\in H^p, \quad p>1,\\ \operatorname{Re}\left[\frac{i\exp(\alpha(z(\zeta))i)\Psi^+(\zeta)}{\rho(z(\zeta))\cdot\sqrt[p]{z'(\zeta)}}\right]f(z(\zeta)), \qquad \zeta\in\gamma.\end{cases} \tag{17}$$

Obviously, if $\Psi(w)$ is a solution of problem (17), then

$$\widetilde{\phi}(z)=\frac{\Psi(w(z))}{\rho(z(\zeta))\cdot\sqrt[p]{z'(\zeta)}}$$

will be a solution of problem (15).

Further, let

$$\Omega(w)=\begin{cases}\Psi(w), & |w|<1,\\ \overline{\Psi(\frac{1}{\overline{w}})}, & |w|>1,\end{cases} \tag{18}$$

$$\Omega_*(w)=\overline{\Omega(\frac{1}{\overline{w}})}, \quad |w|\neq 1, \tag{19}$$

then the function Ω is a solution of the problem

$$\begin{cases}\Omega\in\mathcal{K}_0^p(\gamma), \ \ \Omega_*(w)=\Omega(w), & p>1,\\ \Omega^+(\zeta)=G(\zeta)\Omega^-(\zeta)+g(\zeta), & g\in L^p(\gamma),\end{cases} \tag{20}$$

where

$$G(\zeta)=\frac{\exp[-2\alpha(z(\zeta))]\rho(z(\zeta))\,\sqrt[p]{z'(\zeta)}}{\rho(z(\zeta))\,\sqrt[p]{z'(\zeta)}}, \tag{21}$$

$$g(\zeta)=\frac{2f(z(\zeta))\rho(z(\zeta))\,\sqrt[p]{z'(\zeta)}}{i\exp[\alpha(z(\zeta))i]}. \tag{22}$$

Will be the homogeneous problem, corresponding to (20)

$$\begin{cases}\Omega\in\mathcal{K}_0^p(\gamma), \ \ \Omega_*(w)=\Omega(w), & p>1,\\ \Omega^+(\zeta)=G(\zeta)\Omega^-(\zeta)\end{cases}. \tag{20_0}$$

If $\Omega(w)$ is the solution of problem (20), then the function

$$u(z)=\operatorname{Re}\left[\int\limits_{z_0}^{z}\frac{\Omega(w(\tau))\,d\tau}{\rho(\tau)\cdot\sqrt[p]{z'(w(\tau))}}\right]+M, \quad M\in\mathbb{R} \tag{23}$$

will be the solution of problem (11).

3^0. Suppose

$$y(w)=\exp\left[\frac{1}{2\pi i}\int\limits_{\gamma}\frac{-2\alpha(z(\zeta))\,d\zeta}{\zeta-w}\right],\quad |w|\neq 1. \tag{24}$$

Since $\alpha(z(\zeta))$ is the function continuous on $\gamma\backslash\{a\}$, and at the point a it, according to (9), has the first order discontinuity of size $\pi(1+\nu)$, therefore, as is known (see [6], p. 81) for the close to w points Γ we have

$$y(w)=(w-a)^{-\delta-1}\cdot y_0(w)\cdot y_1(w), \tag{25}$$

where

$$y_0(w)=\exp\left[\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\psi_0(\zeta)}{\zeta-w}\,d\zeta\right],\quad |w|\neq 1,\quad \psi_0\in C(\gamma), \tag{26}$$

and $y_1(w)$ is the analytic function for which

$$0<\inf_w|y_1(w)|\quad\text{and}\quad \sup_w|y_1(w)|<\infty, \tag{27}$$

(relations (27) are understood for the close to w points Γ).

4^0. Let

$$k^*=\delta\Big(\beta+\frac{1}{p}-1\Big)-1 \tag{28}$$

and $k=[k^*]$, i.e., k is the integer, such that

$$k\leq k^*<k+1. \tag{29}$$

Consider the function

$$X(w)=\frac{y(w)X_0(w)}{(w-a)^k},\quad |w|\neq 1, \tag{30}$$

where

$$X_0(w)=\begin{cases}\rho(z(w))\sqrt[p]{z'(w)}, & |w|<1,\\ \overline{\rho(z(w))\sqrt[p]{z'(w)}}, & |w|>1.\end{cases} \tag{31}$$

Taking into consideration representations (12), (13) and (25), we conclude that for the close to w points γ

$$X(w)=(w-a)^{k^*-k-1/p}\cdot y_1(w)\cdot y_2(w), \tag{32}$$

where

$$y_2(w)=\exp\left[\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\psi_2(\zeta)}{\zeta-w}\,d\zeta\right],\quad |w|\neq 1,\quad \psi_2\in C(\gamma), \tag{33}$$

Lemma 1. *If* $k^*\overline{\in}\mathbb{Z}$, *then*

$$X^+ \in W_p(\gamma), \quad X \in \mathcal{K}^p_{-k}(\gamma), \quad \frac{1}{X} \in \mathcal{K}^{p'}_k(\gamma).$$

Proof. In the case under consideration

$$-\frac{1}{p} < k^* - k - 1/p < \frac{1}{p'}, \tag{34}$$

therefore

$$\rho_1(\zeta) = (\zeta - a)^{k^*-k-\frac{1}{p}} \in W_p(\gamma)$$

(see [9], p. 78). It follows from representation (33) that

$$y_2^+(\zeta) \in \cap_{p>1} W_p(\gamma) \tag{35}$$

(see [10]).

Taking into account (27) and applying Stein's interpolation theorem to the (given by equality (32)) boundary value of the function $X(w)$, we conclude that

$$X^+ \in W_p(\gamma). \tag{36}$$

This, in particular, implies the existence of a positive number ε, such that

$$X^+ \in L^{p+\varepsilon}(\gamma), \quad \frac{1}{X} \in L^{p'+\varepsilon}(\gamma) \tag{37}$$

(see [11]).

Next, taking into account (12) and (13), for functions $X_0|_U$ (narrowings of X_0 on U) we obtain

$$X_0\big|_U = (w-a)^{\delta\beta + \frac{\delta-1}{p}} \cdot z_1(w) \cdot z_0(w). \tag{38}$$

This implies

$$X_0\big|_U \in H_{\delta_1}, \quad \frac{1}{X_0}\big|_U \in H_{\delta_1}, \tag{39}$$

for some $\delta_1 > 0$.

Since the function $\frac{y(w)}{(w-a)^k}$ in the circle U belongs likewise to certain Hardy classes, the functions $(X|_U)^\pm$, consequently, belong to the class H_δ for some $\delta > 0$. Moreover, according to $X^+ \in L^{p+\varepsilon}(\gamma)$, $\frac{1}{X^+} \in L^{p'+\varepsilon}$, and therefore using Smirnov's theorem (see [7], p. 448), we conclude that

$$X\big|_U \in H_p, \quad \frac{1}{X}\big|_U \in H_{p'}. \tag{40}$$

Consider now the function

$$\widetilde{X}(w) = \begin{cases} X(w), & |w| < 1, \\ \overline{X(\frac{1}{\overline{w}})}, & |w| > 1 \end{cases}. \tag{41}$$

Since $X|_U \in H_p$, hence $\widetilde{X}(w) \in \mathcal{K}_0^p(\gamma)$ (see [12]).

For $|w| > 1$ we have

$$\widetilde{X}(w) = \overline{X\Big(\frac{1}{\overline{w}}\Big)} = \frac{\overline{y(\frac{1}{\overline{w}})} \cdot \overline{X_0(\frac{1}{\overline{w}})}}{(\frac{1}{\overline{w}-a})^k} == \frac{(-a)^k \cdot w^k \cdot X_0(w)}{(w-a)^k} \times$$
$$\times \exp\Big[\frac{1}{2\pi i}\int\limits_\gamma -2\alpha(z(\zeta))i\Big(\frac{1}{\zeta - w} - \frac{1}{\zeta}\Big)\,d\zeta\Big] =$$
$$= C_k \cdot w^k \cdot X(w), \tag{42}$$

where

$$C_k = (-a)^k \exp\Big[\frac{1}{\pi}\int\limits_\gamma \frac{\alpha(z(\zeta))}{\zeta}\,d\zeta\Big]. \tag{41_0}$$

The function $\widetilde{X}(w) - \widetilde{X}(\infty)$ is representable in the domain $|w| > 1$ by the Cauchy integral. From (42) we have

$$X\Big|_{\mathbb{C}\setminus\overline{U}} = \frac{\widetilde{X}\big|_{\mathbb{C}\setminus\overline{U}}}{C_k w^k}. \tag{43}$$

If $k > 0$, then

$$\frac{1}{w^k} = -\frac{1}{2\pi i}\int\limits_\gamma \frac{1}{\zeta^k}\cdot\frac{d\zeta}{\zeta - w}, \quad |w| > 1,$$

and therefore the function $X|_{\mathbb{C}\setminus\overline{U}}$ is representable by the Cauchy integral (see [9], p. 99) whose density is the function $-\frac{1}{C_k\tau^k}\widetilde{X}(\tau)$ belonging to the class $L^p(\gamma)$. Moreover, $X|_U \in H_p$, therefore for $k > 0$ we have

$$X \in \mathcal{K}_{-k}^p(\gamma). \tag{44}$$

When $k = 0$, the inclusion (44) follows immediately from (43) and (41).

If $k < 0$, then on the basis of (43) for $|w| > 1$ we have

$$C_k \cdot X(w) = w^{-k}\Big[-\frac{1}{2\pi i}\int\limits_\gamma \frac{\widetilde{X}^-(\zeta)}{\zeta - w}\,d\zeta + \text{const}\Big] =$$
$$= \frac{1}{2\pi i}\int\limits_\gamma \frac{\widetilde{X}^-(\zeta)(\zeta^{-k} - w^{-k} - \zeta^{-k})}{\zeta - w}\,d\zeta +$$
$$+ \text{const}\cdot w^{-k} = \sum_{i=0}^{-k-1}\frac{1}{2\pi i}\int\limits_\gamma \widetilde{X}^-(\zeta)\cdot\zeta^{-k-1-j}\,d\zeta\cdot w^j -$$
$$-\frac{1}{2\pi i}\int\limits_\gamma \frac{\widetilde{X}^-(\zeta)\zeta^{-k}}{\zeta - w}\,d\zeta + \text{const}\cdot w^{-k} =$$

$$= \frac{1}{2\pi i} \int\limits_{\gamma} \frac{-\widetilde{X}^{-}(\zeta) \cdot \zeta^{-k}}{\zeta - w}\, d\zeta + Q_{-k}(w). \tag{45}$$

From (45) and (41) follows

$$X \in \mathcal{K}^{p}_{-k}(\gamma).$$

Reasoning as above, we state that $\frac{1}{X} \in \mathcal{K}^{p'}_{k}(\gamma)$. □

In a way, similar to Lemma 1, we prove

Lemma 2. *If $k^* \in \mathbb{Z}$ (in this case $k = k^*$), then for an arbitrarily small number $\varepsilon > 0$*

$$X \in \mathcal{K}^{p-\varepsilon}_{k}(\gamma), \quad \frac{1}{X} \in \cap_{q>1} \mathcal{K}^{q}_{k}(\gamma). \tag{46}$$

Lemma 1 results in

Lemma 3. *If $k^* \overline{\in} \mathbb{Z}$, then $X(w)$ is the factor-function of $G(\zeta)$.*

Since the requirements (1)–(3) from the definition of factor-functions are fulfilled, it suffices to fulfil the equality

$$G(\zeta) = X^{+}(\zeta) \cdot \left[X^{-}(\zeta)\right]^{-1},$$

which can be easily verified.

5^0. Let $\Omega(w)$ be a solution of the problem of linear conjugation (20). Since by Lemma 1 $\frac{1}{X} \in \mathcal{K}^{p'}_{k}(\gamma)$, the function $\widetilde{\Omega} = \frac{\Omega}{X}$ will be a solution of the following problem:

$$\begin{cases} \widetilde{\Omega}^{+} = \widetilde{\Omega}^{-} + \frac{g}{X^{+}}, \\ \widetilde{\Omega} \in \mathcal{K}^{1}_{k}(\gamma), \qquad \frac{g}{X^{+}} \in L^{1}(\gamma). \end{cases} \tag{47}$$

If $\widetilde{\Omega}$ is a solution of problem (47), then for the function $\Omega = \widetilde{\Omega}$ to be a solution of problem (20), it is necessary and sufficient that the conditions (a) $\Omega \in \mathcal{K}^{p}_{0}(\gamma)$ and (b) $\Omega_{*}(w) = \Omega(w)$ be fulfilled.

As is known (see, for e.g., [9], p. 102),

(i) if $k > -1$, then a solution of the homogeneous, corresponding to (47), problem is an arbitrary polynomial of degree, not higher than k, i.e.,

$$\widetilde{\Omega}_{0}(w) = Q_{k}(w), \quad Q_{k}(w) = \sum_{j=0}^{k} a_j \cdot w^j, \tag{48}$$

and a particular solution of (47) is the function

$$\widetilde{\Omega}_{g}(w) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{g(\zeta)}{X^{+}(\zeta)} \cdot \frac{d\zeta}{\zeta - w}. \tag{49}$$

Consequently, a general solution has the form

$$\widetilde{\Omega}(w)=\widetilde{\Omega}_0(w)+\widetilde{\Omega}_g(w)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{g(\zeta)}{X^+(\zeta)}\cdot\frac{d\zeta}{\zeta-w}+Q_k(w). \tag{50}$$

(ii) if $k\geq -2$, then problem (47) for $k=-1$ is, undoubtedly, solvable and its solution is

$$\widetilde{\Omega}(w)=\widetilde{\Omega}_g(w), \tag{51}$$

if $k\geq -2$, then for problem (47) to be solvable, it is necessary and sufficient that conditions

$$\int\limits_{\gamma}\frac{g(\tau)}{X^+(\tau)}\tau^j\,d\tau=0,\quad j=\overline{0,-k-2}. \tag{52}$$

be fulfilled.

If conditions (51) are fulfilled, problem (47) is uniquely solvable and its solution is given by equality (51).

6^0. In this section we will construct a solution of problem (20). Consider four cases: I. $k^*\overline{\in}\mathbb{Z}$, $k=[k^*]\geq 0$; II. $k^*\overline{\in}\mathbb{Z}$, $k<[k^*]$; III. $k^*\in\mathbb{Z}$, $k\geq 0$; IV. $k^*\in\mathbb{Z}$, $k<0$.

I. $k^*\overline{\in}\mathbb{Z}$, $k\geq 0$.

All possible solutions of problem (20) are contained in a set of functions

$$\Omega_0(w)=X(w)\cdot\widetilde{\Omega}_0(w)=X(w)\cdot Q_k(w),\quad Q_k(w)=\sum_{j=0}^{k}a_jw^j. \tag{53}$$

For $\Omega_0(w)$ to be a solution of (20_0), it is necessary that requirements (a) and (b) from Section 5^0 be fulfilled.

By Lemma 1, $X\in\mathcal{K}^p_{-k}(\gamma)$, and therefore $\Omega_0\in\mathcal{K}^p_0(\gamma)$. Further, on the basis if (42) we get

$$(\Omega_0)_*(w)=\overline{\Omega_0\Big(\frac{1}{w}\Big)}=\overline{X\Big(\frac{1}{\overline{w}}\Big)}\cdot\overline{\sum_{j=0}^{k}a_j\cdot\Big(\frac{1}{\overline{w}}\Big)}=C_k\cdot w^k\cdot X(w)\times$$
$$\times\sum_{j=0}^{k}\overline{a_j}\cdot\frac{1}{w^j}=C_k\cdot X(w)\cdot\sum_{j=0}^{k}\overline{a_j}\cdot w^{k-j}. \tag{54}$$

As is seen from (54), for the condition $(\Omega_0)_*=\Omega_0$ to be fulfilled, it is necessary and sufficient that conditions

$$C_k\overline{a_j}=a_{k-j},\quad j=\overline{0;\Big[\frac{k+1}{2}\Big]}, \tag{55}$$

be fulfilled, where C_k is given by formula (42_0).

Relying on Lemma 3 and theorem from [9] (Chapter IV, §2, item 2.4), we find that

$$\Omega_2(w) = \frac{x(w)}{2\pi i} \int\limits_\gamma \frac{g(\zeta)}{x^+(\zeta)} \cdot \frac{d\zeta}{\zeta - w}, \tag{56}$$

is the function of the $\mathcal{K}_0^p(\gamma)$ class and it satisfies the boundary condition of problem (20). The function $(\Omega_2)_*(w) = \overline{\Omega_2(\frac{1}{\overline{w}})}$ possesses the same properties (see, for e.g., [12]).

Consequently, the function

$$\Omega_1(w) = \frac{1}{2}\big(\Omega_2(w) + (\Omega_2)_*(w)\big),$$

satisfying the condition $(\Omega_1)_*(w) = \Omega_1(w)$, is a particular solution of problem (20).

We have

$$\Omega_1(w) =$$
$$= \frac{1}{2}\left[\frac{X(w)}{2\pi i} \int\limits_\gamma \frac{g(\zeta)}{X^+(\zeta)} \cdot \frac{d\zeta}{\zeta - w} + \overline{\frac{X(\frac{1}{\overline{w}})}{2\pi i} \int\limits_\gamma \frac{g(\zeta)}{X^+(\zeta)} \cdot \frac{d\zeta}{\zeta - \frac{1}{\overline{w}}}}\right].$$

By (42), $\overline{X(\frac{1}{\overline{w}})} = C_k w^k X(w)$, and from (22) it follows

$$g(\zeta) = \frac{2f(z(\zeta)) \cdot \rho(z(\zeta)) \cdot \sqrt[p]{z'(\zeta)}}{i \exp[\alpha(z(\zeta))i]}.$$

Therefore

$$\Omega_1(w) = \frac{x(w)}{2\pi i} \int\limits_\gamma \frac{f(z(\zeta))(\zeta - a)^k}{i \exp[\frac{1}{2\pi i}\int_\gamma \frac{-2\alpha(z(\zeta))i}{\tau - \zeta} d\tau]} \Big(1 + \frac{w^{k+1}}{\zeta^{k+1}}\Big) \frac{d\zeta}{\zeta - w}.$$

If we denote

$$u_k(\zeta) = -i(\zeta - a)^k \exp\left(\frac{1}{\pi} \int\limits_\gamma \frac{\alpha(z(\tau))}{\tau - \zeta}\, d\tau\right), \tag{57}$$

then

$$\Omega_1(w) = \frac{x(w)}{2\pi i} \int\limits_\gamma f(z(\zeta)) \cdot u_k(\zeta)\Big(1 + \frac{w^{k+1}}{\zeta^{k+1}}\Big) \cdot \frac{d\zeta}{\zeta - w}. \tag{58}$$

A general solution of problem (20) has the form

$$\Omega(w) = \Omega_1(w) + \Omega_0(w) = \Omega_1(w) + X(w) \cdot Q_k(w), \tag{59}$$

where the coefficients a_k of the polynomial $Q_k(w) = \sum\limits_{j=0}^{k} a_j w^j$ are connected by conditions (55).

II. $k^* \overline{\in} \mathbb{Z}$, $k \leq -1$.

As is mentioned in Section 5^0, problem (47) for $k = -1$ is uniquely solvable and a solution is given by formula (51), while for this problem to be solvable for $k \leq -2$, it is necessary and sufficient that conditions (52) be fulfilled. If they are fulfilled, then (47) is uniquely solvable and a solution is given by formula (51). By Lemma 1, $X^+ \in W^p(\gamma)$ and $X \in \mathcal{K}^p_{-k}(\gamma)$, therefore reasoning just as in case I for $k = -1$, as well as for $k \leq -2$ if conditions (52) are fulfilled, a unique solution of problem (20) is given by equality (58).

When $k^* \in \mathbb{Z}$, problem (20) is unsolvable in a general case. Namely, for $\nu = p$, when Γ is a piecewise Ljapunov curve of the class C^ν_A and $\beta = 0$, the work [1] gives an example of the function $g_0 \in L^p(\gamma)$ construction for which problem (20) is unsolvable; hence problem (11) is likewise unsolvable.

Suppose analogously to [2] that $g(\zeta) \ln |\zeta - a| \in L^p(\gamma)$, i.e.,

$$f(t) \cdot \ln \big|w(t) - w(A)\big| \rho(t) \in L^p(\Gamma) \tag{60}$$

and consider the case

III. $k^* \in \mathbb{Z}$, $k \geq 0$.

Depending on the fact whether the function X belongs to the class $\mathcal{K}^p_{-k}$, there appear different situations.

(i) Let $X \in \mathcal{K}^p_{-k}(\gamma)$. Reasoning just as in case I, we get that a solution of the homogeneous problem (20_0) is the function $\Omega_0(w)$ given by formula (53) in which requirements (55) are fulfilled for the coefficients of the polynomial $Q_k(w)$. Since the narrowing on U of the function $\Omega_2(w)$ belongs to H_q for some $q > 0$, then for $\Omega_2 \in \mathcal{K}^p_0(\gamma)$ to take place, it is sufficient according to Smirnov's theorem (see [7], p. 448) that the function

$$\Omega_2^+(\tau) = \frac{1}{2} g(\tau) + \frac{x^+(\tau)}{2\pi i} \int\limits_\gamma \frac{g(\zeta)}{x^+(\zeta)} \cdot \frac{d\zeta}{\zeta \tau} = \frac{1}{2} g(\tau) + \frac{x^+(\tau)(\tau - a)}{2\pi i} \times$$
$$\times \int\limits_\gamma \frac{g(\zeta)}{(\zeta - a) \cdot x^+(\zeta)} \cdot \frac{\zeta - a}{(\tau - a)(\zeta - \tau)} \, d\zeta,$$

to belong to the class $L^p(\gamma)$. As far as the function $\frac{1}{(\zeta - a)^{1/p'} \ln |\zeta - a|}$ in the neighborhood of the point a is of order $\frac{1}{X^+(\zeta)(\zeta - a) \ln |\zeta - a|}$, it belongs to $L^{p'}$, and hence there exists the integral

$$M = \frac{1}{2\pi i} \int\limits_\gamma \frac{g(\zeta)}{X^+(\zeta) \cdot (\zeta - a)} \, d\zeta. \tag{61}$$

Let us now write the function Ω_2^+ in terms of

$$\Omega_2^+(\tau) = \frac{1}{2} g(\zeta) + M \cdot X(\tau) +$$

$$+\frac{X^+(\tau)(\tau-a)}{2\pi i}\int\limits_\gamma \frac{g(\zeta)}{(\zeta-a)\cdot X^+(\zeta)}\frac{d\zeta}{\zeta-\tau}. \tag{62}$$

As is seen from (32), $X^+(\tau)(\tau-a)=(\tau-a)^{1/p'}\rho_1(\tau)$, where $\rho_1(\tau)=\exp[\frac{1}{2\pi i}\int_\gamma\frac{\psi(\zeta)}{\zeta-\tau}]$, $\psi\in C(\gamma)$. Applying Theorem 4.7 from [3] (p.64), we can conclude that the last summand in (62) belongs to $L^p(\gamma)$. Since in this case $X\in\mathcal{K}^p_{-k}$, the second summand in (62) is likewise belongs to $L^p(\gamma)$. Thus $\Omega_2^+\in L^p(\gamma)$ and hence $\Omega_2^+\in\mathcal{K}^p_0(\gamma)$. Reasoning as in case I, we arrive at the fact that the function $\Omega_1(w)$ is a particular solution of problem (20).

(ii) Let $X\overline{\in}\mathcal{K}^p_{-k}(\gamma)$. This time conditions (55) are insufficient for Ω_0 to be a solution of problem (20_0). Indeed, since the function $X^+(\zeta)$ has in the neighborhood of the point a the form

$$x^+(\zeta)=(\zeta-a)^{-\frac{1}{p}}\cdot y_1^+(\zeta)\cdot y_2(\zeta),$$

and by assumption (ii) does not belong to the class $L^p(\gamma)$, therefore if the polynomial $Q_k(w)$ has no zero at the point a, then the function $X(\zeta)\cdot Q_k(\zeta)$ does not belong to $L^p(\gamma)$, and hence $X\cdot Q_k\overline{\in}\mathcal{K}^p_0(\gamma)$. We have to require additionally the fulfilment of the condition

$$Q_k(a)=a_0+a_1\cdot a+\cdots+a_k\cdot a^k=0. \tag{63}$$

It is clear that if $k=0$, condition (63) is fulfilled only for $Q_k\equiv 0$, i.e., in this case problem (20_0) has only a trivial solution.

In order for the function Ω_2 to be of the class $\mathcal{K}^p_0(\gamma)$, it is necessary, as is seen from (62), that the condition $M=0$, i.e., the condition

$$\int\limits_\gamma \frac{g(\zeta)}{x^+(\zeta)}\cdot\frac{d\zeta}{\zeta-a}=0 \tag{64}$$

be fulfilled.

If this condition is fulfilled, then the function $\Omega_1(w)$ given by equality (58) is a particular solution of problem (20).

IV. $k^*\in\mathbb{Z}$, $k<0$. In this case problem (20_0) has only a trivial solution. In order for (20) to have a solution for $k\le -2$, it is necessary that condition (52) be fulfilled. Moreover, reasoning as in the foregoing case, we obtain that:

(i) if $X\in\mathcal{K}^p_{-k}(\gamma)$ for $k=-1$, then the problem is uniquely solvable and its solution is given by formula (58); for $k\le 2$ the necessary conditions (52) for the solvability of problem (20) become sufficient as well, and the problem is uniquely solvable and its solution is given by formula (58);

(ii) if $X\overline{\in}\mathcal{K}^p_{-k}(\gamma)$, then for the solvability of problem (20) for $k=-1$ it is necessary and sufficient to fulfil condition (64), while if $k\le -2$, then for the solvability of problem (20) it is necessary and sufficient to fulfil conditions (52) and (64). If these conditions are fulfilled, then a solution does exist, its solution is unique and given by equality (58).

7^0. In this section we will formulate the theorem which follows directly from the previous section and the remark at the end of Section 2^0.

Theorem. *Let D be a single-valued finite domain bounded by the curve Γ of the class C_A^ν, $\rho(z)=(z-A)^\beta$, $\beta\in\mathbb{R}$; the numbers k^* and $k=[k^*]$ are defined, respectively, by* (28) *and* (29). *Then:*

I. *If $k^*\in\mathbb{Z}$; $k\geq 0$, then the Neumann problem* (11) *is solvable and its general solution is of the form*

$$u(z)=\operatorname{Re}\Bigg[\int_{z_0}^{z}\Bigg[\frac{Q_k(w(\tau))\cdot y(w(\tau))}{(w(\tau)-w(A))^k}+\frac{y(w(\tau))}{2\pi i(w(\tau)-w(A))^k}\times$$

$$\times\int_\Gamma\frac{f(t)\cdot u_k(w(t))}{w^{k+1}(t)}\big(w^{k+1}(t)+w^{k+1}(\tau)\big)\cdot\frac{w'(t)\,dt}{w(t)-w(\tau)}\Bigg]d\tau\Bigg]+M; \tag{65}$$

$$M\in\mathbb{R},$$

in which $Q_k(w)=\sum\limits_{j=0}^{k}$ and the coefficients a_k are connected by condition (55),

$$y\big(w(\tau)\big)=\exp\bigg[\frac{1}{2\pi i}\int_\Gamma\frac{-2\alpha(t)i}{w(t)-w(\tau)}\cdot w'(t)\,dt\bigg], \tag{66}$$

$$u_k\big(w(\tau)\big)=-i\big(w(t)-w(A)\big)^k\exp\bigg[\frac{1}{\pi}\int_\Gamma\frac{\alpha(\zeta)}{w(\zeta)-w(t)}\cdot w'(t)\,d\zeta\bigg]. \tag{67}$$

II. *If $k^*\overline{\in}\mathbb{Z}$, $k<0$ then for $k=-1$ the problem is undoubtedly solvable, while for its solvability for $k\geq -2$ it is necessary and sufficient to fulfil the conditions*

$$\int_\Gamma f(t)\cdot u_k(w(t))\cdot w^j(t)\cdot w'(t)dt=0,\quad j=\overline{0;-k-2}. \tag{68}$$

In both cases the solution is given by the equality

$$u(z)=\operatorname{Re}\Bigg[\int_{z_0}^{z}\Bigg[\frac{y(w(\tau))}{2\pi i(w(\tau)-w(A))^k}\times$$

$$\times\int_\Gamma\frac{f(t)u_k(w(\tau))(w^{k+1}(t)+w^{k+1}(\tau))w'(t)\,dt}{w^{k+1}(t)(w(t)+w(\tau))}\Bigg]d\tau\Bigg]+M.\ \ M\in\mathbb{R}. \tag{69}$$

When $k^*\in\mathbb{Z}$, the Neumann problem (11) *is in a general case unsolvable. If we additionally suppose that*

$$f(t)\cdot\ln\big|w(t)-w(A)\big|\in L^p(\Gamma,\rho), \tag{70}$$

then

III. *For $k^*\in\mathbb{Z}$, $k\geq 0$ we have:*

(i) *If* $X \in \mathcal{K}^p_{-k}$, *then problem* (11) *is undoubtedly solvable and its general solution is given by formula* (65), *in which the coefficients of the polynomial* Q_k *are connected by condition* (55)*;*

(ii) *if* $X \overline{\in} \mathcal{K}^p_{-k}$, *then for problem* (11) *to be solvable it is sufficient that the condition*

$$\int\limits_{\Gamma} f(t) \cdot u_k(w(t)) \cdot (w(t) - w(A))^{-1} \cdot w'(t) \cdot dt = 0 \tag{71}$$

be fulfilled; when condition (71) *is fulfilled, the solution is given by equality* (65), *in which the coefficients of the polynomial* Q_k *are connected by conditions* (55) *and* (63).

IV. *For* $k^* \in \mathbb{Z}$, $k \leq -1$ *we have:*

(i) *If* $X \in \mathcal{K}_{-k}(\gamma)$, *then the problem is undoubtedly solvable for* $k = -1$*; for its solvability for* $k \leq -2$ *it is necessary and sufficient that conditions* (68) *be fulfilled;*

(ii) *If* $X \in \mathcal{K}_{-k}(\gamma)$, *then for the solvability of the problem for* $k = -1$ *it is necessary and sufficient to fulfil condition* (71) *and additionally conditions* (68) *for* $k \leq -2$.

In both cases the solution is given by formula (69).

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