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CONTACT PROBLEM OF BENDING OF A RECTANGULAR ELASTIC PLATE WITH AN ELASTIC INCLUSION

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Abstract. The contact problem of bending of a rectangular plate with an elastic inclusion of variable rigidity is considered. The integro-differential equation for contact stress is obtained. The uniqueness theorem for the problem under consideration is proved. The obtained equation is, on the one hand, reduced to the equivalent Fredholm integral equation of the second kind and, on the other hand, to the equivalent system of linear algebraic equations. The system is investigated from the point of view of its regularity.

A number of problems on bending of finite or infinite isotropic plates reinforced by thin (rigid or elastic) inclusions have been considered in [1–4]. These problems were reduced to systems of integral equations whose characteristic part had in the general case the form

$$\int_{-1}^{1}\frac{(t-\tau)^2}{2}\Big[a\frac{\operatorname{sgn}(t-\tau)}{2}+\frac{b}{\pi i}\ln\frac{1}{|t-\tau|}\Big]\varphi(\tau)\,d\tau=f(t),\quad |t|<1,\qquad (1)$$

a solution $\varphi(\tau)$ was sought in a class of functions with non-integrable singularities by using the method of regularization of diverging integrals [5]. We investigated contact problems of bending of plates with cover pieces of variable rigidity [6–8]; as for unknown contact stresses, we obtained the

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integro-differential equation whose characteristic part is the Prandtl integro-differential equation. Under certain conditions this equation is studied in [9–11]. But in the case when the coefficient of a singular operator tends at the ends of a line of integration to zero of any order, this equation varies qualitatively, in particular, it is equivalent to a singular integral equation of the third kind. We have investigated equations with such coefficients, obtained efficient solutions in certain conditions and established behaviour of unknown contact stresses at the ends of elastic cover pieces.

Let us consider a rectangular ($|x| \le \frac{a}{2},\ 0 \le y \le b,\ a > 2$) hinged plate inside of which on the segment $y = b/2$, $|x| < 1$ there is an inclusion which causes discontinuity of some of its values. For the sake of simplicity of our discussion we assume that the normal load is applied to the inclusion only, and the deflection obtained under the sagging of the cover piece, $\omega(x, y)$, is even with respect to x. It is required to find contact stress upon interaction of the inclusion and the plate.

The above-posed problem is equivalent to finding of a solution of the homogeneous biharmonic equation

$$\Delta\Delta\omega(x, y) = 0, \quad y \neq \frac{b}{2}, \tag{2}$$

satisfying the boundary conditions

$$\begin{gathered} \omega = \frac{\partial^2\omega}{\partial x^2} = 0, \quad x = \pm a/2, \quad 0 \le y \le b; \\ \omega = \frac{\partial^2\omega}{\partial y^2} = 0, \quad y = 0; \quad b, \quad |x| \le \frac{a}{2}, \end{gathered} \tag{3}$$

and also the conditions imposed on the inclusion:

$$\langle\omega\rangle = \langle\omega'_y\rangle = \langle M_y\rangle = 0, \ \ \langle N_y\rangle = \mu(x), \ \ |x| < 1, \ \ y = \frac{b}{2}. \tag{4}$$

Here the use is made of the following notation: $\langle f\rangle = f(x, \frac{b}{2}-) - f(x, \frac{b}{2}+)$, ω is the plate sagging, ω'_y, M_y and N_y are, respectively, the angle of rotation, bending moment and generalized transversal force in the plate, $\mu(x)$ is an unknown contact stress upon interaction of the inclusion and the plate, provided $\mu(x) \equiv 0$ for $|x| > 1$.

Assuming the ends of the inclusion are free, we obtain for the inclusion sagging $\omega_0(x)$ the following conditions:

$$\begin{aligned} &\frac{d^2}{dx^2} D_0(x) \frac{d^2}{dx^2}\omega_0(x) = q(x) - \mu(x), \ \ |x| < 1, \\ &D_0(x)\omega''_0(x)\big|_{x=\pm1} = 0, \qquad \big[D_0(x)\omega''_0(x)\big]'_{x=\pm1} = 0, \end{aligned} \tag{5}$$

where $D_0(x) = \frac{E_0(x)h_0^3(x)}{12}$ is the bending stiffness factor of the inclusion, $E_0(x)$ is elasticity modulus of the material, $h_0(x)$ is its thickness, $q(x)$ is normal load applied to the inclusion.

Equation of equilibrium of the inclusion has the form

$$\int_{-1}^{1} \big[\mu(x) - q(x)\big]\, dx = 0, \quad \int_{-1}^{1} x\big(\mu(x) - q(x)\big)\, dx = 0. \tag{6}$$

A solution of problem (2–3) can be represented as follows:

$$\omega(x, y) = \sum_{k=1,3,\dots}^{\infty} \cos \alpha_k x Y_k(y), \quad \alpha_k = \frac{\pi k}{a}, \tag{7}$$

where

$$\begin{gathered} Y_k(y) = A_k sh\alpha_k y + \alpha_k B_k y ch\alpha_k y, \ \ 0 \le y < \frac{b}{2}, \\ Y_k(y) = C_k sh\alpha_k(b - y) + D_k \alpha_k (b - y) ch\alpha_k (b - y), \ \ \frac{b}{2} < y \le b. \end{gathered} \tag{8}$$

Realization of conditions (4) allows one to express A_k, B_k, C_k, D_k by $\mu(x)$, and we obtain $A_k = C_k$, $B_k = D_k$ and

$$\sum_{k=1,3\dots}^{\infty} \cos \alpha_k x \Big[\alpha_k ch\frac{\alpha_k b}{2} A_k + \Big(\alpha_k ch\frac{\alpha_k b}{2} + \frac{\alpha_k^2 b}{2} sh\frac{\alpha_k b}{2}\Big) B_k\Big] = 0,$$

$$\sum_{k=1,3\dots}^{\infty} \cos \alpha_k x \Big[\alpha_k^3 ch\frac{\alpha_k b}{2} A_k + \Big(3\alpha_k^3 ch\frac{\alpha_k b}{2} + \frac{\alpha_k^4 b}{2} sh\frac{\alpha_k b}{2}\Big) B_k\Big] = \frac{1}{2D}\mu(x).$$

From the last equalities we get

$$A_k = -\frac{ch\frac{\alpha_k b}{2} + \frac{\alpha_k^2 b}{2} sh\frac{\alpha_k b}{2}}{2\alpha_k^3 ch\frac{\alpha_k b}{2}} \frac{1}{2Da} \int_{-1}^{1} \mu(\xi) \cos \alpha_k \xi \, d\xi,$$

$$B_k = \frac{1}{2\alpha_k^3 ch\frac{\alpha_k b}{2}} \frac{1}{2Da} \int_{-1}^{1} \mu(\xi) \cos \alpha_k \xi \, d\xi,$$

where D is cylindrical rigidity of the plate.

If we substitute these expressions in (7), then the limiting value of the function $\omega(x, y)$ for $y = b/2$ will take the form

$$\omega(x, b/2) = \frac{1}{2Da} \sum_{1,3,\dots}^{\infty} \int_{-1}^{1} \frac{\cos \alpha_k (x - \xi)}{\alpha_k^3} \rho_k \mu(\xi)\, d\xi, \tag{9}$$

where $\rho = th\frac{\alpha_k b}{2} - \frac{\alpha_k b}{2ch^2\frac{\alpha_k b}{2}}$.

In (9) we distinguish the principal part of the integral operator and take into account the asymptotics $\rho_k = 1 + O(\alpha_k e^{\alpha_k})$. Next we use the formula

$$\sum_{k=1,3,\dots}^{\infty} \frac{\cos kt}{k^3} = -\frac{t^2}{4} \ln \frac{1}{|t|} + B_0(t), \quad |t| < \pi, \tag{10}$$

where

$$B_0(t) = -\frac{t^2}{2}\Big(\frac{3}{2} + \ln 2\Big) + \sum_{k=1,3,\dots}^{\infty} \frac{1}{k^3} + \sum_{n=1}^{\infty} \frac{2^{2n-1} - 1}{(2n+2)!2n} B_n t^{2n+2},$$

where B_n are the Bernoulli numbers.

Formula (9) with regard for (10) takes the form

$$\omega(x, b/2) = -\frac{1}{2\pi D} \int_{-1}^{1} \frac{(x-\xi)^2}{4} \ln \frac{1}{|x-\xi|} \mu(\xi)\, d\xi + \int_{-1}^{1} R(x,\xi)\mu(\xi)\, d\xi, \tag{11}$$

where

$$R(x,\xi) = \frac{1}{2Da} B_0(x-\xi) + \frac{1}{2Da} \sum_{k=1,3,\dots}^{\infty} \frac{\cos \alpha_k (x-\xi)}{\alpha_k^3} (\rho_k - 1)$$

is the function infinitely differentiable in the square $-1 \leq x,\ \xi \leq 1$. Note that variation of boundary conditions and the plate shape leads to variation in $R(x,\xi)$.

Realizing the condition of the contact between the inclusion and the plate, $\omega(x, \frac{b}{2}) = \omega_0(x)$, $|x| < 1$ integrating twice condition (5) and introducing the notation $\lambda(x) = \int_{-1}^{x} dt \int_{-1}^{t} (q(\tau) - \mu(\tau)) d\tau$, we obtain the integro-differential equation

$$\lambda(x) - \frac{D_0(x)}{4\pi D} \int_{-1}^{1} \frac{\lambda'(t)dt}{t-x} +$$
$$+ D_0(x) \int_{-1}^{1} R_0(x,t)\lambda(t)dt = f(x)D_0(x), \quad |x| < 1, \tag{12}$$

where

$$R_0(x,t) = \frac{\partial^3 R(x,t)}{\partial x^2 \partial t}, \quad f(x) = \int_{-1}^{1} \Big[\frac{\partial^2 R(x,\xi)}{\partial x^2} - \frac{1}{8\pi D}\Big(2 \ln \frac{1}{|x-\xi|} + 3\Big)\Big] q(\xi)\, d\xi.$$

Conditions (6) provide us with the following boundary conditions:

$$\lambda(\pm 1) = 0, \quad \lambda'(\pm 1) = 0. \tag{13}$$

Theorem 1. *If the above-formulated problem* (2–5) *has a solution, then this solution is unique.*

Indeed, suppose problem (2–5) admits two solutions. Let $\omega_1(x,y)$ be a sagging of the plate, corresponding to the first solution and $\omega_2(x,y)$ be a sagging corresponding to the second one. We make up the "difference" of these solutions, i.e., we assume

$$w(x,y) = \omega_1(x,y) - \omega_2(x,y).$$

It is evident that $w(x,y)$ satisfies the basic equations in the absence of external forces, i.e., $q(x) \equiv 0$, $|x| < 1$.

In the conditions of the above-posed problem, taking (5) and (6) into consideration, the potential energy of deformation for the "difference" of the two solutions takes the form

$$h\int_{-1}^{1} \langle N_y\rangle w\Big(x,\frac{b}{2}\Big)\,dx = h\int_{-1}^{1} \mu(x) w\Big(x,\frac{b}{2}\Big)\,dx =$$

$$= h\int_{-1}^{1} w'\Big(x,\frac{b}{2}\Big)\,d\Big(\int_{-1}^{x}\mu(t)\,dt\Big) = hw\Big(x,\frac{b}{2}\Big)\int_{-1}^{x}\mu(t)\,dt\Big|_{-1}^{1} -$$

$$-h\int_{-1}^{1} w'\Big(x,\frac{b}{2}\Big)\Big(\int_{-1}^{x}\mu(t)\,dt\Big)\,dx = -h\int_{-1}^{1} w'\Big(x,\frac{b}{2}\Big)d\Big(\int_{-1}^{x}dt\int_{-1}^{t}\mu(\tau)\,d\tau\Big) =$$

$$= -hw'\Big(x,\frac{b}{2}\Big)\Big(\int_{-1}^{x}dt\int_{-1}^{t}\mu(t)\,dt\Big)\Big|_{-1}^{1} + h\int_{-1}^{1} w''(x,0)\Big(\int_{-1}^{x}dt\int_{-1}^{t}\mu(\tau)\,d\tau\Big) =$$

$$= -h\int_{-1}^{1} w''^2\Big(x,\frac{b}{2}\Big)D_0(x)\,dx.$$

Since the potential energy of deformation is the positive quadratic form of components of that deformation, we conclude that $w''(x,b/2) = 0$, $|x| < 1$. In its turn this means that in the absence of external forces

$$\mu(x) = 0, \quad |x| < 1. \tag{14}$$

Introduce the notation $\Omega = \{(x,y)||x| < \frac{a}{2},\, 0 < y < l\}$, $S = \Omega\backslash b$, $l = \{(x,y)||x| < 1,\, y = \frac{b}{2}\}$, $L = \partial S$.

For any biharmonic function satisfying the conditions of regularity near the boundary L of domain S the formula

$$\iint_S \Big\{(\Delta w)^2 - (1-\sigma)\Big[\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial^2 y} - \Big(\frac{\partial^2 w}{\partial x \partial y}\Big)^2\Big]\Big\}\,dxdy +$$

$$+\int\limits_{L}\Big(wNw-\frac{dw}{dn}Mw\Big)\,ds=0 \tag{15}$$

is valid, where

$$Mw=\sigma\Delta w+(1-\sigma)\Big[\cos^2 Q\frac{\partial^2 w}{\partial x^2}+\sin^2 Q\frac{\partial^2 w}{\partial y^2}+\sin 2Q\frac{\partial^2 w}{\partial x\partial y}\Big],$$
$$Nw=\frac{d\Delta w}{d_n}+(1-\sigma)\frac{d}{ds}\Big[\cos 2Q\frac{\partial^2 w}{\partial x\partial y}+\frac{1}{2}\sin^2 Q\Big(\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}\Big)\Big],$$

Q is the angle made up of the normal n and the $0x$-axis, σ is the Poisson coefficient, $(0<\sigma<\frac{1}{2})$. By virtue of conditions (3), (4) and (14), the last integral in formula (15) is equal to zero.

Since

$$(\Delta w)^2-(1-\sigma)\Big[\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2}-(\frac{\partial^2 w}{\partial x\partial y})^2\Big]=$$
$$+\Big(\frac{\partial^2 w}{\partial x^2}\Big)^2+\Big(\frac{\partial^2 w}{\partial y^2}\Big)^2+(1+\sigma)\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2}+(1-\sigma)\Big(\frac{\partial^2 w}{\partial x\partial y}\Big)^2$$

represents by itself the positive definite form of second derivatives of the function $w(x,y)$, it follows from (15) that all the second partial derivatives of the function $w(x,y)$ are equal to zero, i.e., $w(x,y)$ is the linear function of its arguments, but since $w=0$ for $x=\pm\frac{a}{2}$ and $w=0$ for $y=0;b$, it is easy to see that $w(x,y)=0$ everywhere in Ω.

Thus we have proved the uniqueness theorem.

In case $D_0(x)=\sqrt{1-x^2}d(x)$, where $d(x)$ is any continuous function on the segment $[-1,1]$, $d(x)>0$, we use the method of reducing the integral differential equation (12–13) to the equivalent Fredholm integral equation of the second kind [11]. Applying the well-known inversion formula to the Cauchy type integrals, from (12) we get

$$\lambda'(x)=-\frac{1}{\sqrt{1-x^2}}\,\frac{\lambda_0}{\pi}\int\limits_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}\,\frac{\lambda(t)}{D_0(t)}\,dt+\frac{1}{\sqrt{1-x^2}}\,\frac{\lambda_0}{\pi}\int\limits_{-1}^{1}\frac{\sqrt{1-t^2}f(t)}{t-x}\,dt+$$
$$+\frac{1}{\sqrt{1-x^2}}\frac{\lambda_0}{\pi}\int\limits_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}\,dt\int\limits_{-1}^{-1}R_0(t,\tau)\lambda(\tau)d\tau+\frac{C}{\sqrt{1-x^2}},$$
$$|x|<1,\ \ \lambda_0=4D,$$

where C is an arbitrary constant. Integrating both parts of the latter equality and taking into account the integral expression

$$\int\frac{dx}{\sqrt{1-x^2}(t-x)}=\frac{1}{2\sqrt{1-t^2}}\ln\frac{1-xt+\sqrt{(1-x^2)(1-t^2)}}{1-xt-\sqrt{(1-x^2)(1-t^2)}},$$

we obtain

$$\lambda(x) = \frac{\lambda_0}{2\pi}\int_{-1}^{1} \ln\frac{1-xt+\sqrt{(1-x^2)(1-t^2)}}{1-xt-\sqrt{(1-x^2)(1-t^2)}}\frac{\lambda(t)}{D_0(t)}\,dt +$$

$$+\frac{\lambda_0}{2\pi}\int_{-1}^{1} \ln\frac{1-xt+\sqrt{(1-x^2)(1-t^2)}}{1-xt-\sqrt{(1-x^2)(1-t^2)}}\left[f(t)-\int_{-1}^{-1} R_0(t,\tau)\lambda(\tau)d\tau\right]dt +$$

$$+C\arcsin x + C_1.$$

Satisfying the boundary conditions $\lambda(\pm 1) = 0$, we find that $C_1 = C = 0$. Consequently, integro-differential equations (12–13) are equivalent to the Fredholm integral equation of the second kind:

$$\lambda(x) - \frac{\lambda_0}{2\pi}\int_{-1}^{1} L(x,t)\lambda(t)\,dt = r(x), \quad |x| < 1, \tag{16}$$

where

$$L(x,t) = \frac{1}{D_0(t)}\ln\frac{1-xt+\sqrt{(1-x^2)(1-t^2)}}{1-xt-\sqrt{(1-x^2)(1-t^2)}} -$$

$$-\int_{-1}^{1}\ln\frac{1-xt+\sqrt{(1-t^2)(1-x^2)}}{1-xt-\sqrt{(1-t^2)(1-x^2)}}R_0(t,\tau)\,d\tau,$$

$$r(x) = \frac{\lambda_0}{2\pi}\int_{-1}^{1}\ln\frac{1-xt+\sqrt{(1-t^2)(1-x^2)}}{1-xt-\sqrt{(1-t^2)(1-x^2)}}f(t)\,dt.$$

Obviously, equation (16) can be considered in the space $L^2(-1,1)$ with weight $\frac{1}{D_0(t)}$, where this equation is integral with the Hilbert-Schmidt kernel.

Now we describe the technique of reducing equation (12) to the equivalent infinite system of linear algebraic equations. Bearing this in mind, we represent a solution of equation (12) as an infinite series

$$\lambda'(x) = \frac{1}{\sqrt{1-x^2}}\sum_{m=0}^{\infty} a_m T_m(x), \quad |x| < 1, \tag{17}$$

with unknown coefficients $a_m (m = 0, 1, 2 \dots)$. Satisfying boundary conditions (13), the above equality results in

$$\lambda(x) = -\sqrt{1-x^2}\sum_{m=1}^{\infty}\frac{a_m}{m}U_{m-1}(x), \quad |x| \le 1, \quad a_0 = 0, \tag{18}$$

where $T_m(x)$ and $U_{m-1}(x)$ and the Chebyshev polynomials of the first and second kind, respectively.

Substituting (17) and (18) in (12) and making use of the relation

$$\frac{1}{\pi}\int\limits_{-1}^{1}\frac{T_m(t)dt}{(t-x)\sqrt{1-t^2}}=U_{m-1}(x),\quad |x|<1,$$

we arrive at the equation

$$-\frac{1}{d(x)}\sum_{m=1}^{\infty}\frac{a_m}{m}U_{m-1}(x)-\frac{1}{\lambda_0}\sum_{m=1}^{\infty}a_mU_{m-1}(x)-$$
$$-\sum_{m=1}^{\infty}\frac{a_m}{m}\int\limits_{-1}^{1}R_0(x,t)\sqrt{1-t^2}U_{m-1}(t)\,dt=f(x),\quad |x|<1.$$

Now we multiply both parts of the equality by the function $\sqrt{1-x^2}U_{k-1}(x)$ and integrate from -1 to 1 with regard for the condition of orthogonality of Chebyshev polynomials of the second kind:

$$\int\limits_{-1}^{1}U_{m-1}(x)U_{k-1}\sqrt{1-x^2}\,dx=\begin{cases}0, & m\neq k\\ \frac{\pi}{2}, & m=k,\quad m,k=1,2\dots.\end{cases}$$

After certain operations for determination of unknown coefficients a_m, we obtain an infinite system of linear equations

$$a_k+\lambda_0\sum_{m=1}^{\infty}R_{mk}a_m+\lambda_0\sum_{m=1}^{\infty}R'_{mk}a_m=f_k,\quad k=1,2,\dots, \tag{19}$$

where

$$\begin{aligned}R_{mk}=&\frac{2}{\pi m}\int\limits_{-1}^{1}U_{m-1}(x)U_{k-1}(x)\frac{\sqrt{1-x^2}}{d(x)}\,dx,\\ R'_{mk}=&\frac{2}{\pi m}\int\limits_{-1}^{1}\sqrt{1-x^2}U_{k-1}(x)dx\int\limits_{-1}^{1}R_0(x,\tau)\sqrt{1-x^2}U_{m-1}(\tau)d\tau\\ f_k=&-\lambda_0\int\limits_{-1}^{1}\sqrt{1-\tau^2}U_{k-1}(x)f(x)\,dx.\end{aligned}\tag{20}$$

It is well-known that the Chebyshev polynomials form a basis in the space $L_2(-1,1)$, series (17) converges in the norm of that space and the corresponding sequences $a=\{a_m\}$ belong to l_2 because of Parseval's equality: $\|\lambda\|_{L_2(-1,1)}=\|a\|_{l_2}$.

If $f(x) \in L_2(-1,1)$, then to any solution $\lambda(x)$ of equation (12) from the class $L_2(-1,1)$ there corresponds the sequence $\{a_m\} \in l_2$, satisfying an infinite system of linear algebraic equations (19), and vice versa.

Theorem 2. *If $f(x) \in L_2(-1,1)$, then the operators appearing in equation* (19) *act in the space l_2 and are completely continuous for all values of the parameter $\lambda_0 > 0$. System* (19) *is uniquely solvable in l_2.*

Indeed, from (20) we have

$$R_{mk} = \frac{2}{\pi m} \int_0^\pi \frac{\sin m\theta \sin k\theta}{d(\cos\theta)}\, d\theta = \frac{1}{\pi m} \int_0^\pi \frac{\cos(m-k)\theta d\theta}{d(\cos\theta)} - \frac{1}{\pi m} \int_0^\pi \frac{\cos(m+k)\theta d\theta}{d(\cos\theta)},$$

whence it follows that $R_{mk} = O(m^{-1})$ as $m \to \infty$, $m = k$ and R_{mk} vanishes as $k \to \infty$, $m \to \infty$ with the rate not less than k^{-1}, m^{-1} respectively, i.e., $\sum\limits_{m,k=1}^{\infty} R_{mk}^2 < \infty$, $S_k = O(k^{-1})$, $k \to \infty$, where $S_k = \sum_{m=1}^\infty |R_{mk}|$.

Let $R'_{mk} = \frac{1}{m} T_{mk}$, where

$$T_{mk} = \frac{2}{\pi} \int_{-1}^{1} \sqrt{1-x^2}\, U_{k-1}(x) dx \int_{-1}^{1} R_0(x,\tau) \sqrt{1-\tau^2}\, U_{m-1}(\tau) d\tau.$$

Note that the coefficients $\{T_{mk}\}_{m,k=1}^\infty$ are the Fourier coefficients of quadratically summable in the square $-1 \le x$, $x \le 1$ function $R_0(x,t)$ with respect to a full orthogonal system of functions $\{U_{k-1}(x) U_{m-1}(t)\}_{m,k=1}^\infty$. Therefore, by virtue of the Bessel inequality, the series $\sum_{m,k=1}^\infty |T_{mk}|^2$ and hence the series $\sum_{k=1}^\infty T_k$, where $T_k = \sum_{m=1}^\infty |T_{mk}|$, converge, whence at least $T_k = O(k^{-(1+\varepsilon)})$, $k \to \infty$, $(\varepsilon > 0)$.

If we assume that $S'_k = \sum_{m=1}^\infty |R'_{mk}| = \sum_{m=1}^\infty \frac{1}{m} |T_{mk}|$, then by means of the Cauchy-Bunjakovsky's inequality we obtain

$$S'_k \le \Big[\sum_{m=1}^\infty \frac{1}{m^2} \Big]^{1/2} \Big[\sum_{m=1}^\infty |T_{mk}|^2 \Big]^{1/2} = \frac{\pi}{\sqrt{6}} \sqrt{T_k}.$$

Thus we have $S'_k = O(k^{-(1+\varepsilon)/2})$, $k \to \infty$.

From this it follows that the operators appearing in (19) act in l_2 and are completely continuous for $\lambda_0 > 0$.

A free term of that system tends to zero as $k \to \infty$ with the rate not less than $k^{-(1+\varepsilon)/2}$. This allows one to claim that the infinite system (19) is quasi-completely regular [12]. Therefore Hilbert's alternative [13] on the solvability of infinite systems is applicable to the system under consideration.

Thus the theorem is proved.

Note that in a particular case, where $d(x) = 1$,

$$R_{mk} = \begin{cases} \frac{1}{k}, & k = m \\ 0, & k \neq m \end{cases}$$

and system (19) takes the form

$$a_k + \sum_{m=1}^{\infty} R^0_{mk} a_m = f^0_k, \quad k = 1, 2, \dots ,$$

where

$$R^0_{mk} = \frac{\lambda_0 k}{\lambda_0 + k} R'_{mk}, \quad f^0_k = \frac{k f_k}{\lambda_0 + k}.$$

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