Proceedings of A.Razmadze
Mathematical Institute,
Vol. **126**, 2001

EXISTENCE RESULTS ON NONCOMPACT INTERVALS FOR FUNCTIONAL DIFFERENTIAL AND INTEGRO-DIFFERENTIAL INCLUSIONS IN BANACH SPACES

M. BENCHOHRA AND S. NTOUYAS

Abstract. In this paper we investigate the existence of mild solutions on an infinite interval to first order initial value problems for functional semilinear differential and integro-differential inclusions in Banach spaces.

1. Introduction

In Section 3 of this paper we study the existence of mild solutions, defined on semiinfinite interval, for initial value problem (IVP for short) for first order functional differential inclusions

$$y' - A(t)y \in F(t, y_t), \quad \text{a.e.} \quad t \in J = [0, \infty), \tag{1.1}$$

$$y_0 = \phi, \tag{1.2}$$

where $F : J \times C(J_0, E) \longrightarrow 2^E$ (here $J_0 = [-r, 0]$) is a bounded, closed, convex multivalued map, $\phi \in C(J_0, E)$, $A(t)$, $t \in J$ a linear closed operator from a dense subspace $D(A(t))$ of E into E and E a real Banach space with the norm $|\cdot|$

For any continuous function y defined on the interval $[-r, \infty)$ and any $t \in J$, we denote by y_t the element of $C(J_0, E)$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in J_0.$$

Here $y_t(.)$ represents the history of the state from time $t - r$, up to the present time t.

2000 *Mathematics Subject Classification.* 34A60, 34G20, 34K10.
Key words and phrases. Initial value problems, convex multivalued map, mild solution, functional differential and integro-differential inclusion, existence, fixed point, abstract space.

Recent results on existence of solutions on compact intervals for functional differential equations, with the aid of topological transversality method of Granas [7], may be found e.g. in the books of Erbe, Qingai and Zhang [8], Henderson [10], and the survey paper of Ntouyas [18], while with monotone iterative method combined with upper and lower solutions [13], see e.g. the papers of Hristova and Bainov [11], Nieto, Jiang and Jurang [17] and Liz and Nieto [15], and the references cited therein.

For other results on differential inclusions on compact intervals, see for instance the papers of Avgerinos and Papageorgiou [1], Papageorgiou [19]and [20], and for differential inclusions on noncompact intervals, see Benchohra [2]. Additional results for differential equations in abstract spaces, and properties of semigroup theory can be found in the book of Goldstein [9].

Recently in [3], we studied existence results on compact intervals, for functional differential and integrodifferential inclusions, by using a fixed point theorem for condensing maps due to Martelli. In this paper, we extend the results in [3] studying existence results on infinite intervals, for functional differential and integrodifferential inclusions, by using a fixed point theorem of Ma [16]. The method we are going to use is to reduce the existence of mild solutions to problem (1.1)–(1.2) to the search for fixed points of a suitable multivalued map on the Fréchet space $C([-r,\infty),E)$. In order to prove the existence of fixed points, we shall rely on a theorem due to Ma [16], which is an extension to multivalued maps between locally convex topological spaces of Schaefer's theorem [21].

In Section 4 we give an existence result for the following IVP

$$y' - A(t)y \in \int_0^t K(t,s)F(s,y_s)ds, \quad t \in J = [0,\infty), \tag{1.3}$$

$$y_0 = \phi, \tag{1.4}$$

where F, ϕ are as in the problem (1.1)–(1.2) and $K : D \longrightarrow \mathbb{R}$, $D = \{(t,s) \in J \times J : t \geq s\}$.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

J_m is the compact real interval $[0,m]$ $(m \in \mathbb{N})$.

$C(J,E)$ is the linear metric Fréchet space of continuous functions from J into E in which the metric (see Corduneanu [4]), is defined by semi-norms

$$\|y\|_m := \sup\{|y(t)| : t \in J_m\}, \quad m \in \mathbb{N}.$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E.

$L^\infty(J,\mathbb{R})$ is the space of essentially bounded measurable functions from J into $\mathbb{R}$.

A measurable function $y : J \longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral see Yosida [22]).

$L^1(J, E)$ denotes the Banach space of measurable functions $y : J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^\infty |y(t)|dt \quad \text{for all} \quad y \in L^1(J, E).$$

U_p denotes the neighbourhood of 0 in $C(J, E)$ defined by

$$U_p := \{y \in C(J, E) : \|y\|_m \le p \ \text{ for each } \ m \in \mathbb{N}\}.$$

The convergence in $C(J, E)$ is the uniform convergence on compact intervals, i.e. $y_j \longrightarrow y$ in $C(J, E)$ if and only if for each $m \in \mathbb{N}$, $\|y_j - y\|_m \longrightarrow 0$ in $C(J_m, E)$ as $j \longrightarrow \infty$.

$M \subseteq C(J, E)$ is a bounded set if and only if there exists a positive function $\varphi \in C(J, \mathbb{R})$ such that

$$|y(t)| \le \varphi(t) \quad \text{for all} \quad t \in J \quad \text{and all} \quad y \in M.$$

Let $(X, |\cdot|)$ be a Banach space. A multivalued map $G : X \longrightarrow 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x \in B} G(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x \in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

G is called upper semicontinuous (u.s.c.) on X if for each $x_* \in X$ the set $G(x_*)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(x_*)$, there exists an open neighbourhood V of x_* such that $G(V) \subseteq B$.

G is said to be *totally bounded* if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If this property holds and G is u.s.c., we say that G is *completely u.s.c.* If G is totally bounded with non-empty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

In the following $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G : J \longrightarrow BCC(E)$ is said to be measurable if for each $x \in E$ the function $Y : J \longrightarrow \mathbb{R}$ defined by

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\}$$

belongs to $L^1(J, \mathbb{R})$. For more details on multivalued maps see the books of Deimling [5]and Hu and Papageorgiou [12].

Let us list the following hypotheses:

(H1) $A(\cdot) : t \longrightarrow A(t),\ t \in J$ is continuous such that

$$A(t)y = \lim_{h \to 0^+} \frac{T(t+h,t)y - y}{h}, \quad y \in D(A(t)),$$

where $T(t,s) \in B(E)$ for each $(t,s) \in \gamma := \{(t,s); 0 \le s \le t < \infty\}$, satisfying

(i) $T(t,t) = I$ (I is the identity operator in E),
(ii) $T(t,s)T(s,z) = T(t,z)$ for $0 \le z \le s \le t < \infty$,
(iii) the mapping $(t,s) \longmapsto T(t,s)y$ is strongly continuous in γ for each $y \in E$;

(H2) $F : J \times C(J_0, E) \longrightarrow BCC(E); (t,u) \longmapsto F(t,u)$ is measurable with respect to t for each $u \in C(J_0, E)$, u.s.c. with respect to u for each $t \in J$ and for each fixed $u \in C(J_0, E)$ the set

$$S_{F,u} = \{g \in L^1(J,E) : \ g(t) \in F(t,u) \text{ for a.e. } \ t \in J\}$$

is nonempty;

(H3) $\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \le p(t)\psi(\|u\|)$ for almost all $t \in J$ and all $u \in C(J_0, E)$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \longrightarrow (0,\infty)$ is continuous and increasing with

$$\int_c^\infty \frac{d\tau}{\psi(\tau)} = \infty;$$

where $c = M\|\phi\|$ and $M \ge 1$ be such that $\|T(t,s)\| \le M$ for each $(t,s) \in \gamma$;

(H4) for each neighbourhood U_p of 0, $u \in U_p$ and $t \in J$ the set

$$\left\{T(t,0)\phi(0) + \int_0^t T(t,s)g(s)ds : g \in S_{F,u}\right\}$$

is relatively compact.

Remark 2.1. (i) If $\dim E < \infty$ and if J is a compact real interval, then for each $u \in C(J_0, E)$, $S_{F,u} \neq \emptyset$ (see Lasota and Opial [14]).

(ii) If $\dim E = \infty$ then $S_{F,u}$ is nonempty if and only if the function $Y : J \longrightarrow \mathbb{R}$ defined by

$$Y(t) := \inf\{|v| : v \in F(t,u)\}$$

belongs to $L^1(J, \mathbb{R})$ (see Papageorgiou [20]).

Definition 2.2. A function $y \in C([-r,\infty),E)$ is said to be a mild solution of (1.1)–(1.2) if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t,y_t)$ a.e. on J, $y(t)=\phi(t)$, $t \in [-r,0]$, and

$$y(t) = T(t,0)\phi(0) + \int_0^t T(t-s)v(s)ds, \quad t \in J.$$

The following lemmas are crucial in the proof of our main theorem.

Lemma 2.3 ([14]). *Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying $(H2)$ and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$, then the operator*

$$\Gamma \circ S_F : C(I,X) \longrightarrow BCC(C(I,X)), \;\; y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Lemma 2.4 ([16]). *Let X be a locally convex space and $N : X \longrightarrow 2^X$ be a compact convex valued, u.s.c. multivalued map such that there exists a closed neighbourhood U_p of 0 for which $N(U_p)$ is a relatively compact set for each $p, m \in \mathbb{N}$. If the set*

$$\Omega := \{y \in X : \lambda y \in N(y) \;\; \textit{for some} \;\; \lambda > 1\}$$

is bounded, then N has a fixed point.

3. First Order Differential Inclusions

Now, we are able to state and prove our main theorem.

Theorem 3.1. *Assume that hypotheses $(H1)$–$(H4)$ are satisfied. Then the IVP (1.1)–(1.2) has at least one mild solution on $[-r,\infty)$.*

Proof. Transform the problem into a fixed point problem. Consider the multivalued map, $N : C([-r,\infty),E) \longrightarrow 2^{C([-r,\infty),E)}$ defined by:

$$N(y) := \left\{ h \in C([-r,\infty),E) : h(t) = \begin{cases} \phi(t), & \text{if } t \in J_0 \\ T(t,0)\phi(0) + \int_0^t T(t,s)g(s)ds, & \text{if } t \in J \end{cases} \right\}$$

where

$$g \in S_{F,y} = \{g \in L^1(J,E) : g(t) \in F(t,y_t) \;\; \text{for a.e.} \;\; t \in J\}. \qquad \square$$

Remark 3.2. It is clear that the fixed points of N are mild solutions to (1.1)–(1.2).

We shall show that $N(U_q)$ is relatively compact for each neighbourhood U_q of $0 \in C(J,E)$ with $q \in \mathbb{N}$ and the multivalued map N has bounded, closed and convex values and it is u.s.c. The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each $y \in C(J,E)$.*

This is obvious, since $N(y)$ is obtained by linear mapping of the convex set.

Step 2: *$N(U_q)$ is bounded in $C([-r,\infty),E)$ for each $q \in \mathbb{N}$.*

Indeed, it is enough to show that there exists a positive constant $\tilde{\ell}_m$ such that for each $h \in N(y), y \in U_q$ one has $\sup\{|h(t)| : t \in [-r,m]\} \leq \tilde{\ell}_m$ for each $m \in \mathbb{N}$.

If $h \in N(y)$, then there exists $g \in S_{F,y}$ such that for each $t \in J_m$ we have

$$h(t) = T(t,0)\phi(0) + \int_0^t T(t,s)g(s)ds.$$

By (H3) we have for each $t \in J_m$

$$\begin{aligned} |h(t)| &\leq |T(t,0)\phi(0)| + \int_0^t \|T(t,s)g(s)\|ds \leq \\ &\leq M\|\phi\| + M \cdot \sup_{t\in J_m} \psi(\|y_t\|)\left(\int_0^t p(s)ds\right) := \ell_m. \end{aligned}$$

Then for each $h \in N(U_q)$ we have

$$\sup\{|h(t)| : t \in [-r,m]\} \leq \tilde{\ell}_m = \max\{\|\phi\|, \ell_m\}.$$

Step 3: *For each $q \in \mathbb{N}$, $N(U_q)$ is equicontinuous for $U_q \in C(J,E)$.*

Let $t_1, t_2 \in J_m$, $t_1 < t_2$ and U_q be a neighbourhood of 0 in $C([-r,\infty),E)$ for $q \in \mathbb{N}$. For each $y \in U_q$ and $h \in N(y)$, there exists $g \in S_{F,y}$ such that

$$h(t) = T(t,0)\phi(0) + \int_0^t T(t,s)g(s)ds.$$

Thus

$$|h(t_2)-h(t_1)| \le |T(t_2,0)\phi(0)-T(t_1,0)\phi(0)|+$$

$$+\left\| \int_0^{t_2} [T(t_2,s)-T(t_1,s)]g(s)ds \right\| + \left\| \int_{t_1}^{t_2} T(t_1,s)g(s)ds \right\| \le$$

$$\le |T(t_2,0)\phi(0)-T(t_1,0)\phi(0)| \left\| \int_0^{t_2} [T(t_2,s)-T(t_1,s)]g(s)ds \right\| +$$

$$+M\int_{t_1}^{t_2} \|g(s)\|ds \le |T(t_2,0)\phi(0)-T(t_1,0)\phi(0)|+$$

$$+\left\| \int_0^{t_2} \big[T(t_2,s)-T(t_1,s)\big] g(s)ds \right\| + M.\sup_{t\in J_m} \psi(\|y_t\|)\left(\int_{t_1}^{t_2} p(s)ds \right).$$

As $t_2 \longrightarrow t_1$ the right-hand side of the above inequality tends to zero.

The equicontinuity for the cases $t_1 < t_2 \le 0$ and $t_1 \le 0 \le t_2$ follows from the uniform continuity of ϕ on the interval J_0 and from the relation

$$|h(t_2)-h(t_1)| = |h(t_2)-\phi(t_1)| \le |h(t_2)-h(0)| + |\phi(0)-\phi(t_1)|$$

respectively.

As a consequence of Step 2, Step 3 and (H4) we can conclude (see [6]) that $N(U_q)$ is relatively compact in $C(J,E)$.

Step 4: *N has a closed graph.*

Let $y_n \longrightarrow y_*$, $h_n \in N(y_n)$, and $h_n \longrightarrow h_*$. We shall prove that $h_* \in N(y_*)$.

$h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that

$$h_n(t) = T(t,0)\phi(0) + \int_0^t T(t,s)g_n(s)ds, \quad t\in J.$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that

$$h_*(t) = T(t,0)\phi(0) + \int_0^t T(t,s)g_*(s)ds, \quad t\in J. \tag{3.1}$$

The idea is then to use the facts that

(i) $h_n \longrightarrow h_*$;

(ii) $h_n - T(t,0)\phi(0) \in \Gamma(S_{F,y_n})$, where

$$\Gamma : L^1([-r,\infty),E) \longrightarrow C([-r,\infty),E) \text{ defined by } (\Gamma g)(t) := \int_0^t T(t,s)g(s)ds.$$

If $\Gamma \circ S_F$ is a closed graph operator, we would be done. But we don't know whether $\Gamma \circ S_F$ is a closed graph operator. So, we cut the functions y_n, h_n, g_n and we consider them defined on the interval $[k, k+1]$ for any $k \in \mathbb{N} \cup \{0\}$. Then, using Lemma (2.3), in this case we are able to affirm that (3.1) is true on the compact interval $[k, k+1]$, i.e.

$$(h_*(t) - T(t,0)\phi(0))\Big|_{[k,k+1]} = \int_0^t T(t,s)g_*^k(s)ds$$

for a suitable L^1-selection g_*^k of $F(t, y_t^*)$ on the interval $[k, k+1]$.

At this point we can paste the functions g_*^k obtaining the selection g_* defined by

$$g_*(t) = g_*^k(t) \quad \text{for} \quad t \in [k, k+1).$$

We obtain then that g_* is a L^1-selection and (3.1) will satisfied.
We give now the details.

By hypothesis we have for each $t \in J_m$ that

$$|(h_n(t) - T(t,0)\phi(0)) - (h_*(t) - T(t,0)\phi(0))| \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

Now, we consider for all $k \in \mathbb{N} \cup \{0\}$, the mapping

$$S_F^k : C([k,k+1],E) \longrightarrow L^1([k,k+1],E)$$
$$u \longmapsto S_{F,u}^k := \{f \in L^1([k,k+1],E) : f(t) \in F(t,u(t)) \text{ for a.e. } t \in [k,k+1]\}.$$

Also, we consider the linear continuous operators

$$\Gamma_k : L^1([k,k+1],E) \longrightarrow C([k,k+1],E)$$
$$g \longmapsto \Gamma_k(g)(t) = \int_0^t T(t,s)g(s)ds.$$

From Lemma (2.3), it follows that $\Gamma_k \circ S_F^k$ is a closed graph operator for all $k \in \mathbb{N} \cup \{0\}$.

Moreover, we have that

$$(h_n(t) - T(t,0)\phi(0))\Big|_{[k,k+1]} \in \Gamma_k(S_{F,y_n^k}).$$

Since $y_n \longrightarrow y^*$, it follows from Lemma (2.3) that

$$(h_*(t) - T(t,0)\phi(0))\Big|_{[k,k+1]} = \int_0^t T(t,s)g_*^k(s)ds$$

for some $g_*^k \in S^k_{F,y^*}$. So the function g_* defined on J by

$$g_*(t) = g_*^k(t) \quad \text{for} \quad t \in [k, k+1)$$

is in S_{F,y^*} since $g_*(t) \in F(t, y_t^*)$ for a.e. $t \in J$.

Step 5: *The set*

$$\Omega := \{y \in C([-r,\infty), E) : \lambda y \in N(y), \quad \textit{for some} \quad \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$y(t) = \lambda^{-1} T(t,0)\phi(0) + \lambda^{-1} \int_0^t T(t,s) g(s) ds, \quad t \in J.$$

This implies by (H3) that for each $t \in J_m$ we have

$$|y(t)| \leq M\|\phi\| + M \int_0^t p(s)\psi(\|y_s\|) ds.$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq m.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, m]$, by the previous inequality we have for $t \in [0, m]$

$$\begin{aligned} \mu(t) &\leq M\|\phi\| + M \int_0^t p(s)\psi(\|y_s\|) ds \leq \\ &\leq M\|\phi\| + M \int_0^t p(s)\psi(\mu(s)) ds. \end{aligned}$$

If $t^* \in J_0$, then $\mu(t) = \|\phi\|$ and the previous inequality holds since $M \geq 1$.

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$v(0) = M\|\phi\|, \quad \mu(t) \leq v(t), \quad t \in [0, m],$$

and

$$v'(t) = M p(t)\psi(\mu(t)), \quad t \in J_m.$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq M p(t)\psi(v(t)), \quad t \in J_m.$$

This implies for each $t \in J_m$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq M \int_0^t p(s)ds < \infty.$$

From (H3) it follows that there exists a constant b such that $v(t) \leq b$, $t \in J_m$, and hence $\mu(t) \leq b$ for each $t \in J_m$ where b depends only on the functions p and ψ. Since for every $t \in J_m$, $\|y_t\| \leq \mu(t)$, we have

$$\sup\{|y(t)| : t \in [-r, m]\} \leq b.$$

This shows that Ω is bounded.

Set $X := C([-r, \infty), E)$. As a consequence of Lemma (2.4) we deduce that N has a fixed point which is a mild solution of (1.1)–(1.2).

4. First Order Integro-differential Inclusions

In this section we give an existence result for the IVP (1.3)–(1.4).

Assume that:

(H5) for each $t \in J_m$ $(m \in \{1, 2, \dots\})$, $K(t, s)$ is measurable on $[0, t]$ and

$$K(t) = \operatorname{ess\,sup}\big\{|K(t, s)|, \quad 0 \leq s \leq t\big\},$$

is bounded on J_m;

(H6) the map $t \longmapsto K_t$ is continuous from J_m to $L^\infty(J_m, \mathbb{R})$; here $K_t(s) = K(t, s)$;

(H7) for each neighbourhood U_p of 0, $u \in U_p$ and $t \in J$ the set

$$\left\{T(t, 0)\phi(0) + \int_0^t T(t, s) \int_0^s K(s, \sigma)g(\sigma)d\sigma ds : g \in S_{F,u}\right\}$$

is relatively compact.

Definition 4.1. A function $y \in C([-r, \infty), E)$ is said to be a mild solution of (1.3)–(1.4) if there exists a function $v \in L^1(J, E)$ such that $v(t) \in F(t, y_t)$ a.e. on J, $y(t) = \phi(t)$, $t \in [-r, 0]$, and

$$y(t) = T(t, 0)\phi(0) + \int_0^t T(t, s) \int_0^s K(s, u)v(u)duds, \ t \in J.$$

Theorem 4.2. *Assume that hypotheses (H1)–(H3), (H5)–(H7) are satisfied. Then the IVP (1.3)–(1.4) has at least one mild solution on $[-r, \infty)$.*

Proof. Transform the problem into a fixed point problem. Consider the multivalued map, $N : C([-r,\infty),E) \longrightarrow 2^{C([-r,\infty),E)}$ defined by:

$$N(y) := \left\{ h \in C([-r,\infty),E) : h(t) = \begin{cases} \phi(t), & \text{if } t \in J_0 \\ T(t,0)\phi(0)+ & \\ +\int_0^t T(t,s)\int_0^s K(s,u)g(u)duds, & \text{if } t \in J \end{cases} \right\},$$

where

$$g \in S_{F,y} = \{g \in L^1(J,E) : g(t) \in F(t,y_t) \text{ for a.e. } t \in J\}.$$

Remark 4.3. It is clear that the fixed points of N are mild solutions to (1.3)–(1.4).

As in Theorem 3.1 we can show that $N(U_q)$ is relatively compact for each neighbourhood U_q of $0 \in C(J,E)$ with $q \in \mathbb{N}$ and the multivalued map N has bounded, closed and convex values and it is u.s.c.

Now, it remains to show that the set

$$\Omega := \{y \in C([-r,\infty),E) : \lambda y \in N(y), \text{ for some } \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$y(t) = \lambda^{-1}T(t,0)\phi(0) + \lambda^{-1}\int_0^t T(t,s)\int_0^s K(s,u)g(u)duds, \quad t \in J.$$

This implies by (H3) and (H5) that for each $t \in J_m$ we have

$$\begin{aligned} |y(t)| &\leq M\|\phi\| + M\left\|\int_0^t\int_0^s K(s,u)g(u)duds\right\| \leq \\ &\leq M\|\phi\| + M\int_0^t\int_0^s |K(s,u)|p(u)\psi(\|y_u\|)duds \leq \\ &\leq M\|\phi\| + Mm\sup_{t\in J_m} K(t)\int_0^t p(s)\psi(\|y_s\|)ds. \end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq m.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0,m]$, by the previous inequality we have for $t \in [0,m]$

$$\begin{aligned}\mu(t) &\leq M\|\phi\| + Mm \sup_{t\in J_m} K(t) \int_0^t p(s)\psi(\|y_s\|)ds \leq \\ &\leq M\|\phi\| + Mm \sup_{t\in J_m} K(t) \int_0^t p(s)\psi(\mu(s))ds.\end{aligned}$$

If $t^* \in J_0$ then $\mu(t) = \|\phi\|$ and the previous inequality holds, since $M \geq 1$.

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$v(0) = M\|\phi\|, \quad \mu(t) \leq v(t),\ t \in J_m,$$

and

$$v'(t) = Mm \sup_{t\in J_m} K(t)p(t)\psi(\mu(t)),\ t \in J_m.$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq Mm \sup_{t\in J_m} K(t)p(t)\psi(v(t)),\ t \in J_m.$$

This implies for each $t \in J_m$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq Mm \sup_{t\in J_m} K(t) \int_0^t p(s)ds < \infty.$$

From (H7) it follows that there exists a constant b such that $v(t) \leq b,\ t \in J_m$, and hence $\mu(t) \leq b$ for each $t \in J_m$ where b depends only on the functions p and ψ. Since for every $t \in J_m, \|y_t\| \leq \mu(t)$, we have

$$\sup\{|y(t)| : t \in [-r,m]\} \leq b.$$

This shows that Ω is bounded.

Set $X := C([-r,\infty),E)$. As a consequence of Lemma (2.4) we deduce that N has a fixed point which is a mild solution of (1.3)–(1.4). □

ACKNOWLEDGEMENT

The authors thanks the referee for his helpful comments.

REFERENCES

1. E. P. Avgerinos and N. S. Papageorgiou, On quasilinear evolution inclusions, *Glas. Mat. Ser.* III **28, 48**(1993), No. 1, 35–52.

2. M. Benchohra, Existence of mild solutions on infinite intervals to first order initial value problems for a class of differential inclusions in Banach spaces, *Discuss. Math. Differential Incl.*, **19**(1999), No. (1-2), 111–121.

3. M. Benchohra and S. K. Ntouyas, Existence results for functional differential and integro-differential inclusions in Banach spaces. *Indian J. Pure Appl. Math.* **32**(2001), 665–675.

4. C. Corduneanu, Integral Equations and Applications. *Cambridge Univ. Press, New York,* 1990.

5. K. Deimling, Multivalued Differential Equations. *Walter de Gruyter, Berlin-New York,* 1992.

6. K. Deimling, Nonlinear Functional Analysis, *Springer Verlag, New York,* 1985.

7. J. Dugundji and A. Granas, Fixed Point Theory, *Monografie Mat. PWN, Warsaw,* 1982.

8. L. H. Erbe, Q. Kong and B. G. Zhang, Oscillation Theory for Functional Differential Equations, *Lecture Notes in Pure and Appl. Math., Marcel Dekker, New York,* 1994.

9. J. A. Goldstein, Semigroups of Linear Operators and Applications, *Oxford, New York,* 1985.

10. J. Henderson, Boundary Value Problems for Functional Differenial Equations, *World Scientific, Singapore,* 1995.

11. S. G. Hristova and D. D. Bainov, Application of the monotone iterative techniques of Lakshmikantham to the solution of the initial value problem for functional differential equations, *J. Math. Phys. Sci.* **24**(1990), 405–413.

12. Sh. Hu and N. Papageorgiou, Handbook of Multivalued Analysis, **I**, *Theory, Kluwer, Dordrecht, Boston, London,* 1997.

13. G. S. Ladde, V. Lakshmikantham and A. S. Vatsala, Monotone Iterative Techniques for Nonlinear Differential Equations, *Pitman, Boston,* 1985.

14. A. Lasota and Z. Opial, An application of the Kakutani-Ky-Fan theorem in the theory of ordinary differential equations. *Bull Polish. Acad. Sci. Math. Astronom. Phys.* **13**(1965), 781–786.

15. E. Liz and J. J. Nieto, Periodic BVPs for a class of functional differential equations, *J. Math. Anal. Appl.* **200**(1996), 680–686.

16. T. W. Ma, Topological degrees for set-valued compact vector fields in locally convex spaces, *Dissertatione Math.* **92**(1972), 1–43.

17. J. J. Nieto, Y. Jiang and Y. Jurang, Monotone iterative method for functional differential equations, *Nonlinear Anal.* **32**(1998), No. 6, 741–749.

18. S. K. Ntouyas, Initial and boundary value problems for functional differential equations via the topological transversality method: *Bull. Greek Math. Soc.* **40**(1998), 3–41.

19. N. S. Papageorgiou, Mild solutions of semilinear evolution inclusions, *Indian J. Pure Appl. Math.* **26**(1995), No. 3, 189–216.

20. N. S. Papageorgiou, Boundary value problems for evolution inclusions, *Comment. Math. Univ. Carolin.* **29**(1988), 355–363.

21. H. Schaefer, Über die methode der a priori schranken, *Math. Ann.* **129**(1955), 415–416.

22. K. Yosida, Functional Analysis, 6$^{\text{th}}$ edn. *Springer-Verlag, Berlin,* 1980.

(Received 10.12.2000)

Authors' addresses:

S. K. Ntouyas
Department of Mathematics
University of Ioannina
451 10 Ioannnina
Greece

M. Benchohra
Département de Mathématiques,
Université de Sidi Bel
Abbès, BP 89,
22000 Sidi Bel Abbès, Algérie