Proceedings of A. Razmadze
Mathematical Institute,
126, 2001

ON ONE APPROACH TO THE INTERSECTION PROBLEM OF PEETRE

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Abstract. In the present paper we consider the intersection problem of Peetre, i.e., we investigate the question whether the equality

$$(A, A_0 \cap A_1)_{\theta,q} = (A, A_0)_{\theta,q} \cap (A, A_1)_{\theta,q}$$

is fulfilled.

Let F be an interpolation functor (see [1], [2]) and $\{A, A_0, A_1\}$ be an interpolation triple (see Definition 1).

In [3], J. Peetre deals with the question: when the equality

$$F(A, A_0 \cap A_1) = F(A, A_0) \cap F(A, A_1) \tag{1}$$

is valid?

Considering as functor F the functor of real interpolation (see [1], [2]), J. Peetre pointed out the sufficient conditions for the fulfilment of the equality

$$(A, A_0 \cap A_1)_{\theta,q} = (A, A_0)_{\theta,q} \cap (A, A_1)_{\theta,q} \tag{2}$$

in terms of “quasi-linearization” of the corresponding interpolation pairs. Here and in the sequel, by symbol $(,)_{\theta,q}$ we denote interpolation spaces obtained by the method of real interpolation (see [1], [2]).

Equality (2) has been considered by various authors (see [3],[4],[5],[6] and references in [1],[2]).

2000 *Mathematics Subject Classification.* 46B70, 46M35.
Key words and phrases. Intersection problem of Peetre, interpolation function, interpolation pair, Banach space.

Noticing that in equality (1) besides the functor F there takes place the intersection functor, we can formulate the problem, replacing the intersection functor by the sum functor, and then consider whether the equality

$$(A, A_0 + A_1)_{\theta,q} = (A, A_0)_{\theta,q} + (A, A_1)_{\theta,q} \tag{3}$$

is valid.

One can generalize the statement of Peetre's problem by replacing in equality (1) the intersection functor by an interpolation functor G and then consider whether the equality

$$F(A, G(A_0 A_1)) = G(F(A, A_0), F(A, A_1)) \tag{4}$$

is valid.

Using in (4) as the functors F and G the sum and intersection functors, we arrive at the following question: when equalities

$$A \cap (A_0 + A_1) = (A \cap A_0) + (A \cap A_1), \tag{5}$$
$$A + (A_0 \cap A_1) = (A + A_0) \cap (A + A_1) \tag{6}$$

are valid?

Equalities (2),(3) and (5),(6) turn out to be connected (see Theorem 5).

1. On Equalities (5),(6).

Recall some basic notions concerning the interpolation theory (see [1], [2]).

Definition 1. Let A_0 and A_1 be the two Banach spaces, linearly and continuously embedded in a linear topological space T (here and in the sequel, embeddings are understood in a set-theoretic and topological sense). A pair $\{A_0, A_1\}$ of such spaces will be called an interpolation pair. In a similar way we define an interpolation triple $\{A, A_0, A_1\}$.

Definition 2. For the interpolation pair $\{A_0, A_1\}$ we define respectively an intersection and a sum

$$\Delta \equiv A_0 \cap A_1 = \big\{a \in T;\ \|a\|_\Delta = \max(\|a\|_{A_0}, \|a\|_{A_1}) < \infty\big\},$$
$$\Sigma \equiv A_0 + A_1 = \big\{a \in T;\ a = a_0 + a_1,\ a_0 \in A_0,\ a_1 \in A_1\big\}.$$

The norm in Σ is introduced as follows:

$$\|a\|_\Sigma = \inf_{\substack{a=a_0+a_1 \\ a_j \in A_j}} \big(\|a_0\|_{A_0} + \|a_1\|_{A_1}\big).$$

Definition 3. The set A is said to be intermediate with respect to the interpolation pair $\{A_0, A_1\}$, if the embeddings $\Delta \subset A \subset \Sigma$ take place.

Theorem 1. *Let $\{A, A_0, A_1\}$ be an interpolation triple. If one of the triple spaces is embedded one into another, then equalities* (5), (6) *hold for that triple.*

Proof. It is not difficult to see that regardless the condition of the theorem, for any interpolation triple the embeddings

$$A \cap (A_0 + A_1) \supset (A \cap A_0) + (A \cap A_1), \tag{7}$$

$$A + (A_0 \cap A_1) \subset (A + A_0) \cap (A + A_1) \tag{8}$$

take place.

Embeddings inverse to (7), (8) can be easily verified, if one of the embeddings $A_0 \subset A_1$, $A_1 \subset A_0$ holds. Moreover, the embedding inverse to (7) is obvious for $A \subset A_0$ (or for $A \subset A_1$), and the embedding inverse to (8) is obvious for $A_0 \subset A$ (or for $A_1 \subset A$).

Let $A_0 \subset A$. Then the embedding inverse to (7) has the form

$$A \cap (A_0 + A_1) \subset A_0 + (A \cap A_1). \tag{9}$$

Let $a \in A \cap (A_0 + A_1)$. Then $a \in A$ and $a = a_0' + a_1'$; $a_0' \in A_0$, $a_1' \in A_1$. For $(A_0 \subset A)$ we have

$$\|a_1'\|_A \leq \|a\|_A + \|a_0'\|_A \leq c\big(\|a\|_A + \|a_0'\|_{A_0}\big) < \infty.$$

That is, $a_1' \in A$. Then $a_1' \in A \cap A_1$, and hence $a \in A_0 + (A \cap A_1)$. Thus we have proved the set-theoretic embedding (9). Let us now prove that this embedding is continuous. For $(A_0 \subset A)$ we have

$$\|a\|_{A_0 + (A \cap A_1)} = \inf_{\substack{a = a_0 + a_1 \\ a_0 \in A_0 \\ a_1 \in A \cap A_1}} \big(\|a_0\|_{A_0} + \|a_1\|_{A \cap A_1}\big) \leq$$

$$\leq c\big(\|a_0'\|_{A_0} + \|a_1'\|_A + \|a_1'\|_{A_1}\big) \leq c'\big(\|a\|_A + \|a_0'\|_{A_0} + \|a_1'\|_{A_1}\big).$$

Taking the lower bound with respect to all expansions of the element a; $a = a_0' + a_1$; $a_0' \in A_0$, $a_1' \in A_1$, we arrive at (9).

Equality (5) is proved for $A_0 \subset A$ (or for $A_1 \subset A$). Analogously we prove the inverse to (8) embedding for $A \subset A_0$ (or for $A \subset A_1$). □

Thus the theorem is proved.

Theorem 2. *Let* $\{A, A_0, A_1\}$ *be an interpolation triple.*

(*a*) *If for* $\{A, A_0, A_1\}$ *one of equalities* (5), (6) *is fulfilled, then the other is fulfilled too.*

(*b*) *If for* $\{A, A_0, A_1\}$ *equalities* (5), (6) *are fulfilled, then the analogues of these equalities are fulfilled for the triple* $\{A_1, A, A_0\}$ *as well.*

Proof. Since $A_1 \subset A_0 + A_1$, we apply to the triple $\{A_0 + A_1,\ A, A_1\}$ Theorem 1. We have

$$(A_0 + A_1) \cap (A + A_1) = \big[(A_0 + A_1) \cap A\big] + \big[(A_0 + A_1) \cap A_1\big] =$$

$$= \big[(A_0 + A_1) \cap A\big] + A_1. \tag{10}$$

Let for the triple $\{A, A_0, A_1\}$ equality (5) be fulfilled. Then from (5), (10) we get

$$(A_0 + A_1) \cap (A + A_1) = (A_0 \cap A) + (A_1 \cap A) + A_1 = A_1 + (A_0 \cap A),$$

i.e., for the triple $\{A_1, A, A_0\}$ equality (6) is fulfilled. Thus, if for the triple $\{A, A_0, A_1\}$ equality (5) is fulfilled, then for the triple $\{A_1, A, A_0\}$ equality (6) is fulfilled as well. This case we can write briefly as follows:

$$\{A, A_0, A_1\}_{(5)} \Rightarrow \{A_1, A, A_0\}_{(6)}. \tag{11}$$

Analogously, since $A \cap A_0 \subset A$, applying Theorem 1 to the triple

$$\{A \cap A_0, A_0, A_1\}$$

we get

$$(A \cap A_0) + (A_0 \cap A_1) = \big[(A \cap A_0) + A\big] \cap \big[(A \cap A_0) + A_1\big] = A_0 \cap \big[(A \cap A_0) + A_1\big].$$

Using now (11), we obtain

$$(A \cap A_0) + (A_0 \cap A_1) = A_0 \cap (A + A_1) \cap (A_0 + A_1) = A_0 \cap (A + A_1),$$

i.e.,

$$\{A_1, A, A_0\}_{(6)} \Rightarrow \{A_0, A_1, A\}_{(5)}. \tag{12}$$

Thus we have proved a chain of Corollaries (see (11), (12)) that

$$\{A, A_0, A_1\}_{(5)} \Rightarrow \{A_1, A, A_0\}_{(6)} \Rightarrow \{A_0, A_1, A\}_{(5)} \Rightarrow \{A, A_0, A_1\}_{(6)}.$$

In a similar way we prove that

$$\{A, A_0, A_1\}_{(6)} \Rightarrow \{A, A_0, A_1\}_{(5)}.$$

Thus statement (a) is proved. Statement (b) follows from statement (a) and corollaries (11) and (12).

So the theorem is proved. □

Remark 1. The condition of Theorem 1 is not necessary for equalities (5) and (6) to be valid. For the triple $\{(A_0, A_1)_{\theta,q}, A_0, A_1\}$, equalities (5) and (6), as is shown in [7], hold, and for that triple take the form (see [6], [7]):

$$(A_0, A_1)_{\theta,q} = \big[(A_0, A_1)_{\theta,q} \cap A_0\big] + \big[(A_0, A_1)_{\theta,q} \cap A_1\big] = \\ = (A_0, \Delta)_{\theta,q} + (\Delta, A_1)_{\theta,q}, \tag{13}$$

$$(A_0, A_1)_{\theta,q} = \big[(A_0, A_1)_{\theta,q} + A_0\big] \cap \big[(A_0, A_1)_{\theta,q} + A_1\big] = \\ = (A_0, \Sigma)_{\theta,q} \cap (\Sigma, A_1)_{\theta,q}, \tag{14}$$

where $0 < \theta < 1$, $1 \le q \le \infty$.

Remark 2. It is obvious that from the point of view of the fulfilment of equalities (5) and (6), the second and the third spaces of the triple are symmetric. Statement (b) of Theorem 2 shows that from that point of view, all three spaces of the triple are symmetric, i.e., if (5) and (6) are fulfilled for one or another sequence of spaces, then these equalities will be fulfilled for any other sequence of spaces.

2. Specification of One Classical Statement

It is well known (see statement (f) of Theorem 1.3.3 from [1] and statement (e) of Theorem 3.4.1 from [2]) that if $A_1 \subset A_0$, then $(A_0, A_1)_{\theta,q} \subset (A_0, A_1)_{\eta,q}$ for $0 < \eta < \theta < 1$, $1 \le q \le \infty$. It turns out that the inverse statement is likewise valid. Moreover, the second subscripts of interpolation spaces may fail to coincide.

Theorem 3. *Let* $0 < \eta < \theta < 1$, $1 \le q, r \le \infty$. *For the embedding*

$$(A_0, A_1)_{\theta,q} \subset (A_0, A_1)_{\eta,q} \tag{15}$$

to be fulfilled, it is necessary and sufficient that $A_1 \subset A_0$.

The statement of the theorem remains valid for $\theta = 1$ *and* $q = 0$, *if we replace* $(A_0, A_1)_{\theta,q}$ *and* $(A_0, A_1)_{\eta,q}$ *by* A_1 *and* A_0, *respectively.*

Proof. Let first $0 < \eta < \theta < 1$, $q = r$. Taking into account statement (e) of Theorem 3.4.1 from [2], it is sufficient to prove that embedding (15) implies the embedding $A_1 \subset A_0$.

Using [6] and embedding (15), we have

$$(\Delta, A_1)_{\theta,q} = A_1 \cap (A_0, A_1)_{\theta,q} \subset A_1 \cap (A_0, A_1)_{\eta,q} = (\Delta, A_1)_{\eta,q}. \tag{16}$$

Further, using statement (a) of Theorem 3.4.1 from [2] and taking into account that $1 - \theta < 1 - \eta$ and $\Delta \subset A_1$, we have

$$(\Delta, A_1)_{\theta,q} = (A_1, \Delta)_{1-\theta,q} \supset (A_1, \Delta)_{1-\eta,q} = (\Delta, A_1)_{\eta,q}. \tag{17}$$

Embeddings (16) and (17) result in

$$(\Delta, A_1)_{\theta,q} = (\Delta, A_1)_{\eta,q}, \quad \eta < \theta.$$

It follows from Statement 21, §3.13 of [2] that $\Delta = A_1$, i.e., $A_1 \subset A_0$.

Let us convince ourselves that the statement of the theorem is valid for $\theta = 1$ or $\eta = 0$. Assume, for example, that $\theta = 1$. Then embedding (15) takes the form

$$A_1 \subset (A_0, A_1)_{\eta,r}. \tag{18}$$

Clearly, (18) follows from the embedding $A_1 \subset A_0$. The fact that the inverse statement is also valid can be easily seen if we apply the reiteration theorem (see Theorem 3.5.2 from [2]). We have

$$(A_0, A_1)_{\alpha,p} \subset \big(A_0, (A_0, A_1)_{\eta,r}\big)_{\alpha,p} = (A_0, A_1)_{\eta\alpha,p},$$

where $0 < \alpha < 1$, $1 \le p \le \infty$.

Now the embedding $A_1 \subset A_0$ follows from our proof.

Thus the statement of the theorem is proved for $q = r$.

Let now $0 < \eta < \theta < 1$, $1 \le q$, $r \le \infty$. Suppose that embedding (15) holds. Analogously to what we have done above, on the basis of the reiteration theorem we have ($0 < \alpha < 1$, $1 \le p \le \infty$)

$$(A_0, A_1)_{\theta\alpha,p} = \big(A_0, (A_0, A_1)_{\theta,q}\big)_{\alpha,p} \subset \big(A_0, (A_0, A_1)_{\eta,r}\big)_{\alpha,p} = (A_0, A_1)_{\eta\alpha,p}.$$

From the proven above it follows that $A_1 \subset A_0$.

Consider now the last case. Let $A_1 \subset A_0$. From the reiteration theorem follows

$$\big((A_0, A_1)_{\eta,r}, (A_0, A_1)_{\theta,q}\big)_{\alpha,q} = (A_0, A_1)_{\beta,q}, \tag{19}$$

where $\beta = (1 - \alpha)\eta + \theta\alpha$.

Since $\beta < \theta$, from the embedding $A_1 \subset A_0$ (see Theorem 3.4.1 from [2]) we have

$$(A_0, A_1)_{\theta,q} \subset (A_0, A_1)_{\beta,q}. \tag{20}$$

If we put $B_0 = (A_0, A_1)_{\eta,r}$ and $B_1 = (A_0, A_1)_{\theta,q}$, then from (19) and (20) we arrive at

$$B_1 \subset (B_0, B_1)_{\alpha,q}.$$

From the above proven (case $\theta = 1$, $\eta = \alpha$) it follows that $B_1 \subset B_0$, i.e., embedding (15) holds. □

Thus the theorem is proved.

3. On the Intersection Problem of Peetre

In this section we will investigate the question of validity of equalities (2), (3) for the triple $\{A, A_0, A_1\}$, where the space A is intermediate with respect to the interpolation pair $\{A_0, A_1\}$.

In this case it turns out that for the fulfilment of equalities (2) and (3) it is sufficient that at least one of equalities (5), (6) be fulfilled (the second equality in this case is fulfilled automatically).

We begin our consideration with the proof of formulas for the sum and intersection of interpolation spaces.

Theorem 4. *Let $\{A_0, A_1\}$ be an interpolation pair, $0 < \eta < \theta < 1$, $1 \le q$, $r \le \infty$. Then the equalities*

(*a*) $(A_0, A_1)_{\theta,q} \cap (A_0, A_1)_{\eta,r} = (A_0, \Sigma)_{\eta,r} \cap (\Sigma, A_1)_{\theta,q}$;

(*b*) $(A_0, A_1)_{\theta,q} \cap (A_0, A_1)_{\eta,r} = (A_0, \Delta)_{\theta,q} + (\Delta, A_1)_{\eta,r}$;

(*c*) $(A_0, A_1)_{\theta,q} + (A_0, A_1)_{\eta,r} = (A_0, \Delta)_{\eta,r} + (\Delta, A_1)_{\theta,q}$;

(*d*) $(A_0, A_1)_{\theta,q} + (A_0, A_1)_{\eta,r} = (A_0, \Sigma)_{\theta,q} \cap (\Sigma, A_1)_{\eta,r}$;

hold.

Proof. Let us first prove statement (a). We have (explanations will be given below)

$$\begin{gathered}(A_0, A_1)_{\theta,q} \cap (A_0, A_1)_{\eta,r} = \\ = (A_0, \Sigma)_{\theta,q} \cap (\Sigma, A_1)_{\theta,q} \cap (A_0, \Sigma)_{\eta,r} \cap (\Sigma, A_1)_{\eta,r} = \\ = (\Sigma, A_0)_{1-\theta,q} \cap (\Sigma, A_1)_{\theta,q} \cap (\Sigma, A_0)_{1-\eta,r} \cap (\Sigma, A_1)_{\eta,r} = \\ = (\Sigma, A_1)_{\theta,q} \cap (\Sigma, A_0)_{1-\eta,r} = (A_0, \Sigma)_{\eta,r} \cap (\Sigma, A_1)_{\theta,q}.\end{gathered}$$

The first equality is written on the basis of (14), the second and the fourth on the basis of statement (a) of Theorem 3.4.1 from [2], and in the third equality we apply Theorem 3.

Statement (c) of our theorem can be proved analogously. By virtue of (13), statement (a)of Theorem 3.4.1 from [2] and our Theorem 3 we have

$$\begin{gathered}(A_0, A_1)_{\theta,q} + (A_0, A_1)_{\eta,r} = (A_0, \Delta)_{\theta,q} + (\Delta, A_1)_{\theta,q} + (A_0, \Delta)_{\eta,r} + \\ +(\Delta, A_1)_{\eta,r} = (A_0, \Delta)_{\theta,q} + (A_1, \Delta)_{1-\theta,q} + (A_0, \Delta)_{\eta,r} + (A_1, \Delta)_{1-\eta,r} = \\ = (A_0, \Delta)_{\eta,r} + (A_1, \Delta)_{1-\theta,q} = (A_0, \Delta)_{\eta,r} + (\Delta, A_1)_{\theta,q}.\end{gathered}$$

Let us prove statement (b). According to (13) we have

$$\begin{gathered}(A_0, A_1)_{\theta,q} \cap (A_0, A_1)_{\eta,r} = \big[(A_0, \Delta)_{\theta,q} + (\Delta, A_1)_{\theta,q}\big] \cap \\ \cap \big[(A_0, \Delta)_{\eta,r} + (\Delta, A_1)_{\eta,r}\big].\end{gathered}$$

Owing to Theorem 3, since $(A_0, \Delta)_{\theta,q} \subset (A_0, \Delta)_{\eta,r}$, using Theorem 1 we can see that for the triple

$$\big\{(A_0, \Delta)_{\theta,q}, (\Delta, A_1)_{\theta,q}, (A_0, \Delta)_{\eta,r} + (\Delta, A_1)_{\eta,r}\big\}$$

formula (5) holds. Then (the explanation will be given below)

$$\begin{gathered}(A_0, A_1)_{\theta,q} \cap (A_0, A_1)_{\eta,r} = \big[(A_0, \Delta)_{\theta,q} + (\Delta, A_1)_{\theta,q}\big] \cap (A_0, \Delta)_{\eta,r} + \\ +\big[(A_0, \Delta)_{\theta,q} + (\Delta, A_1)_{\theta,q}\big] \cap (\Delta, A_1)_{\eta,r} = (A_0, \Delta)_{\theta,q} \cap (A_0, \Delta)_{\eta,r} + \\ +(\Delta, A_1)_{\theta,q} \cap (A_0, \Delta)_{\eta,r} + (A_0, \Delta)_{\theta,q} \cap (\Delta, A_1)_{\eta,r} + \\ +(\Delta, A_1)_{\theta,q} \cap (\Delta, A_1)_{\eta,r} = (A_0, \Delta)_{\theta,q} + (\Delta, A_1)_{\theta,q} \cap (A_0, \Delta)_{\eta,r} + \\ (A_0, \Delta)_{\theta,q} \cap (\Delta, A_1)_{\eta,r} + (\Delta, A_1)_{\eta,r}.\end{gathered} \tag{21}$$

The first and the second equalities are written on the basis of formula (5) (see Theorems 1 and 3), and in the third equality we apply Theorem 3.

It has been proved in [6] that

$$(A_0, \Delta)_{\alpha,p} = A_0 \cap (A_0, A_1)_{\alpha,p}, \quad (\Delta, A_1)_{\alpha,p} = A_1 \cap (A_0, A_1)_{\alpha,p}, \tag{22}$$

where $0 < \alpha < 1$, $1 \le p \le \infty$.

Using these equalities, it is not difficult to see that the second and the third summands in the right-hand side of (21) coincide with Δ. Statement (b) follows now from (21).

Finally, statement (d) is the consequence of the chain (explanations will be given below)

$$
\begin{gathered}
(A_0, A_1)_{\theta,q} + (A_0, A_1)_{\theta,q} = \big[(\Sigma, A_1)_{\theta,q} \cap (A_0, \Sigma)_{\theta,q}\big] + \\
+\big[(\Sigma, A_1)_{\eta,r} \cap (A_0, \Sigma)_{\eta,r}\big] = \\
= \Big(\big[(\Sigma, A_1)_{\theta,q} \cap (A_0, \Sigma)_{\theta,q}\big] + (\Sigma, A_1)_{\eta,r}\Big) \cap \\
\cap\Big(\big[(\Sigma, A_1)_{\theta,q} \cap (A_0, \Sigma)_{\theta,q}\big] + (A_0, \Sigma)_{\eta,r}\Big) = \\
= \big[(\Sigma, A_1)_{\theta,q} + (\Sigma, A_1)_{\eta,r}\big] \cap \big[(A_0, \Sigma)_{\theta,q} + \\
+(\Sigma, A_1)_{\eta,r}\big] \cap \big[(\Sigma, A_1)_{\theta,q} + (A_0, \Sigma)_{\eta,r}\big] \cap \big[(A_0, \Sigma)_{\theta,q} + \\
+(A_0, \Sigma)_{\eta,r}\big] = (\Sigma, A_1)_{\theta,q} \cap (A_0, \Sigma)_{\eta,r}.
\end{gathered}
$$

The first equality is written on the basis of formula (5). The second and the third equalities are written on the basis of formula (6) with regard for Theorems 1 and 3. The last equality is written with the help of Theorem 3 and proven in [7] equalities

$$
(A_0, \Sigma)_{\alpha,p} = A_0 + (A_0, A_1)_{\alpha,p}, \quad (\Sigma, A_1)_{\alpha,p} = A_1 + (A_0, A_1)_{\alpha,p}, \tag{23}
$$

where $0 < \alpha < 1$, $1 \le p \le \infty$.

The above-obtained results allow one to prove the statement, pertaining to the intersection problem of Peetre. □

Theorem 5. *Let $\{A_0, A_1\}$ be an interpolation pair and A be an intermediate space with respect to that pair.*

If for the triple $\{A, A_0, A_1\}$ one of the equalities (5) *or* (6) *is fulfilled, then for that triple equalities* (2) *and* (3) *are fulfilled for all θ and q; $0 < \theta < 1$, $1 \le q \le \infty$.*

Proof. Note that for an arbitrary space A the embeddings

$$
\begin{gathered}
(A, \Delta)_{\theta,q} \supset (A \cap A_0 + A \cap A_1, \Delta)_{\theta,q} \supset (A \cap A_0, \Delta)_{\theta,q} + \\
+(A \cap A_1, \Delta)_{\theta,q} = (A \cap A_0, \Delta)_{\theta,q} + (\Delta, A \cap A_1)_{1-\theta,q}
\end{gathered} \tag{24}
$$

hold. If we put $B_0 = A \cap A_0$, $B_1 = A \cap A_1$, then

$$
B_0 \cap B_1 = A \cap A_0 \cap A_1 = \Delta, \tag{25}
$$

$$
B_0 + B_1 = A \cap A_0 + A \cap A_1 = A \cap (A_0 + A_1) = A. \tag{26}
$$

In the first equality of (25) we have used the embedding $A \supset \Delta$. In equality (26) we have used both the embedding $A \subset \Sigma$ and equality (5).

Using now statements (a)–(c) of Theorem 5 and equalities (25) and (26), from (24) we obtain

$$\begin{gathered}(A,\Delta)_{\theta,q}\supset(B_0,\Delta)_{\theta,q}+(\Delta,B_1)_{1-\theta,q}=\\=(B_0,B_0+B_1)_{1-\theta,q}\cap(B_0+B_1,B_1)_{\theta,q}=\\=(B_0,A)_{1-\theta,q}\cap(A,B_1)_{\theta,q}=(A,B_0)_{\theta,q}\cap(A,B_1)_{\theta,q}=\\=(A,A\cap A_0)_{\theta,q}\cap(A,A\cap A_1)_{\theta,q}=A\cap(A,A_0)_{\theta,q}\cap(A,A_1)_{\theta,q}.\end{gathered}\tag{27}$$

The last step of the given chain is written by means of equality (22).

Using equality (6), we have

$$A=A+(A_0\cap A_1)=(A+A_0)\cap(A+A_1)\supset(A,A_0)_{\theta,q}\cap(A,A_1)_{\theta,q}.\tag{28}$$

Relations (27) and (28) result in

$$(A,\Delta)_{\theta,q}\supset(A,A_0)_{\theta,q}\cap(A,A_1)_{\theta,q}.$$

Since the inverse embedding is obvious, we arrive at (2).

Let us prove equality (3). We have (explanations are given below)

$$\begin{gathered}(A,\Sigma)_{\theta,q}\subset\big((A+A_0)\cap(A+A_1),\Sigma\big)_{\theta,q}\subset\\\subset(A+A_0,\Sigma)_{\theta,q}\cap(A+A_1,\Sigma)_{\theta,q}=\\=(A+A_0,\Sigma)_{\theta,q}\cap(\Sigma,A+A_1)_{1-\theta,q}=(A+A_0,A)_{1-\theta,q}+(A,A+A_1)_{\theta,q}=\\=A+(A_0,A)_{1-\theta,q}+(A,A_1)_{\theta,q}=(A,A_0)_{\theta,q}+(A,A_1)_{\theta,q}.\end{gathered}$$

The first two embeddings are obvious. The third equality is written on the basis of statement (a) of Theorem 3.4.1 from [2]. The fourth equality is given by Theorem 4 and equality (6), and the fifth one by equality (23). The last equality is the consequence of the chain (see (5))

$$A=A\cap(A_0+A_1)=A\cap A_0+A\cap A_1\subset(A,A_0)_{\theta,q}+(A,A_1)_{\theta,q}.$$

Thus

$$(A,\Sigma)_{\theta,q}\subset(A,A_0)_{\theta,q}+(A,A_1)_{\theta,q}.$$

Since the inverse embedding is obvious, we arrive at (3), and consequently, the theorem is proved. □

Theorems 1 and 5 result in the following

Theorem 6. *Let $\{A_0,A_1\}$ be an interpolation pair. If the space A is such that one of the following four two-sided embeddings*

$$\Delta\subset A\subset A_0,\ \ \Delta\subset A\subset A_1,\ \ A_0\subset A\subset\Sigma,\ \ A_1\subset A\subset\Sigma,$$

holds, then for the triple $\{A,A_0,A_1\}$ equalities (2) *and* (3) *take place for all θ and q; $0<\theta<1$, $1\le q\le\infty$.*

Proof. Clearly, if any of the embeddings is fulfilled, then the space A will be intermediate with respect to the pair $\{A_0, A_1\}$. Moreover, the embeddings $A \subset A_j$ or $A_j \subset A$ $(j = 0, 1)$ ensure the fulfilment of equalities (5) and (6) (see Theorem 1).

The statement of the theorem follows now from Theorem 5.

Thus the theorem is proved. □

Remark 3. In [8], H. Triebel presented an example of the triple $\{A, A_0, A_1\}$ for which equality (2) does not hold. However, the space A is that example is not intermediate with respect to the pair $\{A_0, A_1\}$. This fact explains partly our interest to the case considered above.

It would be interesting to have (if possible) an example of the triple $\{A, A_0, A_1\}$, such that A is intermediate with respect to the pair $\{A_0, A_1\}$ and equality (2) does not hold.

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(Received 10.10.2001)

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