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SYMMETRIC DIFFERENTIABILITY OF FUNCTIONS OF TWO VARIABLES

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Abstract. We introduce the notions of symmetric differentiability and symmetric partial derivatives in the strong and in the angular sense. These notions are connected.

Introduction

A symmetric derivative has been considered by M.V. Ostrogradsky ([1], pp. 166–167) and by Du-Buiss-Reymond ([2], pp. 122–123). It is widely used now. According to the Lebesgue method ([3], pp.321), the symmetric derivative is connected with the radial limit of the derivative of the Poisson integral ([3], 99–100) and with the summation of trigonometric series. Moreover, if the function has finite right and left derivatives at the given point, then it has a symmetric derivative at the same point which is equal to the half-sum of unilateral derivatives. The inverse proposition is invalid.

In the present paper we introduce the notions of a symmetric total differential, symmetric partial derivatives in the strong and in the angular sense, symmetric gradient in the strong sense and its connection with symmetric differentiability, symmetric gradient in the angular sense and its connection with symmetric differentiability.

In the sequel the function to be considered receives only finite values and is measurable.

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Symmetric Total Differential of Functions of Two Variables

As is known, the measurable function $f(x)$ possesses a symmetric derivative at the inner point x_0 of the interval (a,b) if there exists the limit

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$$

which sometimes is denoted by the symbol $f^{(\prime)}(x_0)$. It should be noted that the ratio $\frac{f(x_0+h)-f(x_0-h)}{2h}$ is an even function with respect to h. Therefore for definition of the symmetric derivative we write $h\to 0+$. This does not imply the restriction.

1. **Definition 1.** A function $f(x,y)$ is said to possess at the point (x_0,y_0) a symmetric partial derivative with respect to x (with respect to y) if there exists the limit

$$\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0-h,y_0)}{2h} \tag{1}$$

(correspondingly, $\lim\limits_{k\to 0}\frac{f(x_0,y_0+k)-f(x_0,y_0-k)}{2k}$ $(1')$), which is denoted by the symbol $f_x^{(\prime)}(x_0,y_0)$ (correspondingly, $f_y^{(\prime)}(x_0,y_0)$). Obviously, if there exists a partial derivative with respect to x $f'_x(x_0,y_0)$ (with respect to $f'_y(x_0,y_0)$), then there exists a symmetric partial derivative with respect to x $f_x^{(\prime)}(x_0,y_0)$ (correspondingly, with respect to $f_y^{(\prime)}(x_0,y_0)$), and the equality $f'_x(x_0,y_0)=f_x^{(\prime)}(x_0,y_0)$ $f_y^{(\prime)}$ (correspondingly, $[f'_y(x_0,y_0)=f_y^{(\prime)}(x_0,y_0)]$) is fulfilled.

Proposition 1. *If there exists a finite limit $f_x^{(\prime)}(x_0,y_0)$ $(f_y^{(\prime)}(x_0,y_0))$, then $f(x,y)$ is symmetrically continuous with respect to x (correspondingly, with respect to y) at the point (x_0,y_0). Moreover, there exists everywhere continuous, and hence everywhere separately symmetrically continuous function, not possessing symmetric derivatives.*

Proof. Indeed, if the limit (1) is finite, then the relation $f(x_0+h,y_0)-f(x_0-h,y_0)=2hf_x^{(\prime)}(x_0,y_0)+\varepsilon h\to 0$ as $h\to 0$ is obvious.

As an example, we take a continuous function $\varphi(x,y)=\sqrt[3]{x}+\sqrt[3]{y}$, which does not possess symmetric derivatives at the point $(0,0)$. The function $\varphi(x,y)$ is continuous at every point, but $\varphi_x^{(\prime)}(0,0)$ $[\varphi_y^{(\prime)}(0,0)]$ does not exist. □

2. We now introduce the basic definitions.

Defintion 2. A function $f(x,y)$ of two variables (x_0,y_0) is said to be symmetrically totally differentiable (or simply, symmetrically differentiable) at the point (x_0, y_0), if there exist finite numbers $A = A(x_0, y_0)$ and $B = B(x_0, y_0)$ such that the equality

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0-h, y_0-k) - 2Ah - 2Bk}{|h| + |k|} = 0 \quad (2)$$

is fulfilled. In this case the value $Ah + Bk$ is called a symmetric total differential of the function $f(x, y)$ at the point (x_0, y_0). We denote it by $d^{\text{sym}} f(x_0, y_0; h, k)$, or simply, by $d^{\text{sym}} f(x_0, y_0)$.

The following theorem is valid.

Theorem 1. *If the function $f(x, y)$ is symmetrically differentiable at the point (x_0, y_0), then there exist symmetric partial derivatives at the same point and $f_x^{(\prime)}(x_0, y_0) = A$, $f_y^{(\prime)}(x_0, y_0) = B$.*

Proof. If in equality (2) we take $k = 0$, then

$$\lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0-h, y_0) - 2Ah}{|h|} = 0, \quad \text{so}$$
$$\lim_{h\to 0} \frac{2h}{|h|} \frac{f(x_0+h, y_0) - f(x_0-h, y_0) - 2Ah}{2h} = 0.$$

From this we have $\lim\limits_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0-h,y_0)}{2h} = A$, which implies the equality $f_x^{(\prime)}(x_0, y_0) = A$.

Analogously, $f_y^{(\prime)}(x_0, y_0) = B$. □

Remark 1. It is known that the analogue of Theorem 1 does not take place for an approximate differentiability ([4], p. 300).

Proposition 2. *There is a function possessing symmetric partial derivatives at the point* $(0, 0)$ *which is not symmetrically differentiable at the same point. Such is the function*

$$U(x, y) = \begin{cases} \frac{x|y|}{x^2+y^2}, & \text{if } \ (x, y) \neq (0, 0) \\ 0, & \text{if } \ (x, y) = (0, 0). \end{cases} \quad (3)$$

Thus we have $u_x^{(\prime)}(0, 0) = 0 = u_y^{(\prime)}(0, 0)$, and the ratio

$$\frac{u(h, k) - u(-h, -k) - 2h \cdot 0 - 2k \cdot 0}{|h| + |k|} = \frac{2h|k|}{(h^2 + k^2)(|h| + |k|)}$$

is equal to $\frac{2k^2}{2k^2 \cdot 2k} = \frac{1}{2k}$, if $h = k > 0$, and tends to $+\infty$ as $k \to 0$. Hence $u(x, y)$ is not symmetrically differentiable at the point $(0, 0)$.

Remark 2. The existence of finite approximate partial derivatives at any point causes approximate differentiability at the same point ([4], p.300).

The following definition is well-known.

Definition. A function of two real variables is said to be differentiable (in the Stolz sense) at the point (x_0, y_0), if there exist numbers $A(x_0, y_0)$ and $B(x_0, y_0)$ such that to each $\varepsilon > 0$ there corresponds $\delta = \delta(\varepsilon, x_0, y_0) > 0$ such that $|h| < \delta$ and $|k| < \delta$. We have the relation

$$|f(x_0 + h, y_0 + k) - f(x_0, y_0) - hA(x_0, y_0) - kB(x_0, y_0)| < \varepsilon(|h| + |k|).$$

The theorem below is valid.

Theorem 2. *If the function $f(x, y)$ is Stolz differentiable at the point (x_0, y_0), then $f(x_0, y)$ is symmetrically differentiable at the same point and not vice versa.*

Proof. We have the equality

$$\frac{f(x_0 + h, y_0 + k)f(x_0 - h, y_0 - k) - 2hf'_x(x_0, y_0) - 2kf'_y(x_0, y_0)}{|h| + |k|} =$$
$$= \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - hf'_x(x_0, y_0) - kf'_y(x_0, y_0)}{|h| + |k|} -$$
$$- \frac{f(x_0 - h, y_0 - k) - f(x_0, y_0) - (-h)f'_x(x_0, y_0) - (-k)f'_y(x_0, y_0)}{|-h| + |-k|}.$$

If the function $f(x, y)$ is Stolz differentiable at the point (x_0, y_0), the each of the fractions tends to 0 as $(h, k) \to (0, 0)$.

The inverse proposition is invalid. Consider the function

$$\chi(x, y) = \sqrt{|xy|}. \tag{4}$$

We have $\chi'_x(0, 0) = \chi'_y(0, 0) = \chi_x^{(\prime)}(0, 0) = \chi_y^{(\prime)}(0, 0) = 0$. Thus

$$\frac{\chi(h, k) - \chi(-h, -k) - 2h \cdot 0 - 2k \cdot 0}{|h| + |k|} = 0,$$

and $\chi(x, y)$ is symmetrically differentiable at the point $(0, 0)$. Let us prove that $\chi(x, y)$ is not differentiable at the point $(0, 0)$. Indeed,

$$\frac{\chi(h, k) - \chi(0, 0)}{|h| + |k|} = \frac{\sqrt{|hk|}}{|h| + |k|}.$$

For $h = k$, this function is equal to $\frac{|k|}{2|k|} = \frac{1}{2} \nrightarrow 0$ as $k \to 0$. □

Proposition 3. *If the function $f(x, y)$ is symmetrically differentiable at the point (x_0, y_0), then $f(x, y)$ is symmetrically continuous at the same point, and not vice versa. There exists a symmetrically continuous function with symmetric partial derivatives at the given point which is not symmetrically differentiable at the same point.* (*For the notion of symmetric continuity see* [5]).

Indeed, since the function $f(x,y)$ is symmetrically differentiable at the point (x_0,y_0), there are finite numbers $A=A(x_0,y_0)$ and $B=B(x_0,y_0)$ such that equality (2) is fulfilled.

Thus

$$\begin{gathered} ||f(x_0+h,y_0+k)-f(x_0-h,y_0-k)| \le \\ \le |f(x_0+h,y_0+k)-f(x_0-h,y_0-k)-2hA-2kB| + \\ +|2hA+2kB| \to 0, \quad \text{as} \quad (h,k)\to(0,0). \end{gathered}$$

To prove the second part of our proposition, we consider the function

$$\psi(x,y)=\begin{cases} \frac{x|y|}{|x|+|y|}, & \text{if} \quad (x,y)\neq(0,0),\\ 0, & \text{if} \quad (x,y)=(0,0). \end{cases}$$

This function is symmetrically continuous at the point $(0,0)$, because $|\psi(h,k) - \psi(-h,-k)| = \frac{2|h||k|}{|h|+|k|} \to 0$ as $(h,k)\to(0,0)$.

At the same time, $\psi(x,y)$ is not symmetrically differentiable at the point $(0,0)$, although $\psi_x^{(\prime)}(0,0)=\psi_y^{(\prime)}(0,0)=0$.

Consider the ratio

$$\frac{\psi(h,k)-\psi(-h,-k)-2h\cdot 0-2k\cdot 0}{|h|+|k|} = \frac{2h|k|}{(|h|+|k|)^2},$$

which is equal to $\frac{2|h|^2}{(|h|+|k|)^2}=\frac{1}{2}$ for $h=k>0$. Consequently, $\psi(x,y)$ is not symmetrically differentiable at the point $(0,0)$.

Proposition 4. *There exists the function possessing symmetric partial derivatives but which is not symmetrically continuous at the point* $(0,0)$. *Such is the function* $u(x,y)$ *defined by equality* (3).

Indeed, $u_x^{(\prime)}(0,0)=\lim\limits_{h\to 0}\frac{u(h,0)-u(-h,0)}{2h}=0=u_y^{(\prime)}(0,0)$, but $u(x,y)$ is not symmetrically continuous at the point $(0,0)$ (see [6]).

3. Let us introduce the notion of a symmetric partial derivative in the strong sense.

Defintion 3. A function $f(x,y)$ is said to possess at the point (x_0,y_0) a symmetric strong partial derivative with respect to x (with respect to y, if there exists the limit

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h,y_0+k)-f(x_0-h,y_0+k)}{2h} \equiv f_{[1]}^{(\prime)}(x_0,y_0) \qquad (5)$$

(correspondingly,

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h,y_0+k)-f(x_0+h,y_0-k)}{2k} \equiv f_{[2]}^{(\prime)}(x_0,y_0)) \qquad (5')$$

Proposition 5. *If there exists* $f_{[1]}^{(\prime)}(x_0,y_0)$ *(or* $f_{[2]}^{(\prime)}(x_0,y_0)$*), then there exists* $f_x^{(\prime)}(x_0,y_0)$ *(correspondingly,* $f_y^{(\prime)}(x_0,y_0)$*), and not vice versa.*

Indeed, if in equality (5) (or in equality (5')) we mean the $k=0$ (correspondingly, $h=0$), then we get equality (1) (correspondingly, (1')).

Consider the function

$$W(x,y)=\begin{cases}\frac{xy}{x^2+y^2} & \text{as } (x,y)\neq(0,0)\\ 0, & \text{as } (x,y)=(0,0).\end{cases} \tag{6}$$

We have $W_x^{(\prime)}(0,0)=0=W_y^{(\prime)}(0,0)$, but

$$W_{[1]}^{(\prime)}(0,0)=\lim_{(h,k)\to(0,0)}\frac{W(h,k)-W(-h,k)}{2h}=$$
$$\lim_{(h,k)\to(0,0)}\frac{k}{h^2+k^2}=\begin{cases}0, & \text{as } k=0, h\to 0\\ \infty, & \text{as } k=h\to 0,\end{cases}$$

Thus $W_{[1]}^{(\prime)}(0,0)$ does not exist. The same is valid for y.

Theorem 3. *If there exist* $f_{[1]}^{(\prime)}(x_0,y_0)$ *and* $f_{[2]}^{(\prime)}(x_0,y_0)$*, then* $f(x,y)$ *is symmetrically differentiable at the point* (x_0,y_0) *and not vice versa.*

We have

$$\begin{aligned}f(x_0+h,y_0+k)-f(x_0-h,y_0-k)-2hf_{[1]}^{(\prime)}(x_0,y_0)-2kf_{[2]}^{(\prime)}(x_0,y_0)=\\ =[f(x_0+h,y_0+k)-f(x_0-h,y_0+k)-2hf_{[1]}^{(\prime)}(x_0,y_0)]+\\ +[f(x_0-h,y_0+k)-f(x_0-h,y_0-k)-2kf_{[1]}^{(\prime)}(x_0,y_0)]=\\ =[M]+[N].\end{aligned} \tag{7}$$

According to the term of our theorem, for every number $\varepsilon>0$ there exists the number $\delta_1=\delta_1(\varepsilon,x_0,y_0)>0$ such that the inequalities $|h|<\delta_1$ and $|k|<\delta_1$ imply

$$|M|<\varepsilon|h|.$$

Analogously, for the same number $\varepsilon>0$ there exists the number $\delta_2=\delta_2(\varepsilon,x_0,y_0)>0$, such that if $|h|<\delta_2$ and $|k|<\delta_2$, then the inequality $|N|<\varepsilon|k|$ is fulfilled. If $\delta\equiv\min(\delta_1,\delta_2)$, then owing to the above-given inequalities, the absolute value in the left-hand side of equality (7) is less than $\varepsilon(|h|+|k|)$, since $|h|<\delta$, $|k|<\delta$. Thus the function $f(x,y)$ is symmetrically differentiable at the point (x_0,y_0) and not vice versa.

Consider the function $W(x,y)$ defined by equality (6). As we have seen, $W_{[1]}^{(\prime)}(0,0)$ and $W_{[2]}^{(\prime)}(0,0)$ do not exist. Let us now prove that $W(x,y)$

is symmetrically differentiable at the point $(0,0)$. As is already known, $W_x^{(\prime)}(0,0) = 0 = W_y^{(\prime)}(0,0)$, and hence

$$\frac{W(h,k) - W(-h,-k) - 2\cdot 0 - 2k\cdot 0}{|h|+|k|} = 0.$$

Defintion 4. A function $f(x,y)$ is said to possess at the point (x_0,y_0) a symmetric strong gradient, if there exist symmetric strong partial derivatives $f_{[1]}^{(\prime)}(x_0,y_0)$ and $f_{[2]}^{(\prime)}(x_0,y_0)$.

A symmetric strong gradient of the function $f(x,y)$ at the point (x,y) will be denoted by $\operatorname{strgrad}^{(\prime)} f(x_0,y_0)$:

$$\operatorname{strgrad}^{(\prime)} f(x_0,y_0) = (f_{[1]}^{(\prime)}(x_0,y_0), f_{[2]}^{(\prime)}(x_0,y_0)). \tag{8}$$

Theorem 3 can now be reformulated as follows.

Theorem 4. *The finiteness of the symmetric strong gradient of the function $f(x,y)$ at the point (x_0,y_0) causes the symmetric differentiability of the function $f(x,y)$ at the same point and not vice versa.*

In conclusion of the first part of Theorem 4 there naturally arises the question as to whether the finiteness of the symmetric strong gradient of the function $f(x,y)$ at the point (x_0,y_0) can through parting cause differentiability (in the sense of Stolz) of the function $f(x,y)$ at the same point. The answer is negative. Moreover, the following theorem is valid.

Theorem 5. *The differentiability in the sense of Stolz of the function $f(x,y)$ at the point (x_0,y_0) and the existence of the symmetric strong gradient at the same point are non-comparable properties.*

Proof. Consider the function $\chi(x,y)$ defined by equality (4). As we have seen above, the function $\chi(x,y)$ is not differentiable at the point $(0,0)$, but we have $\chi_{[1]}^{(\prime)}(0,0) = 0 = \chi_{[2]}^{(\prime)}(0,0)$. So the function $\chi(x,y)$ has the symmetric strong gradient at the point $(0,0)$ and at the same time $\chi(x,y)$ it is not differentiable at the same point.

On the other hand, differentiability of the function at the given point does not cause the existence of symmetric strong partial derivatives at the same point. Such is the function

$$F(x,y) = x^{\frac{1}{3}}y^{\frac{4}{3}} + x^{\frac{4}{3}}y^{\frac{1}{3}}.$$

Indeed, $F_x'(0,0) = 0 = F_y'(0,0)$. Consider now the ratio

$$\left|\frac{F(h,k) - F(0,0) - 0\cdot h - 0\cdot k}{|h|+|k|}\right| < 2|h|^{\frac{1}{3}}|k|^{\frac{1}{3}} \to 0, \quad \text{as} \quad (h,k)\to(0,0).$$

Thus the function $F(x,y)$ is differentiable at the point $(0,0)$. Let us prove that $F^{(\prime)}_{[1]}(0,0)$ and $F^{(\prime)}_{[2]}(0,0)$ do not exist. Indeed,

$$F^{(\prime)}_{[1]}(0,0)=\lim_{(h,k)\to(0,0)}\frac{F(h,k)-F(-h,k)}{2h}=\lim_{(h,k)\to(0,0)}\frac{h^{\frac{1}{3}}k^{\frac{4}{3}}}{h}=$$
$$=\lim_{(h,k)\to(0,0)}\Big(\frac{k^2}{h}\Big)^{2/3}=\begin{cases}0, & \text{as} \quad h=k\to 0,\\ 1, & \text{as} \quad h=k^2\to 0.\end{cases}$$

The same will be valid for $F^{(\prime)}_{[2]}(0,0)$. $\square$

4. Introduce now the notion of a symmetric angular derivative for the functions of two variables.

Defintion 5. The function $f(x,y)$ is said to possess at the point (x_0,y_0) a symmetric angular partial derivative with respect to x (with respect to y), if there exists a c-independent limit

$$\lim_{\substack{(h,k)\to(0,0)\\ |h|\le c|k|}}\frac{f(x_0+h,y_0+k)-f(x_0-h,y_0+k)}{2h}\equiv f^{(\prime)}_{[1]}(x_0,y_0) \tag{9}$$

(correspondingly,

$$\lim_{\substack{(h,k)\to(0,0)\\ |h|\le c|k|}}\frac{f(x_0+h,y_0+k)-f(x_0+h,y_0-k)}{2k}\equiv f^{(\prime)}_{[2]}(x_0,y_0)) \tag{8'}$$

for every constant $c>0$.

Proposition 6. *If there exists $f^{(\prime)}_{[1]}(x_0,y_0)$ (or $f^{(\prime)}_{[2]}(x_0,y_0)$), then there exists $f^{(\prime)}_{\widehat{1}}(x_0,y_0)$ (correspondingly, $f^{(\prime)}_{\widehat{2}}(x_0,y_0)$ and they are equal. The converse is not true.*

Indeed, if there exists $f^{(\prime)}_{[1]}(x_0,y_0)$ (or $f^{(\prime)}_{[2]}(x_0,y_0)$), we have that there exists the limit (5) (correspondingly, the limit (5')) as $(h,k)\to(0,0)$ even, in particular, if $(h,k)\to(0,0)$ for $|k|\le c|h|$ (correspondingly, for $|h|\le c|k|$) for every constant $c>0$. Consequently, there exists $f^{(\prime)}_{\widehat{1}}(x_0,y_0)[f^{(\prime)}_{\widehat{2}}(x_0,y_0)]$.

The reverse proposition is not true. Consider now the function

$$\varphi(x,y)=\begin{cases}\frac{x^4}{y}+\frac{y^4}{x}, & \text{as} \quad x\cdot y\neq 0,\\ 0, & \text{as} \quad x\cdot y=0.\end{cases} \tag{10}$$

We have $\frac{\varphi(h,k)-\varphi(-h,k)}{2h}=\frac{k^4}{h^2}$.

Let us show that $\varphi^{(\prime)}_{[1]}(0,0)$ does not take place, but there exists $f^{(\prime)}_{\widehat{1}}(0,0)$. An absolute value ob the obtained fraction is less than $\frac{h^4}{h^2}c^4=c^4h^2\to 0$ as $h\to 0$ for any constant $c>0$ with $|k|\le c|h|$; hence $\varphi^{(\prime)}_{\widehat{1}}(0,0)=0$. But if we take $h = k^4$, then the same fraction will be equal to $\frac{2k^4}{k^8} = \frac{2}{k^4} \to \infty$ as $k \to 0$. Hence $\varphi_{[1]}^{(\prime)}(0,0)$ does not exist.

Defintion 6. The function $f(x,y)$ is said to possess at the point (x_0,y_0) a symmetric angular gradient, if there exist $f_{\widehat{1}}^{(\prime)}(x_0,y_0)$ and $f_{\widehat{2}}^{(\prime)}(x_0,y_0)$. The symmetric angular gradient of the function $f(x,y)$ at the point (x_0,y_0) will be denoted by $\operatorname{anggrad}^{(\prime)} f(x_0,y_0)$:

$$\operatorname{anggrad}^{(\prime)} f(x_0,y_0) = (f_{\widehat{1}}^{(\prime)}(x_0,y_0), f_{\widehat{2}}^{(\prime)}(x_0,y_0)).$$

Now we reformulate Proposition 6 in the form of

Proposition 7. *If the symmetric strong gradient of the function $f(x,y)$ is finite at the point (x_0,y_0), then the symmetric angular gradient of that function is also finite at the same point, and they are equal. The converse is not true.*

Proposition 8. *If there exists $f_{\widehat{1}}^{(\prime)}(x_0,y_0)$ (or $f_{\widehat{2}}^{(\prime)}(x_0,y_0)$), then there exists $f_x^{(\prime)}(x_0,y_0)$ (correspondingly, $f_y^{(\prime)}(x_0,y_0)$), and they are equal. The converse is not true.*

Indeed, if for every constant $c > 0$ there exists the c-independent limit (8) (or c-independent limit (8')), then this limit exists even if $k = 0$ (or $h = 0$). Thus there exists $f_x^{(\prime)}(x_0,y_0)$ (correspondingly, $f_y^{(\prime)}(x_0,y_0)$) and not vice versa.

Consider the function

$$F(x,y) = \begin{cases} \frac{xy}{|x|+|y|} & \text{as} \quad (x,y) \neq (0,0), \\ 0 & \text{as} \quad (x,y) = (0,0). \end{cases} \tag{11}$$

It is obvious that $F_x^{(\prime)}(0,0) = 0$. But if the function $\frac{F(h,k)-F(-h,k)}{2h} = \frac{k}{|h|+|k|}$ has the c-independent limit for every constant $c > 0$ for $|k| \leq c|h|$, then the same proposition holds even if $|k| = c|h|$. But in case $|k| = c|h|$ we have

$$\Big|\frac{k}{|h|+|k|}\Big| = \frac{c|h|}{c|h|+|h|} = \frac{c}{1+c}.$$

It is easy to see that the results depends on c. Thus $F_{\widehat{1}}^{(\prime)}(0,0)$ does not exist.

5. Consider now the relation between the symmetric differentiability of the function $f(x,y)$ and the symmetric derivability of the same function in the angular sense with respect to each of the variable. The following theorem is valid.

Theorem 6. *The symmetric differentiability and the finiteness of the symmetric angular gradient at the given point are non-comparable properties.*

Proof. As above, the function

$$W(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, & \text{as } (x,y)\neq(0,0)\\ 0, & \text{as } (x,y)=(0,0)\end{cases}$$

is symmetrically differentiable at the point $(0,0)$. Let us prove that $W(x,y)$ does not possess the symmetric angular derivative with respect to every variable at the point $(0,0)$.

Indeed,

$$\frac{W(h,k)-W(-h,k)}{2h}=\frac{k}{h^2+k^2}.$$

Take any constant $c>0$ and let $|k|=c|h|$. Then this fraction is equal to $\frac{ch}{h^2+c^2h^2}\to\infty$, as $h\to 0$. The same is valid for $W^{(\prime)}_{\widehat{2}}(0,0)$.

As an example, we take the function which has symmetric angular partial derivatives, but is not symmetrically differentiable at the point $(0,0)$. Such is the function $\varphi(x,y)$ defined by equality (9). As we have seen above, the function $\varphi(x,y)$ is symmetrically derivable in the angular sense at the point $(0,0)$ with respect to each variable. It is evident that $\varphi^{(\prime)}_x(0,0)=0=\varphi^{(\prime)}_y(0,0)$. Consider the ratio

$$\frac{\varphi(h,k)-\varphi(-h,-k)-2h\cdot 0-2k\cdot 0}{|h|+|k|}=\frac{2(h^5+k^5)}{hk(|h|+|k|)}.$$

In case $h=k^3$, the fraction will be equal to

$$\frac{2(k^{15}+k^5)}{k^3\cdot k(k^3+k)\to 2}, \quad \text{as} \quad k\to 0.$$

Thus the function $\varphi(x,y)$ is not symmetrically differentiable at the point $(0,0)$. $\square$

Remark 3. It is known that the differentiability in the sense of Stolz is equivalent to the finiteness of angular partial derivatives (see [6]).

Theorem 7. *If the function $f(x,y)$ is symmetrically differentiable at the point (x_0,y_0) and there exists finite $f^{(\prime)}_{[1]}(x_0,y_0)$ (or finite $f^{(\prime)}_{[2]}(x_0,y_0)$), then there exists $f^{(\prime)}_{\widehat{2}}(X_0,Y_0)$ (correspondingly, $f^{(\prime)}_{\widehat{1}}(x_0,y_0)$).*

Proof. We have

$$\begin{gathered}f(x_0+h_0,y_0+k)-f(x_0+h,y_0-k)-2kf^{(\prime)}_y(x_0,y_0)=\\ =[f(x_0+h,y_0+k)-f(x_0-h,y_0-k)-2hf^{(\prime)}_x(x_0,y_0)-2kf^{(\prime)}_y(x_0,y_0)]-\\ -[f(x_0+h,y_0-k)-f(x_0-h,y_0-k)-2hf^{(\prime)}_x(x_0,y_0)]=B-M.\end{gathered}$$

and $f(x,y)$ is symmetrically differentiable at the point (x_0,y_0); for any number $\delta_1=\delta_1(\varepsilon_1,x_0,y_0)$ there exists the number $|B|<2\varepsilon_1(|h|+|k|)$ such that $|h|<\delta_1$, if $|k|<\delta_1$, while for the same number $\varepsilon_1>0$ there exists the number $\delta = \delta_2(\varepsilon_1, x_0, y_0) > 0$ such that $|M| < 2\varepsilon_1|h|$ if $|h| < \delta_2$. In case $\delta \equiv \min(\delta_1, \delta_2)$ we will have

$$|f(x_0+h, y_0+k) - f(x_0+h, y_0-k) - 2kf_y^{(\prime)}(x_0, y_0)| <$$
$$< 2\varepsilon_1(|h| + |k| + 2\varepsilon_1|h|, \quad \text{for} \quad |h| < \delta|, \ |k| < \delta.$$

Thus

$$\Big|\frac{f(x+h, y_0+k) - f(x_0+h, y_0-k)}{2k} - f_y^{(\prime)}(x_0, y_0)\Big| <$$
$$< \varepsilon_1\Big(\frac{2|h|}{|k|} + 1\Big).$$

Taking any constant $c > 0$ and letting $(h,k) \to (0,0)$ for $|h| \le c|k|$, we obtain

$$|k| < \delta \quad \text{and} \quad |h| \le c|k|, \quad \text{where} \quad \varepsilon_1 = \frac{\varepsilon}{2c+1}.$$

This means that there exists finite $f_{\underset{2}{\sim}}^{(\prime)}(x_0, y_0)$ which is equal to $f_y^{(\prime)}(x_0, y_0)$. $\square$

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