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ON CONFORMAL MAPPING OF A UNIT DISK ONTO A FINITE DOMAIN

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Abstract. A method for approximate construction of a function mapping conformally a unit disk to a finite simply-connected domain is given. The mapping function is sought in the form of polynomial. The method is simple and has high accuracy. Examples are considered.

Let in the plane $z = x + iy$ be given a finite simply-connected domain D with a piece-wise smooth boundary S $(S = \bigcup\limits_{j=1}^{l} S_j)$, which has no multiple points and let G be a unit disk with boundary γ in the plane $\zeta = \xi + i\eta$. Further under $x = x^j(t),\ y = y^j(t)$ $(\alpha_j \le t \le \beta_j)$ we will mean parametric equations of the lines S_j.

It is known [1]-[3] that many methods, which give possibility to solve plane stationary boundary problems of mathematical physics for a certain simply-connected domain D with boundary S of given form are based on the idea of conformal mapping. At this it is assumed that the functon $z = \omega(\zeta)$, mapping conformally a unit disk G to the domain D is given and moreover, has a convenient form in certain sense. The application of such methods for solving practical problems was organically connected with development of methods for construction of conformally mapping function for initial domains.

The urgency of the subject is evident on the example of the N. I. Muskhelishvili method [3], based on Cauchy type integrals and allowing to get an efficient solution of the basic problems of plane theory of elasticity for the case when the elastic isotropic medium fills in the plane $z = x + iy$ a simply connected domain D bounded by simple closed contour S, if the

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rational function $z = \omega(\zeta)$ mapping conformally the disc $|\zeta| < 1$ to the domain D is known. Most simply the solution is being built in the case when the function $z = \omega(\zeta)$ is a polynomial. N. I. Muskhelishvili [3] and his followers showed that if the function $z = \omega(\zeta)$ is known,then for the domain D the following problems of plane theory of elasticity can be brought to the numerical result: the basic mixed problem, problem of contact with a rigid profile, a wide class of problems of bending thin plates, of calculation of stress in composite mediums, of concentration of stresses near holes, a number of problems of strained condition in anisotropic bodies etc.

Since indication of conformally mapping functions in explicit form is possible only for a rather narrow class of domains, approximate methods are used usually. Due to it the number of various approximate methods for construction of mapping functions arose, namely, so-called analytic, graphical-analytic, experimentally analytic, in which the approximate expression of the mapping function is constructed in the polynomial form — $z = \omega_n(\zeta)$. The neccessity of construction of conformally mapping functions in this form is stipulated by the fact that the boundary problem is easily solved in the case, when the mapping function is a polynomial.

The basic statements of the mentioned above methods for construction of conformally mapping functions $z = \omega(\zeta)$ can be found in monographs of L. V. Kantorovich and V. I. Krylov [2], A. G. Ugodchikov [4], P. F. Filchakov [5], K. Amano [6], in papers T. K. DeLillo [7]-[8] etc.

The realization of existing methods of construction of approximate expression of mapping functions in the polynomial form is connected with rather huge calculations even for contours of simple form and are effective usually for domains close to disk. The main difficulty in these methods is connected with definition of nodes z_j $(j = 1, 2, \ldots, n)$ on S, which are used for construction of the unknown polynomial $\omega_n(\zeta)$ and which are the images of equidistant points τ_j $(j = 1, 2, \ldots, n)$ on $|\zeta| = 1$ (it is proved further [4] that $\omega_n(\zeta) \to \omega(\zeta)$ for $n \to \infty$).

For approximate definition of nodes electrical modelling [4]-[5], graphical ways in method of P. V. Melentev and integral equations [6]-[7] are used mainly.

We should stress that the above mentioned methods for definition of nodes are usually realizable practically in a complicated way, which exposures, in particular, also at composing of the universal programmes of the problem. Besides, as many calculating experiments show, they don't have high accuracy always. In spite of such difficulties during last years usage of method of conformal mappings for solution of practical problems strengthens continuously. Consequently, in various scientific centers a big work is being done on construction of conformal mappings with help of modern computers.

Below we offer one scheme for approximate construction of the function conformally mapping a unit disk G to the given simply connected domain D. The offered method, realizing the definition of the unknown function in the polynomial form is quite simple for numerical realization and has accuracy which is practically sufficient for many problems.

Note that one of the essential moments in the offered here approach is the fact that by the corresponding scheme on the initial step the function $\zeta = f_N(z)$ is found, which realizes approximate conformal mapping of the domain D to the disk G and then with its help the unknown polynomial $z = \omega_n(\zeta)$ is constructed. This circumstance also appears to be convenient enough at solving practical problems by method of conformal mapping. Besides the fact that at this it is possible to compute the values of solution of the given problem at any point of the initial domain, we can also observe the error

$$\delta_{\max} = \max_{\tau\in\gamma} |\omega(\tau) - \omega_n(\tau)|,$$

and also get a certain idea about the dynamics of mapping the domain D to the disk G and backwards. When constructing approximately the functions $f_N(z)$ and $\omega_n(\zeta)$ we rely on the following rather useful theorem (W. F. Osgood) [1], [3]:

Theorem. *Let G be a finite or infinite simply connected domain on the plane ζ, bounded by simple closed contour γ, and let $\omega(\zeta)$ be a function, analytic in G and continuous up to the contour. Let further a point, defined by equality $z = \omega(\zeta)$, describe on the plane z (moving always in one and the same direction) some simple closed contour S, when ζ discribes a contour γ. Then the relation $z = \omega(\zeta)$ gives conformal mapping of the domain G to the domain D, enclosed inside S and conversely.*

It should be observed that this theorem is generalized also for the case of multiply connected domains [3].

On the basis of the formulated above theorem, if the approximate expression of the function, mapping the $|\zeta| < 1$ to the domain D, is sought in the polynomial form

$$z = \omega_n(\zeta) = \sum_{k=0}^{n} C_k \zeta^k,$$

practically it means the neccessity of finding such values of coefficients $C_k = C_k' + i\,C_k''$ $(k = 0, 1, \dots, n)$ for which the line S_n with parametric equation $z = \omega_n(e^{i\varphi})$ $(0 \le \varphi \le 2\pi)$ would not have, for the first, double (multiple) points (and in the case of boundary problems of theory of elasticity also cusps [3]) and, for the second, the deflection of the line S_n from the given boundary S of the domain D would be in accetable limits.

The function $z = \omega(\zeta)$ conformally mapping the unit disk G to the domain D is analytic in the disk G and continuous in $\overline{G}$, hence, it can be represented in this disk by its Tailor series:

$$\omega(\zeta) = \sum_{k=0}^{\infty} C_k \zeta^k. \tag{1}$$

Since the function $z = \omega(\zeta)$ is analytic in the disk G and continuous in $\overline{G}$, due to the Cauchy integral formula the coefficients of the series (1) are defined by formula [1]

$$C_k = \frac{\omega^{(k)}(0)}{k!} = \frac{1}{2\pi i} \int_{\gamma} \frac{\omega(\tau)\, d\tau}{\tau^{k+1}}, \quad (k = 0, 1, \dots). \tag{2}$$

Theoretically [1] any closed simple contour γ^* lying in the disk G and containing the origin can be taken in the role of γ. But in numerical experiment certain calculational inconvenience stipulated by the fact that $|\tau| < 1$ for $\tau \in \gamma^*$ must be taken into account.

By virtue of the principal of correspondence of boundaries at conformal mapping the series (1) is convergent on γ and in G the series (1) converges uniformly. Evidently, via the convergence of the series (1), $C_k \to 0$ when $k \to \infty$. If now instead of the series (1) we take its piece consisting of $n+1$ terms

$$\omega_n(\zeta) = \sum_{k=0}^{n} C_k \zeta^k,$$

the relation $z = \omega_n(e^{i\varphi})$ will map the circle $|\zeta| = 1$ generlly not to the contour S, but to the some contour S_n. Since for the points $\zeta \in \overline{G}$, $\omega_n(\zeta) \to \omega(\zeta)$ for $n \to \infty$, hence $S_n \to S$ for $n \to \infty$.

Thus, if we take n sufficientely large, the deflection of the contour S_n from the given contour S can be made arbitraryly small. After finding the coefficients C_k $(k = 0, 1, \dots, n)$ the contours S_n must be drawn by graphical constructor or sufficiently big number of points of this contour must be drawn and fulfilment of the indicated conditions must be controlled. If these conditions are not fulfilled, the number n, i.e. the polynomial order, must be increased.

Let $\zeta = f(z)$ be a function conformally mapping the domain D to the unit disk G under normalization conditions

$$f(z_0) = \zeta_0, \qquad f(z_1) = \zeta_1, \tag{3}$$

where $z_0, z_1, \zeta_0, \zeta_1$ are fixed points and $z_0 \in D, z_1 \in S, \zeta_0 \in G, \zeta_1 \in \gamma$. Without loss of generality we may mean that $\zeta_0 = 0$ and $\zeta_1 = 1$. According to Riemann theorem the function $f(z)$ exists under conditions (3), is determined uniquely and on the basic of conformality of the mapping has inverse function $z = \omega(\zeta)$, which is determined uniquely, maps conformally the disk G to the domain D under conditions $\omega(\zeta_0) = z_0$, $\omega(\zeta_1) = z_1$. Otherwise, the functions f and ω are inverse.

As to the question of construction of the function $\zeta = f(z)$, more precisely, construction of its approximate expression $f_N(z)$, it is reasonable to realize it using the known (at this simple enough and convenient) scheme, which is given in [9]-[11]. In these papers the function $f_N(z)$ is sought in the following form:

$$f_N(z) = (z - z_0)\exp\{u_N(z) + iv_N(z)\}, \quad z \in \bar{D},$$

where $u_N(z)$ and $v_N(z)$ ara conjugate hormonic functions in $\bar{D}$. The function $u_N(z)$ is the approximate solution of to the Dirichlet problem for Laplace's equation:

$$\Delta u(z) = 0, \quad z \in D,$$
$$u(z) = -\ln|z - z_0|, \quad z \in S$$

and Kupradze-Aleksidze's method (method of fundamental solutions -[16]) is applied for its solution. Assuming that the function $f_N(z)$ is constructed, granting this, further we will deal with approximate construction of the function $z = \omega_n(\zeta)$.

In formula (2) under the integral sign there are boundary values of the functions $z = \omega(\zeta)$ and $\zeta = f(z)$, hence at numerical integrating relating the corresponding integral some circumstances, important from the numerical point of view should be taken into account. Depending on the geometry of the contour S and normalization conditions (3), it is not excluded that the images τ_j $(j = 1, 2, \dots, n)$ of the points z_j $(j = 1, 2, \dots, n)$ which are equidistant with respect to the parameter t on boundaries S will be situated not even non-uniformly, but may be strongly overcrowded at certain points. Therefore in order to prevent the stopping of the computer due to possible devision by the machine zeros and to achieve high accuracy using the proper quadrature formulas it is reasonable in the integral (2) to pass from the contour γ to the contour S. Consequently, after construction of the function $\zeta = f_N(z)$ we will have a scheme

$$C_k^N = \frac{1}{2\pi i}\int\limits_S \frac{z \cdot f'_N(z)\,dz}{[f_N(z)]^{k+1}} = \frac{1}{2\pi i}\sum_{j=1}^{l}\int\limits_{\alpha_j}^{\beta_j} \frac{z^j(t)\,f'_N(z^j(t))\,(z^j(t))'\,dt}{[f_N(z^j(t))]^{k+1}} \tag{4}$$

for approximate values of coefficients C_k, where $z = z^j(t)$ $(\alpha_j \le t \le \beta_j)$ are corresponding parameter equations of smooth curves S_j. The possibility of such passage is evident by virtue of analiticity of the function $f_N(z)$ in $\overline{D}$ (see [9]-[10]).

It is easy to show that if the function $f_N(z)$ is constructed within given ε $(\varepsilon > 0)$, i.e. $|f_N(z) - f(z)| < \varepsilon$ for $\forall z \in S$, then $\left|C_k - C_k^N\right| = O(\varepsilon)$ and $C_k^N \to C_k$ when $N \to \infty$, since the sequence of the functions $f_N(z)$ converges uniformly on $\overline{D}$ to the function $f(z)$. Thus using the formula (4) coefficients of the series (1) are determined within ε.

It should be noted that we can increase the accuracy of determination of coefficient C_k^N, if in the integral (4) we take $\dfrac{f_N(z)}{|f_N(z)|}$ as $f_N(z)$. Indeed, if the function $f_N(z)$ is constructed within sufficiently small ε, by virtue of the inequality

$$\left|\frac{f_N(z)}{|f_N(z)|} - f(z)\right| \le |f_N(z) - f(z)|\,, \quad z \in S,$$

accuracy of determination of C_k^N increases. Practically the drift of the points $f_N(z)$ $(z \in S)$ to the circle γ along normal to the contour γ is being done. Thus, finally for approximate determination of the coefficients C_k^N we will have expression

$$C_k^N = \frac{1}{2\pi i}\sum_{j=1}^{l}\int\limits_{\alpha_j}^{\beta_j}\frac{f_N'(z^j(t))\cdot z^j(t)\cdot (z^j(t))'}{\exp\left[\arg f_N(z^j(t))\right]^{k+1}}\,dt. \tag{5}$$

instead of (4).

Numerical examples of construction of conformally mapping functions.

For comparison with known methods we will first of all start from the examples considered by L. V. Kantorovich, P. V. Melentev, P. F. Filchakov [2], [5]. The carried out numerical experiments showed that higher accuracy of conformal mapping is achieved by method given by us than by other ones.

In all considered below examples the coefficients C_k^N were found from (5) with help of Simpson's quadrature formula, which, as known, has exactness up to $O(h^4)$ with respect to h. In our case the fatal error of calculation of the coefficients C_k^N is represented in the form $R = \sum\limits_{j=1}^{l} R_j$ where $R_j = O(h_j^4)$ $\left(h_j = \dfrac{\beta_j - \alpha_j}{M_j}\right)$ is an error on the interval $[\alpha_j, \beta_j]$, $M_j = 2m_j$ is a number of devision (m_j is natural number). Finally, the total error has accuracy rate $R = O(h^4)$, where $h = \max\limits_{1\le j\le l} |h_j|$.

In the tables below k is the number of coefficient C_k^N, n is an order of polynomial $z = \omega_n(\zeta)$, m is a number of points of quadrature formula, $\varepsilon_{\max}$ and $\delta_{\max}$ are a posteriori estimates of error construction of functions $\zeta = f_N(z)$ and $z = \omega_n(\zeta)$ respectively:

$$\varepsilon_{\max} = \max_{z\in S} \big\||f_N(z)| - |f(z)|\big\| \approx \max \big|\,|f_N(z_j)| - 1\big|,$$
$$\delta_{\max} = \max_{\tau\in\gamma} |\omega(\tau) - \omega_n(\tau)| \approx \max |z_j - \omega_n(\tau_j)|,$$

where the points z_j $(j = 1, 2, \dots, M)$ are situated uniformly with respect to the parameter t on the boundary S and $\tau_j = f_N(z_j)$. Evidently, when the number M is large enough, the values $\varepsilon_{\max}$ and $\delta_{\max}$ practically represent the deflections of contours $\zeta = f_N(z)$ $(z \in S)$ from γ and S_n $(z = \omega_n(\tau))$ from S respectively. In numerical experiments we took $M = 10^5$ and the calculations were carried out with double precision.

For illustration we will give the results of calculations of the following two examples.

Example 1. We map the unit disk $|\zeta| < 1$ to the interior of ellipse $x = a\cos t$, $y = b\sin t$ $(0 \leq t \leq 2\pi)$ with semiaxis $a = 473.8\,\mathrm{mm}$ and $b = 330.2\,\mathrm{mm}$ so that the center turns into center and the real axis — into real axis, i.e. under normalization conditions $\omega(0) = 0$ and $\omega(1) = a$.

This example was considered by P. F. Filchakov (see [5], p. 467). In order to construct the function $f_N(z)$, in the role of auxilary contour (see [9]-[10]) was taken an ellipse with semiaxis $a^* = 2a$ and $b^* = 2b$, and the number of collocation points $N = 100$.

It is easy to show that by the vertue of Shwarz symmetry principal [1], [5] since the axis x is the symmetry axis of ellipse, all $C_k = C_k' + iC_k''$ $(k = 1, 2 \ldots)$ will be real numbers, i.e. $C_k'' = 0$. Similarly, since the axis y is also the symmetry axis, all C_k with even indexes must be equal to zero: $C_{2k} = 0$ $(k = 1, 2, \ldots)$. On the basis of normalization we have $C_0 = 0$. The indicated circumstance holds also in numerical experiments. Thus, series (1) in the given case has the form $z = \sum\limits_{k=0}^{\infty} C_{2k+1}\zeta^{2k+1}$. In tables, in order to be short, the corresponding values are written with exponent E, i.e. E plays the role of base 10.

Table 1

$n = 100,\ m = 1000,\ \varepsilon_{\max} = 0.1E-13,\ \delta_{\max} = 0.23E-09$			
k	C_k^N	k	C_k^N
1	$0.37750536249143E+03$	51	$0.19075728331783E-04$
3	$0.61599832659979E+02$	53	$0.11400132833796E-04$
5	$0.19802952234511E+02$	55	$0.68271760996726E-05$
7	$0.79084344633363E+01$	57	$0.40964819988685E-05$
9	$0.35259894741960E+01$	59	$0.24624226723281E-05$
11	$0.16812139978011E+01$	61	$0.14826688435666E-05$
13	$0.83878613640720E+00$	63	$0.89414780559177E-06$
15	$0.43240220147562E+00$	65	$0.54002742579447E-06$
17	$0.22849219208538E+00$	67	$0.32660645628501E-06$
19	$0.12310398500485E+00$	69	$0.19778841507521E-06$
21	$0.67366858599086E-01$	71	$0.11992610778445E-06$
23	$0.37341356285524E-01$	73	$0.72800362556222E-07$
25	$0.20921613379774E-01$	75	$0.44241495381622E-07$
27	$0.11829303122945E-01$	77	$0.26913812764633E-07$
29	$0.67410509708266E-02$	79	$0.16389426900218E-07$
31	$0.38677210075781E-02$	81	$0.99896617966048E-08$
33	$0.22324335339712E-02$	83	$0.60939723147158E-08$
35	$0.12953815334149E-02$	85	$0.37207605288359E-08$
37	$0.75519967647549E-03$	87	$0.22733371071126E-08$
39	$0.44213919179278E-03$	89	$0.13902088971918E-08$
41	$0.25984171609399E-03$	91	$0.85068522385730E-09$
43	$0.15323442278565E-03$	93	$0.52075895534169E-09$
45	$0.90649875888337E-04$	95	$0.31869397200050E-09$
47	$0.53780470552245E-04$	97	$0.19598519979051E-09$
49	$0.31990867474614E-04$	99	$0.11946028299311E-09$

If we take a piece of the series with 200 first ferms, we will have $\delta_{\max} = 0.6E-10$.

Example 2. The given by us method of finding of mapping function may be applied also in the case when a strong overcrowding has place at mapping. As a corresponding example we will solve a problem of mapping a disk $|\zeta| < 1$ to the interior of ellipse with semiaxis $a = 5, b = 1$ and normalization conditions $\omega(0) = 0, \omega(1) = 5$.

In the given case in order to construct the function $f_N(z)$ in the role of auxilary contour was taken an ellipse with semiaxis $a^* = 5.5$ and $b^* = 1.5$, and $N = 400$. For such parameters $\varepsilon_{\max} = 0.5E-14$. The calculations show that the strong overcrowding has place in the neighborhood of points (1;0) and (-1;0). For example: $\arg f_N(z_2) \approx 0.9E-05$, $\arg f_N(z_3) \approx 0.9E-04$, $\arg f_N(z_4) \approx 0.9E-03$, where $z_2, z_3, z_4 \in S$ and $z_2 \approx 4.99+i\cdot 0.056$, $z_3 \approx$

$4.78 + i \cdot 0.28$, $z_4 \approx 4.17 + i \cdot 0.56$.

In existing methods [2], [4], [5], [7] in order to construct the function $z = \omega_n(\zeta)$, as known,the points z_j $(j = 1, 2, \ldots, n)$, images of points τ_j $(j = 1, 2, \ldots, n)$, situaded uniformly on the contour γ must be found and afterwards the coefficients C_k $(k = 1, 2, \ldots, n)$ of the polynomial $z = \omega_n(\zeta)$ must be determined. However, in order for one of the points z_j to belong to the arc z_1z_4, n must be greater than $2\pi E + 03$ $(z_1 = a = 5)$ to the arc z_1z_3 — $n > 2\pi E + 04$, to the arc z_1z_2 — $n > 2\pi E + 05$. Consequently, if we take into account the difficulties of indicated methods, it is evident that for such big n their numerical realization is practically difficult, when with the help of given here method we can without difficulties calculate any number of coefficients C_k^N.

It should be noted that since in this example at mapping the disk $|\zeta| < 1$ to the interior of ellipse there is a strong dispersion in the neighborhood of the points (5;0) and (-5;0) and therefore for approximation of the function $z = \omega(\zeta)$ with high accuracy a big number of terms in the series (1) must be taken. In other words, this is expressed in the fact that the coefficients C_k^N for $k \to \infty$ tend to zero rather slowly, and this was showed by experiments as well. The indicated fact does not hold in the example 1, since there is not strong crowding.

In the Table 2 the dynamics of deflection $\delta_{\max}$ (at points (5;0), (-5;0)) and coefficients C_k^N for various number of the first $n+1$ terms of the series (1) are given. For the big k in (5) under the integral sign we have quickly oscillating function and therefore it is naturally that for high accuracy numerical integration of such functions a big number of knots is needed; in our case we took $m = 10^5$.

Table 2

n	$\delta_{\max}$	k	C_k^N
100	1.59	1	$0.126E + 01$
500	0.9	499	$0.147E - 02$
1000	0.68	999	$0.634E - 03$
1500	0.56	1499	$0.383E - 03$
2000	0.48	1999	$0.265E - 03$
5000	0.28	4999	$0.781E - 04$
10000	0.15	9999	$0.283E - 04$

Finally, the conclusion can be done that in order to improve the rate of convergence of coefficients C_k^N $(k \to \infty)$ to zero we must map (if possible) the interior of ellipse to the disk G and conversely so that the phenomena of strong crowding should be removed.

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