Proceedings of A.Razmadze
Mathematical Institute,
Vol. **122**, 2000

STABILITY OF ORTHOTROPIC SHELLS OF REVOLUTION, CLOSE TO CYLINDRICAL ONES, WITH ELASTIC FILLER UNDER TORSION

S. KUKUDZHANOV

Abstract. The stability of orthotropic shells of revolution, close by their form to cylindrical ones, with elastic filler under the action of torque moments applied to the shell ends, is considered.

The present paper deals with the investigation of the problem of stability of orthotropic shells of revolution, which by their form are close to cylindrical ones, with an elastic filler under the action of torque moments M applied to the shell ends. The filler is considered as a solid elastic isotropic body getting into contact over its outer surface with a shell. The filler is assumed to be of slipping type, i.e., when tangential stresses on the contact surface are negligibly small and there takes place only radial interaction. We consider shells of positive as well as of negative Gaussian curvature. For shells of revolution whose form of a mid-surface generatrix is described by a parabolic function we obtained in a dimensionless form the formulas and universal curves of dependence of critical shearing stress on the elastic orthotropic parameters, on dimensionless rigidity of an elastic filler and on dimensionless deflection from the cylindrical form. Deviations of order of thickness are considered. For the shells of positive and negative Gaussian curvature we obtained different degrees of dependence of the elastic parameters on the critical load. It is found that critical stresses for shells of positive and negative Gaussian curvature under small values of rigidity of the elastic filler may essentially differ, where as this difference decreases as rigidity of the elastic filler increases. The conditions under which one can neglect the influence of the boundary conditions on the critical load, are given.

1991 *Mathematics Subject Classification.* 35J60.

Key words and phrases. The least eigen-value, Gaussian curvature, wave formation parameter, orthotropic shell, critical stress, stability.

1. We consider the shell whose mid-surface is formed by the rotation of some sufficiently smooth curve around the z-axis of the rectangular system of coordinates x, y, z with the origin in the mid-section of the axis of rotation (Fig. 1). Note that radius of the mid-surface cross section of the shell is defined by the equality

$$R = r_0 + \delta_0 F(\xi), \quad \xi = z/r_0 \tag{1.1}$$

where $F(\xi)$ is the positive function given on the interval $(-l/r_0, l/r_0)$, such that $F(\pm l/r_0) = 0$, $\max F(\xi) = 1$, $|F'(\xi)| \lesssim 1$; $L = 2l$ is the length of the shell; r_0 is radius of the face cross section; δ_0 is the small parameter characterizing maximal deflection from the cylindrical form. For $\delta_0 > 0$, the mid-surface generatrix is convex and for $\delta_0 < 0$ it is concave. We consider both the long and the mid-length shells (see [2]), and it is assumed that

$$(\delta_0/r_0)^2, \quad (\delta_0/l)^2 \ll 1. \tag{1.2}$$

The mid-surface equation in its parametric representation is of the form

$$x = R(\xi)\cos\varphi, \quad y = R(\xi)\sin\varphi, \quad z = \xi r_0,$$

where φ is the angular coordinate. From this equation we find that the coefficients of the mid-surface first quadratic form are

$$A^2 = r_0^2 + \delta_0^2 (F')^2 \quad \text{and} \quad B^2 = R(\xi)^2.$$

On the basis of the above assumptions we can neglect in the expression A^2 the second term. Hence

$$A \approx r_0, \quad B = R(\xi). \tag{1.3}$$

Principal radii of curvature have the form:

$$K_1 = 1/R_1 = -R''/r_0^2, \quad K_2 = 1/R_2 = 1/R(\xi). \tag{1.4}$$

For the shells under consideration as well as for cylindrical ones the more typical are the forms of the stability loss with a large exponent of variability in the circumferential rather than in the axial direction. Hence the relation

$$\frac{\partial^2 U}{\partial \varphi^2} \gg \frac{\partial^2 U}{\partial \xi^2} \quad (U = u, v, w) \tag{1.5}$$

is valid, where u, v and w are respectively the axial, circumferential and radial components of additional displacement which characterizes the form of stability loss. Therefore, according to Novozhilov's statement (see [8], p. 236), one can take as principal equations of stability the corresponding equations of Vlasov's semi-momentless theory. As a result, a system of equations

of stability of orthotropic shells supplemented with a term characterizing interaction with a filler, takes the form

$$\frac{\partial^4 u}{\partial\varphi^4} = \nu_1\frac{\partial^3 w}{\partial\xi^3} - \frac{\partial^3 w}{\partial\xi\partial\varphi^2}, \tag{1.6$_1$}$$

$$\frac{\partial^4 v}{\partial\varphi^4} = \frac{\partial^3 w}{\partial\varphi^3} + \Big(\frac{E_1}{G} - \nu_1\Big)\frac{\partial^3 w}{\partial\xi^2\partial\varphi} \tag{1.6$_2$}$$

$$\varepsilon_0\Big(\frac{\partial^2 w}{\partial\varphi^8} + 2\frac{\partial^6 w}{\partial\varphi^6} + \frac{\partial^4 w}{\partial\varphi^4}\Big) + \frac{E_1}{E_2}\Big\{\frac{\partial^4 w}{\partial\xi^4} + \frac{\partial^2}{\partial\xi^2}\Big[\Big(-\frac{R''}{r_0}\Big)\frac{\partial^2 w}{\partial\varphi^2}\Big] +$$
$$+\Big(-\frac{R''}{r_0}\Big)\frac{\partial^4 w}{\partial\xi^2\partial\varphi^2} + \Big(\frac{R''}{r_0}\Big)^2\frac{\partial^4 w}{\partial\varphi^4}\Big\} -$$
$$-\frac{T_1^0}{E_2h}\Big(\frac{\partial^6 w}{\partial\xi^2\partial\varphi^4} + \frac{\partial^4 w}{\partial\xi^2\partial\varphi^2}\Big) - \frac{T_2^0}{E_2h}\Big(\frac{\partial^6 w}{\partial\varphi^6} + \frac{\partial^4 w}{\partial\varphi^4}\Big) - \frac{2S^0}{E_2h}\Big(\frac{\partial^6 w}{\partial\xi\partial\varphi^5} + \frac{\partial^4 w}{\partial\xi\partial\varphi^3}\Big) +$$
$$+\Big(\frac{\partial^4}{\partial\varphi^4} + \frac{\partial^2}{\partial\varphi^2}\Big)\frac{q_r r_0}{E_2h} = 0, \quad \varepsilon_0 = \frac{h^2}{12r_0^2(1-\nu_1\nu_2)} \tag{1.6$_3$}$$

where T_1^0 and T_2^0 are, respectively, the axial and circumferential normal stresses; S^0 is a tangential stress of the initial pre-critical state; q_r is a radial component of filler's reaction; E_1, E_2, ν_1 and ν_2 are, respectively, module of elasticity and Poisson coefficients in the axial and circumferential directions ($E_1\nu_2 = E_2\nu_1$); G is the shear modulus; h is the shell thickness. Taking into account inequalites (1.2) and (1.5), the above estimates allow one to neglect in equations (1.6$_1$), (1.6$_2$)the terms with variable coefficients. The pre-critical state is assumed to be momentless. On the basis of the corresponding solution and owing to inequality (1.2), it is not difficult to show that

$$T_1 = T_2 = 0, \quad S^0 \approx M/2\pi r_0. \tag{1.7}$$

2. Consider a shell whose mid-surface generatrix is defined by the parabolic function

$$F(\xi) = 1 - \zeta^2(r_0/l)^2. \tag{2.1}$$

In this case the resolving equation (1.6) takes the form

$$\varepsilon_0\Big(\frac{\partial^8 w}{\partial\varphi^8} + 2\frac{\partial^6 w}{\partial\varphi^6} + \frac{\partial^4 w}{\partial\varphi^4}\Big) + \frac{E_1}{E_2}\Big(\frac{\partial^4 w}{\partial\xi^4} + 4\delta\frac{\partial^4 w}{\partial\xi^2\partial\varphi^2} + 4\delta^2\frac{\partial^4 w}{\partial\varphi^4}\Big) -$$
$$-\frac{2S^0}{E_2h}\Big(\frac{\partial^6 w}{\partial\xi\partial\varphi^5} + \frac{\partial^4 w}{\partial\xi\partial\varphi^3}\Big) + \Big(\frac{\partial^4}{\partial\varphi^6} + \frac{\partial^2}{\partial\varphi^4}\Big)\frac{q_r r_0}{E_2h} = 0, \quad \delta = \frac{\delta_0 r_0}{l^2}. \tag{2.2}$$

The additional term of that equation, in comparison with the equation presented by V. Darevsky ([3], p. 236) for isotropic case, is the sixth term which, by virtue of inequality (1.5), is of the same order as the fifth term of the equation.

First we consider long shells. Taking into account the fact that the stress-strained state of sufficiently long solid elastic bodies of revolution, close by

their form to cylindrical ones, is close to that of the corresponding cylindrical body, we consider the filler as a solid elastic cylinder.

The equations of equilibrium of the filler in displacements have the form [6]:

$$\frac{1}{1-2\nu_3}\frac{\partial\theta}{\partial x}+\Delta U=0,\quad \frac{1}{1-2\nu_3}\frac{\partial\theta}{r\partial\varphi}+\Delta V-\frac{V}{r^2}+\frac{2}{r^2}\frac{\partial W}{\partial\varphi}=0,$$
$$\frac{1}{1-2\nu_3}\frac{\partial\theta}{\partial r}+\Delta W+\frac{W}{r^2}-\frac{2}{r^2}\frac{\partial V}{\partial\varphi}=0 \tag{2.3}$$
$$\theta=\frac{\partial W}{\partial r}+\frac{W}{r}+\frac{\partial V}{r\partial\varphi}+\frac{\partial U}{\partial x},\quad \Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial x^2},$$

where x,φ,r are cylindrical coordinates; U, V, W are displacement components; ν_3 and E_3 are, respectively, the Poisson coefficient and the elasticity modulus of the filler.

Not distinguishing the mid-surface radius of the shell and that of the outer surface of the filler, we write the conditions of conjugation of the shell and the filler for $r=r_0$ as follows:

$$u=U,\quad v=V,\quad w=W,\quad \tau_{rx}=0,\quad \tau_{r\varphi}=0,\quad \sigma_r=-q_r. \tag{2.4}$$

Consider first the twisting of sufficiently long shells. Neglecting the influence of the boundary conditions on the critical load, the expression for displacements satisfying a system of equations (1.6_1), (1.6_2) and (2.2) takes the form:

$$u=A\cos(\mu\xi-n\varphi),\quad v=B\cos(\mu\xi-n\varphi),\quad w=C\sin(\mu\xi-n\varphi), \tag{2.5}$$

where μ is the parameter characterizing the wave formation along the generatrix; n is a number of waves in the circumferencial direction; A, B, C are the constants.

Taking into consideration that the filler is solid, the solution of equation (2.3) can be represented as

$$U=\Big[A_1I_n(\rho)+B_1\rho\frac{\partial I_n(\rho)}{\partial\rho}\Big]\cos(\mu\xi-n\varphi)$$
$$V=\Big[-\frac{A_1}{\rho}I_n(\rho)+\frac{B_1}{\rho}4(1-\nu_3)I_n(\rho)-C_1\frac{\partial I_n(\rho)}{\partial\rho}\Big]n\cos(\mu\xi-n\varphi) \tag{2.6}$$
$$W=\Big\{A_1\frac{\partial I_n(\rho)}{\partial\rho}+B_1\Big[\rho I_n(\rho)-4(1-\nu_3)\frac{\partial I_n(\rho)}{\partial\rho}\Big]+C_1n^2\frac{I_n(\rho)}{\rho}\Big\}\sin(\mu\xi-n\varphi),$$

where I_n is the Bessel's first kind modified function of the n-th order; $\rho=\mu r/r_0$; A_1, B_1, C_1 are the constants.

If we subject the solution (2.6) to the conditions of conjugation (2.4), we will be able to find the reaction of radial interaction

$$q_r = \frac{E_3}{1+\nu_3}\Big(n^2 - 1 + \frac{\mu F_1}{F_2}\Big)\frac{w}{r_0} \tag{2.7}$$

$$F_1 = \mu^3(2f-\mu)(1-f^2) -$$
$$- n^2[3\mu^2 - 2(1-\nu_3) - 2(1-2\nu_3)\mu f - 2(2-\nu_3)\mu^2 f^2] - 2n^4$$

$$F_2 = 2(1-\nu_3)\mu^3(2f-\mu) -$$
$$- n^2\mu[4(1-\nu_3)f + \mu(1-f^2)] - n^4; \quad f = \frac{1}{I_n(\mu)}\frac{\partial I_n(\mu)}{\partial \mu}.$$

Condition (1.5) for the solution (2.5) has the form $n^2 \gg \mu^2$. We apply this condition to the expansion of Bessel's function

$$I_n(\mu) = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!}\Big(\frac{\mu}{2}\Big)^{n+2k}.$$

Then restricting ourselves in that expansion to the first two terms, expression (2.7) takes the form

$$q_r = \frac{E_3(n^2-1)}{(1+\nu_3)[2(1-\nu_3)[2(1-\nu_3)n - 1 + 2\nu_3]}\frac{w}{r_0}. \tag{2.8}$$

Thus the problem of the stability is reduced to the finding of the least eigen number of the resolving equation (2.2), which by itself is a critical shearing stress.

Substituting expressions (2.5) and (2.8) in equation (2.2), we obtain the following characteristic equation for determining the critical stress

$$2s^0 =$$
$$= \frac{\varepsilon_0 n^4(n^2-1)^2 + \frac{E_1}{E_2}(\mu^4 + 4\delta\mu^2 n^2 + 4\delta^2 n^4) + \omega n^2(n^2-1)^2(n-\varkappa)^{-1}}{\mu n^3(n^2-1)}, \tag{2.9}$$

where

$$\omega = \frac{1}{2(1-\nu_3^2)}\frac{r_0}{h}\frac{E_3}{E_2}, \quad \varkappa = \frac{1-2\nu_3}{2(1-\nu_3)}. \tag{2.10}$$

To determine the critical value it is necessary to find a minimum s^0 by the parameters μ and n. Minimizing expression (2.9) by the parameter μ, we obtain the biquadratic equation with respect to μ. The positive root of that equation has the form

$$\mu^2 = \frac{2}{3}\varepsilon_0^{1/2}\mu_*^2, \quad \mu_*^2 =$$
$$= -\frac{\delta}{\sqrt{\varepsilon_0}}n^{-2} + \sqrt{4\frac{\delta^2}{\varepsilon_0}n^{-4} + \frac{3}{4}\frac{E_1}{E_2}\Big[\frac{(n^2-1)^2}{n^4} - \frac{\omega}{\varepsilon_0}\frac{(n^2-1)^2}{n^2(n-\varkappa)}\Big]}. \tag{2.11}$$

Substituting (2.11) in (2.9) we get the following, depending only on n, dimensionless expression:

$$\frac{s^0}{s^0_*} = \frac{n^3\mu_*^{-1}}{3\sqrt[4]{12}(n^2-1)} \times$$
$$\times\Big\{2,25\Big[\frac{(n^2-1)^2}{n^4}+\frac{8}{3}\frac{E_1}{E_2}\frac{\delta_*\mu_*^2}{n^2}+4\frac{E_1}{E_2}\frac{\delta_*^2}{n^4}+\frac{\omega(n^2-1)^2}{\varepsilon_0 n^6(n-\varkappa)}\Big]+\frac{E_1}{E_2}\mu_*^4\Big\} \quad (2.12)$$
$$s^0_* = \frac{1}{3\sqrt{2}}(1-\nu_1\nu_2)^{-3/4}(\frac{h}{r_0})^{3/2}, \quad \delta_* = \frac{\delta}{\sqrt{\varepsilon_0}}, \quad s^0 = \frac{S^0}{E_2 h}. \quad (2.13)$$

We represent E_1, E_2 in the dimensionless form

$$E_1 = \alpha_1 E, \quad E_2 = \alpha_2 E, \quad (2.14)$$

where α_1, α_2 are independent parameters. Then formula (2.12) takes the form

$$\frac{s^0}{s^0_*} = \frac{\alpha_2 n^3\mu^{-1}}{3\sqrt[4]{12}(n^2-1)} \times$$
$$\times\Big\{2,25\Big[\frac{(n^2-1)^2}{n^4}+\frac{8}{3}\frac{\alpha_1}{\alpha_2}\frac{\delta_*\mu_*^2}{n^2}+4\frac{\alpha_1}{\alpha_2}\frac{\delta_*^2}{n^4}+\frac{\omega_0(n^2-1)^2}{\alpha_2\varepsilon_0 n^6(n-\varkappa)}\Big]+\frac{\alpha_1}{al_2}\mu_*^4\Big\} \quad (2.15)$$
$$s = \frac{S^0}{Eh}, \quad \omega_0 = \frac{1}{2(1-\nu_3^2)}\frac{r_0}{h}\frac{E_3}{E}.$$

For $\delta = 0$ (i.e., for the cylindrical shell), from formulas (2.11) and (2.15) we get

$$\mu = \frac{1}{\sqrt[4]{3}}\Big[\frac{\alpha_2}{\alpha_1}\varepsilon_0 n^4(n^2-1)^2+\frac{\omega_0 n^2(n^2-1)^2}{\alpha_2(n-\varkappa)}\Big]^{1/4},$$
$$\frac{s}{s^0_*} = \frac{\alpha_1^{1/4}\alpha_2^{3/4}}{\sqrt{3}}(n^2-1)^{1/2}\Big[1-\frac{\omega_0}{\varepsilon_0\alpha_2 n^2(n-\varkappa)}\Big]^{3/4}. \quad (2.16)$$

The above-obtained formula (2.16) for the cylindrical shell unlike the formula given in [4] is more exact, since in the resolving equation (2.2), unlike the equation given in [4], the terms corresponding to the sixth and fourth derivatives in the first parentheses, the mixed fourth derivative in the second parentheses, as well as the second derivative in the last term are taken into account and they may be important for the long shells. In particular, from formula (2.16) for $\omega = 0$, $n = 2$, $\alpha_1 = \alpha_2 = 1$, $\nu_1 = \nu_2 = \nu$ unlike [4], we obtain the well-known Timoshenko's formula ([9], p. 427) $s = s^0_*$ for the critical shearing stress of long cylindrical shells.

To determine the influence of elastic orthotropic parameters on the critical stress s of the cylindrical shell ($\delta = 0$) according to the rigidity of the elastic filler ω, it is necessary to find a least value of expression (2.16), when $n = 2, 3, 4, \ldots$ for fixed values α_1, α_2.

In Fig. 1(a), for the cylindrical shell we can see the curves of the dependence of the dimensionless critical shearing stress on the rigidity of the elastic filler for the cases $\alpha_1 = \alpha_2 = 1(0)$, $\alpha_1 = 2$, $\alpha_1 = 1(1)$, $\alpha_1 = 1$, $\alpha_2 = 2(2)$; the corresponding curves are denoted by (0), (1) and (2); note that ω_0/ε_0 is plotted on the abscissa and s_*/s_*^0 on the ordinate, where s_* is the critical value of s and s_*^0 is defined by formula (2.13). Comparing these curves with the results obtained in [4], we find that if for $\omega_0/\varepsilon_0 = 0$ the divergence between the critical values is about 40%, then as ω_0/ε_0 increases this divergence decreases and vanishes practically for $\omega_0/\varepsilon_0 \gtrsim 80$. This happens because of the fact that as ω_0/ε_0 increases, the critical number n_* of waves also increases, and in this case one can neglect unity compared with n_*^2.

To determine the critical value s for the shells of revolution (for different values of δ_*, α_1, α_2) depending on $\frac{\omega_0}{\varepsilon_0}$, it is necessary to find the minimal value s depending on n by formula (2.15); the corresponding critical value s we denote by s_*. In Figs. 1(b) and (c) are drawn the curves denoted by (0), (1) and (2) for the shells of positive ($\delta = 1$) and negative ($\delta = -1$) Gaussian curvature for the cases (0), (1) and (2). Comparing these curves for small values of rigidity of the elastic filler, we can easily notice that the elastic parameters in the axial and circumferential directions affect to a certain extent the critical load of shells of positive and negative Gaussian curvature. However, as ω_0/ε_0 increases, the difference between critical stresses for these shells and for the corresponding cylindrical shell essentially decreases.

As the shell length decreases, the boundary conditions begin to affect the critical load, and it is very interesting to find the conditions under which this influence is inessential.

Considering in the sequel the shells of mid-length, analogously to cylindrical ones, the principal boundary conditions

$$w(l/r_0, \varphi) = 0, \quad w(-l/r, \varphi) = 0 \tag{2.17}$$

were fulfilled. The fulfilment of these conditions gives, as is known, sufficiently good approximation of values of critical load for freely supported ends ([2], [10]).

It was assumed that the influence of the filler on the shell was realized by the law (2.8) and the solution was sought in the form of the series

$$w = \sum_{mn} \cos \lambda_m \xi [A_{mn} \sin n(\varphi - \gamma\xi) + B_{mn} \cos n(\varphi - \gamma\xi)], \tag{2.18}$$

where $\lambda_m = m\pi r_0/2l$ ($m = 2p + 1, p = 0, 1, 2, \dots$). In this expansion the terms $C_{mn} \sin \lambda_m \xi \sin n(\varphi - \gamma\xi)$ and $D_{mn} \sin \lambda_m \xi \cos n(\varphi - \gamma\xi)$ are absent due to the boundary conditions (2.17).

Represent expression (2.18) as follows:

$$w=\sum_{mn}0,5A_{mn}(\sin[n(\varphi-\gamma\xi)-\lambda_m\xi]+\sin[n(\varphi-\gamma\xi)+\lambda_m\xi])+$$
$$+0,5B_{mn}(\cos[n(\varphi-\gamma\xi)-\lambda_m\xi]+\cos[n(\varphi-\gamma\xi)+\lambda_m\xi]). \quad (2.19)$$

Substituting (2.19) in (2.2), we obtain

$$\sum_{mn}\{A_{mn}F(n,-m)\sin[n(\varphi-\gamma\xi)-\lambda_m\xi]+$$
$$+A_{mn}F(n,m)\sin[n(\varphi-\lambda\xi)+\lambda_m\xi]+B_{mn}F(n,-m)\cos[n(\varphi-\gamma\xi)-\lambda_m\xi]+$$
$$+B_{mn}F(n,m)\cos[n(\varphi-\gamma\xi)+\lambda_m\xi]\}=0,$$
$$F(n,\pm m)=\varepsilon_0 n^4(n^2-1)^2+\frac{E_1}{E_2}(\mu_\pm^4+4\delta\mu_\pm^2n^2+4\delta^2n^4)+$$
$$+2s^0\mu_\pm n^3(n^2-1)+\omega\frac{n^2(n^2-1)^2}{n-\varkappa},$$
$$\mu_\pm=-n\gamma\pm\lambda_m, \quad (2.20)$$

whence

$$A_{mn}[F(n,-m)+F(n,m)]=0,\quad A_{mn}[F(n,-m)-F(n,m)]=0,$$
$$B_{mn}[F(n,-m)+F(n,m)]=0,\quad B_{mn}[F(n,-m)-F(n,m)]=0.$$

Consequently, for the existence of a non-trivial solution of equation (2.2) under the above-mentioned boundary conditions, it is necessary and sufficient, that the integers m and n satisfy the conditions

$$F(n,m)=0,\quad F(n,-m)=0. \quad (2.21)$$

Relations (2.21) are by themselves the conditions

$$\varepsilon_0(n^2-1)^2+\frac{E_1}{E_2}\Big[\Big(-\gamma+\frac{m\pi r_0}{nL}\Big)^4+4\delta\Big(-\gamma+\frac{m\pi r_0}{nL}\Big)^2+4\delta^2\Big]+$$
$$+2s^0(n^2-1)\Big(-\gamma+\frac{m\pi r_0}{nL}\Big)+\omega\frac{(n^2-1)^2}{n^2(n-\varkappa)}=0$$
$$\varepsilon_0(n^2-1)^2+\frac{E_1}{E_2}\Big[\Big(-\gamma-\frac{m\pi r_0}{nL}\Big)^4+4\delta\Big(-\gamma-\frac{m\pi r_0}{nL}\Big)^2+4\delta^2\Big]+$$
$$+2s^0(n^2-1)\Big(-\gamma-\frac{m\pi r_0}{nL}\Big)+\omega\frac{(n^2-1)^2}{n^2(n-\varkappa)}=0.$$

Taking into account that for the mid-length shells one can neglect unity

compared with n^2, as well as $\varkappa$ compared with n, we obtain

$$\varepsilon_0 n^4 + \frac{E_1}{E_2}\Big[\Big(-\gamma + \frac{m\pi r_0}{nL}\Big)^4 + 4\delta\Big(-\gamma + \frac{m\pi r_0}{nL}\Big)^2 + 4\delta^2\Big] + \\ +2s^0 n^2(-\gamma + \frac{m\pi r_0}{nL}) + \omega n = 0 \tag{2.22}$$

$$\varepsilon_0 n^4 + \frac{E_1}{E_2}\Big[\Big(-\gamma - \frac{m\pi r_0}{nL}\Big)^4 + 4\delta\Big(-\gamma - \frac{m\pi r_0}{nL}\Big)^2 + 4\delta^2\Big] + \\ +2s^0 n^2(-\gamma - \frac{m\pi r_0}{nL}) + \omega n = 0. \tag{2.23}$$

It is not difficult to show that the least value s^0, depending on m, will be for $m = 1$. Therefore in what follows we will assume $m = 1$.

Introduce the notation $\rho = \lambda_1 \varepsilon_*^{-1/4}/n$, $\theta = \gamma \varepsilon_*^{-1/4}$, $\delta_* = \delta \varepsilon_*^{-1/2}$, $\varepsilon_* = hr_0/L^2(1-\nu_1\nu_2)^{1/2}$.

$$\Omega = \omega_0 \lambda_1 \varepsilon_*^{-5/4}/\pi^2, \quad \lambda_1 = \pi r_0/L, \quad S = s/s_k,$$
$$s = S^0/Eh, \quad s_k = 0,74(1-\nu_1\nu_2)^{-5/8}(h/r_0)^{5/4}(r_0/L)^{1/2}. \tag{2.24}$$

Then equations (2.22) and (2.23), with regard for designations (2.14) and (2.24), take the form

$$\frac{\pi^2\rho^{-2}}{12} + \frac{1}{\pi^2}\frac{\alpha_1}{\alpha_2}[\rho^2(-\theta \pm \rho)^4 + 4\delta_*\rho^2(-\theta \pm \rho)^2 + 4\delta_*\rho^2] + \\ +1,48\frac{S}{\alpha_2}(-\theta \pm \rho) + \frac{\Omega}{\alpha_2}\rho = 0. \tag{2.24$_1$}$$

Adding and subtracting equalities (2.24$_1$) and making then some simplifications, we get the following equations:

$$1,48\alpha_2^{-1}S = \frac{\pi^2}{12}\frac{\theta}{\rho^2(\theta^2-\rho^2)} + \\ +\frac{\alpha_1}{\pi^2\alpha_2}\Big[\rho^2\theta(\theta^2+3\rho^2) + 4\delta_*\rho^2\theta + 4\delta_*^2\frac{\rho\theta}{\theta^2-\rho^2}\Big] + \frac{\alpha_2^{-1}\Omega\rho\theta}{\theta^2-\rho^2} \tag{2.25}$$

$$\frac{\pi^2}{12}\frac{\theta}{\rho^2(\theta^2-\rho^2)} = \\ = \frac{\alpha_1}{\pi^2\alpha_2}\Big[\rho^2\theta(3\theta^2+\rho^2) + 4\delta_*\rho^2\theta - 4\delta_*^2\frac{\rho\theta}{\theta^2-\rho^2}\Big] - \frac{\alpha_2^{-1}\Omega\rho\theta}{\theta^2-\rho^2}. \tag{2.26}$$

Substituting (2.26) in (2.25), we have

$$S = \frac{2,703}{\pi^2}\alpha_1\rho^3\Lambda[\rho^2(1+\Lambda^2) + 2\delta_*], \quad \Lambda = \theta/\rho. \tag{2.27}$$

Equality (2.26) implies

$$3\Lambda^4 - B\Lambda^2 - C = 0 \tag{2.28}$$

$$B = 2 - 4\delta_*\rho^{-2}, \quad C = 1 + 4\delta_*\rho^{-2} + 4\delta_*^2\rho^{-4} + \frac{\pi^2\alpha_2}{\alpha_1}\Big(\frac{\pi^2\rho^{-8}}{12} + \alpha_2^{-1}\Omega\rho^{-5}\Big).$$

Since $\rho > 0$, $\gamma \geq 0$ of special interest for us are only positive roots of equation (2.28). Moreover, taking into account that $C > 0$, we find that the positive root of that equation is of the form

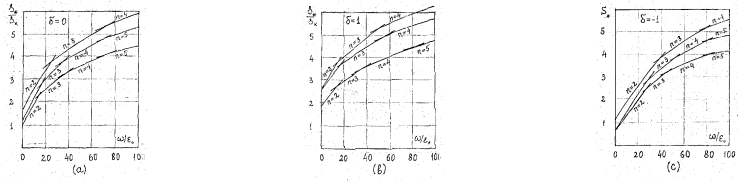
$$\Lambda = 0,408(\sqrt{B^2 + 12C} + B)^{1/2}. \tag{2.29}$$

According to equality (2.29), substituting the expression Λ in formula (2.27), we obtain S in terms of the function of one dimensionless variable ρ and four dimensionless parameters α_1, α_2, δ_*, Ω. Defining the least value S of ρ for fixed α_1, α_2, δ_*, Ω we obtain the corresponding critical values of S which we denote by S_*.

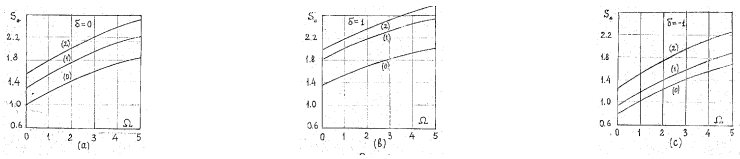
In Figs. 2(a)–(c) are drawn universal curves of the dependence of the dimensionless values S_* on the dimensionless rigidity of the filler Ω for the above-mentioned cases (0), (1) and (2) for $\delta = 0, 1, -1$.

In the limiting case $\delta_* = 0$, $\Omega = 0$ from the expressions (2.27) and (2.29) there follows the well-known formula for the critical shearing stress of the cylindrical orthotropic shell $S_* = \alpha_1^{3/8}\alpha_2^{5/8}$ [7]. For $\delta_* \neq 0$, $\Omega = 0$ we obtain the value of critical shearing stresses which coincide with the results obtained in [5].

To illustrate the influence of the shell length on the critical stress, we present in Figs. 3 and 4 the dependencies $T = s_*(1 - \nu_1\nu_2)^{5/8}$ on the relative length of the shell L/r_0 for different parameters $K = \omega(1 - \nu_1\nu_2)^{5/8}$ and δ_* for $r_0/h = 200$; $\alpha_1 = 2$, $\alpha_2 = 1$. It is not difficult to obtain similar curves for the other values of parameters α_1, α_2, δ_*, K. Moreover, it is easily seen that as L/r_0 and ω_0/ε_0 increase, the influence of the length on the critical stress significantly decreases, and for some value L/r_0 the quantity s_* does not practically depend on the length. Our calculations show that for $(E_2/E_1)^{3/8}\Omega \gtrsim 5$ and $\omega_0/\varepsilon_0 \gtrsim 100$ the values s_* calculated by formula (2.27), coincide practically with those of s_* calculated by formula (2.15) (for sufficiently long shells). Thus we obtain that for $(E_2/E_1)^{3/8}(L/r_0)^{3/2}(h/r_0)^{3/4}(\omega_0/\varepsilon_0)(1 - \nu_1\nu_2)^{-3/8} \gtrsim 60\pi$ for mid-length shells with comparatively rigid filler $\omega_0/\varepsilon_0 \gtrsim 100$ the influence of the boundary conditions on the critical load can be neglected.



Pic. 1



Pic. 2

Pic. 3

References

1. V. Z. Vlasov, Some new problems of relative mechanics of shells of thin-walled constructions. (Russian) *Izv. Acad. Nauk SSSR, Ser. Otd. Techn. Nauk* **1**(1947), 27–52.

2. A. S. Vol'mir, Stability of deformable systems. (Russian) *Fizmatgiz, Moscow*, 1967.

3. V. M. Darevsky, Stability of shells near by their form to cylindrical ones. Problems of calculation of spatial constructions. (Russian) *MISI, Moscow*, 1980, 35–45.

4. A. V. Karasev and I. S. Maljutin, Stability of glass-reinforced plastic cylindrical shell with elastic filler under torsion. (Russian) *Mech. Polym.* **6**(1970), 1082–1086.

5. S. N. Kukudzhanov, On the influence of orthotropy on the stability of shells of revolution, near to cylindrical ones. (Russian) *Izv. Ross. Acad. Nauka.* 1, 158–165, *Mekh. Tverd. Tela.* 1998.

6. A. N. Lur'e, Spatial problems of elasticity. (Russian) *Gostekhizdat, Moscow,* 1955.

7. Kh. M. Mushtari and K. Z. Galimov, Nonlinear theory of elastic shells. (Russian) *Tetknigoizdat, Kazan,* 1957.

8. V. V. Novozhilov, Theory of thin shells. (Russian) *Sudpromgiz, Leningrad,* 1962.

9. S. P. Timoshenko, Stability of elastic systems. (Russian) *Gostekhizdat, Moscow-Leningrad,* 1936.

10. P. E. Tovstik, To the problem of stability of a cylindrical shell. (Russian) *Appl. Mech.* **16**(1980), No. 9, 132–134.

(Received 01.02.1999; revised 21.04.1999)

Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia