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SETS OF UNIQUENESS OF THE POWER OF THE CONTINUUM

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Abstract. By means of the axiom of choice, the authors show that there exist sets of the power of the continuum that are sets of uniqueness for spherical convergence and sets of uniqueness for spherical Abel summability.

Consider the multiple trigonometric series

$$S(x) = \sum_{n \in \mathbb{Z}^d} c_n e^{inx},$$

where $n = (n_1, ..., n_d), x = (x_1, ..., x_d) \in \mathbb{T}^d = [-\pi, \pi)^d$ and nx denotes the scaler product of n and x, $n_1 x_1 + ... + n_d x_d$. Say that $S(x)$ converges spherically to the complex number s at x if

$$\lim_{R \to \infty} S_R(x) = \lim_{R \to \infty} \sum_{|n| \le R} c_n e^{inx} = s,$$

where $|n| = \sqrt{n_1^2 + ... + n_d^2}$, and say that $S(x)$ is spherically Abel summable to s at x if

$$\lim_{t \to 0^+} S(x,t) = \lim_{t \to 0^+} \left(\lim_{R \to \infty} \sum_{|n| \le R} c_n e^{inx - |n|t} \right) = s.$$

Write $S(x) \equiv 0$ if every coefficient c_n is 0. Let E be a subset of $\mathbb{T}^d$.

Definition 1. *Say that E is a set of uniqueness, or U-set if whenever $S(x)$ converges spherically to 0 at every $x \notin E, S(x) \equiv 0$.*

Definition 2. *Say that E is a set of spherical Abel uniqueness of the first type, or U_A-set if whenever $S(x)$ is spherically Abel summable to 0 at every*

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$x \notin E, S(x) \equiv 0$, *provided the coefficients of* $S(x)$ *satisfy* $c_n \to 0$ *as* $|n| \to \infty$ *and*

$$\sum_{2^{2k-2} \le |n|^2 < 2^{2k}} |c_n|^2 = o\left(2^{2k}\right). \tag{0.1}$$

.

Definition 3. *Say that* E *is a set of spherical Abel uniqueness of the second type, or* U_{A^*}*-set if whenever* $S(x)$ *is spherically Abel summable to* 0 *at every* $x \notin E, S(x) \equiv 0$, *provided the coefficients of* $S(x)$ *satisfy condition (*0.1*).*

The following theorem is proved in [AW].

Theorem 1. *Every countable subset of* T^d *is a* U*-set; a* U_A*-set; and, when the dimension* $d \ge 2$, *a* U_{A^*}*-set.*

Our goal is to produce "larger" sets of uniqueness. A perfect set is a set which contains all its limit points. A nonempty perfect set has the power of the continuum, that is, it has c points, where c is the cardinality of the real numbers.(See Proposition 1 below.) In [AW], it is proved there that there exist a nonempty perfect set (called an H set) that is both a U-set and a U_A-set. If $d = 2$, the set $Y = \{(0, x_2) : -\pi \le x_2 < \pi\}$ turns out to be an H set. But Y is not a U_{A^*}-set. To see this, consider the series $\sum e^{in_1x_1} = 1+2\sum_{n_1=1}^{\infty} \cos n_1x_1$, which is spherically Abel summable to 0 on the complement of Y, and whose coefficients satisfy $\sum_{2^{2k-2} \le |n|^2 < 2^{2k}} |c_n|^2 = O\left(2^k\right)$, and hence also condition (0.1). Incidentally, this example, when specialized to dimension 1 shows that the definition of U_{A^*}-set is not a very good one in dimension 1, because even the singleton $\{0\}$ is not a U_{A^*}-set there. This means that U_{A^*}-sets are pretty scarce, at least compared to U-sets and U_A-sets. In fact, we do not know of a non-empty perfect U_{A^*}-set.

Our main result is the following.

Theorem 2. *For each dimension* d, *there is a set contained in* $\mathbb{T}^d$ *that has the power of the continuum that is a* U*-set, a* U_A*-set, and, if* $d \ge 2$, *a* U_{A^*}*-set.*

Notice that the first two thirds of this theorem prove (in a different way) some things that are already known. The proof depends on the axiom of choice, but not on the continuum hypothesis.

Proof. By the axiom of choice there is a set $A \subset T^d$ of cardinality c containing no perfect subset.(See Theorem 4 below.) Let $S(x)$ be a multiple series which is spherically convergent to 0 at all points that do not belong to A and let $C \subset A$ be the set of all points where $S(x)$ does not converge to 0. Then C is a $G_{\delta\sigma}$ set (see Lemma 2 below), so that if C were uncountable, C would contain a non-empty perfect subset P(see Lemma 1 below). But $P \subset C \subset A$ implies $P \subset A$, contrary to the way that A was chosen. So C is countable, whence $S(x) \equiv 0$ by Theorem 1. Thus A is a U-set.

Similarly, let the dimension $d \geq 2$, let $S(x)$ be a multiple series which is spherically Abel summable to 0 at all points that do not belong to A and suppose that the coefficients of $S(x)$ satisfy condition (0.1) and let $C \subset A$ be the set of all points where $S(x)$ is not spherically Abel summable to 0. Then C is a $G_{\delta\sigma}$ set (see Lemma 2 below), so that if C were uncountable, C would contain a non-empty perfect subset P(see Lemma 1 below). But $P \subset C \subset A$ implies $P \subset A$, contrary to the way that A was chosen. So C is countable, whence $S(x) \equiv 0$ by Theorem 1. Thus A is a U_{A^*}-set.

The proof that A is a U_A-set is almost identical $\square$

It was shown in [AW] that a union of countably many closed U-sets is a U-set, a union of countably many closed U_A-sets is a U_A-set, and a union of countably many closed U_{A^*}-sets is a U_{A^*}-set.

Theorem 3. *A union of two U-sets may not be a U-set. A union of two U_A-sets may not be a U_A-set. A union of two U_{A^*}-sets may not be a U_{A^*}-set.*

Proof. The proof of Theorem 2 shows that both of the sets A and B constructed in the proof of Theorem 4 below are U-sets, U_A-sets and U_{A^*}-sets, but $A \cup B = \mathbb{T}^d$, which is neither a U-set, a U_A-set, nor a U_{A^*}-set, since *any* not identically 0 trigonometric series converges to 0 on $\emptyset = \left(\mathbb{T}^d\right)^c$. $\square$

Remark 1. *There is a history to the methods used in this paper. All of the previous results that we will mention here use "uncountable" in place of c and are for dimension 1 only. Thus our main theorem is new, even in dimension 1. Theorem 4 below is due to Bernstein* [B]. *The arguments leading to Theorems 2 and 3 can be found respectively on pages 344 and 349 of Zygmund.*[Z]

What follows is included to make this note self contained.

Proposition 1. *Every non-empty perfect set P contains c points. There are exactly c perfect subsets of $\mathbb{T}^d$.*

Proof. The set P must contain at least two points: a point, and an approaching point. Put disjoint closed balls about each of these two points. Proceed recursively, finding 2 disjoint closed balls in each ball. Continue forever, producing a complete binary tree of closed balls. The paths to infinity correspond to the real numbers between 0 and 1 written in binary and each such path produces at least one point of P because of the nested compact set theorem.

To count the perfect sets, start with the countable base of $\mathbb{T}^d$ consisting of all open discs with rationally coordinated centers and rational radii . Each open subset corresponds to an element of the power set of this base, so there are at most $2^{\aleph_0} = c$ open sets. Matching closed sets with their complements shows that the number of closed sets is at most c. Every perfect set is closed so there are at most c perfect sets. That there are at least c is clear from considering the family of closed intervals $\{[r, 2] \times \mathbb{T}^{d-1} : 0 \leq r \leq 1\}$. $\square$

Lemma 1. *Every uncountable $G_{\delta\sigma}$ set has cardinality c and contains a perfect set.*

Proof. The proof of cardinality c is the same as the preceding proof, except that you use points of condensation rather than points of accumulation.([H],p.205) $\square$

Lemma 2. *The set where a multiple trigonometric series does not spherically converge to* 0 *is a $G_{\delta\sigma}$. If the coefficients of a multiple trigonometric series satisfy the condition (*0.1*), then the set where it is not Abel summable to* 0 *is a $G_{\delta\sigma}$.*

Proof. First let $C = \left\{x \in T^d : \lim_{R\to 0} S_R(x) = 0 \text{ is false}\right\}$. Then

$$C = \bigcup_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \left(\bigcup_{R^2=k}^{\infty} \left\{ x : |S_R(x)| > \frac{1}{m} \right\} \right)$$

is a $G_{\delta\sigma}$, since the quantity in parentheses is an open set for each choice of k.

Now suppose that condition (0.1) holds and let

$$C = \left\{ x \in T^d : \lim_{t\to 0^+} S(x,t) = 0 \text{ is false} \right\}.$$

Condition (0.1) implies that $|c_n| \leq c\,|n|^2$ for some constant c. So by the fact that $\sum |n|^2 e^{-|n|t}$ converges and the Weierstrass M test, $S(x,t)$ is a continuous function for each $t > 0$. Thus again

$$C = \bigcup_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \left(\bigcup_{r=k}^{\infty} \left\{ x : |S(x,1/r)| > \frac{1}{m} \right\} \right)$$

is a $G_{\delta\sigma}$. $\square$

Theorem 4. *There is a partition of $\mathbb{T}^d = [0,2\pi)^d$ into two sets of cardinality c, neither of which contains a perfect subset.*

Proof. Let Ω be the first ordinal number associated with the cardinal number c. (For example, ω is the smallest ordinal associated with $\aleph_0$, but there are others, such as $\omega+1$.) This means that we can well-order the family $\{P \subset \mathbb{T}^d : P \text{ is perfect and non-empty}\}$, which has cardinality c, with indices that are all the ordinals less than Ω. We have the ordering $P_0, P_1, ..., P_\omega, ..., P_\xi, ...$ $(\xi < \Omega)$.

We choose two distinct points, $x_0, y_0 \in P_0$. If, for $0 < \eta < \Omega$, all the points x_ξ and y_ξ $(\xi < \eta)$ have already been defined and if we call the sets formed by them A_η and B_η, then $A_\eta \cup B_\eta$ has cardinality strictly less than c, because its index has ordinality $< \Omega$, and thus there remain c points of P_η that do not belong to $A_\eta \cup B_\eta$. We choose x_η and y_η as two distinct such points. It follows from this construction that if $\xi < \eta < \Omega$, then the four points x_ξ, y_ξ, x_η, and y_η are all distinct.

Let $A = \{x_\xi : \xi < \Omega\}$ and let $B = \mathbb{T}^d \setminus A$. Of course we have $A \cup B = \mathbb{T}^d$ and $A \cap B = \emptyset$. Also A has cardinality c, because $\{\xi : \xi < \Omega\}$ has ordinality Ω and consequently cardinality c. Similarly, B has cardinality c, because $\{y_\xi : \xi < \Omega\} \subset B$. Finally, if P is a perfect set, there is an index $\xi < \Omega$ such that $P = P_\xi$, so $P \nsubseteq A$ since $y_\xi \in P \setminus A$. Similarly $P \nsubseteq B$ since $x_\xi \in P \setminus A$. $\square$

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