Proceedings of A.Razmadze
Mathematical Institute,
Vol. **119**, 1999

DG HOPF ALGEBRAS WITH STEENRODS I-TH COPRODUCTS

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Abstract. In this article we introduce the notion of DG-Hopf algebra with Steenrods ∇_i coproducts $(C, d, \mu, \nabla_0, \nabla_1, \dots)$, where (C, d, μ) is a DG-algebra and $\{\nabla_i : C \to C \otimes C\}$ are chain cooperations, with properties dual to Steenrods $\smile_i$-products, satisfying certain conditions of compatibility with the multiplication $\mu : C \otimes C \to C$. As an application the constructions of $\nabla_i -$s on the singular cubical chain complex $CU_*(X)$ and on the cobar construction $\Omega \bar{C}_*(X)$ of the singular chain complex of an 1-connected space X are given.

In this article we study objects of type

$$(C; d; \nabla_0, \nabla_1, \nabla_2, \dots\ ; \mu)$$

where $(C; d; \nabla_0)$ is a DG-coalgebra (with $\deg d = -1$), $(C; d; \mu)$ is a DG-algebra and $\nabla_i : C \to C \otimes C,\ i > 0$, are cooperations, dual to Steenrods $\smile_i$ products, i. e. they satisfy the conditions

$$\deg \nabla_i = i,\ (d \otimes 1 + 1 \otimes d)\nabla_i + \nabla_i d = \nabla_{i-1} + (-1)^i T \nabla_{i-1}.$$

here $T : C \otimes C \to C \otimes C$ is the permutation map.

We are interested what kind of compatibility of ∇_i-s with the multiplication μ should be required. The requirement for the coproduct ∇_0 is well known: $\nabla_0 : C \to C \otimes C$ should be a multiplicative map (this means, that $(C; d; \nabla_0; \mu)$ is a DG-Hopf algebra). The requirement for ∇_1 is known too (see [3]): ∇_1 should be a $(\nabla_0, T\nabla_1)$-derivation, i. e.

$$\nabla_1 \mu = \mu_{C \otimes C}(\nabla_1 \otimes \nabla_0 + T\nabla_0 \otimes \nabla_1).$$

1991 *Mathematics Subject Classification.* 55S05.
Key words and phrases. Derivation, Hopf algebra, Steenrod products, Cubical set.

Below we introduce the notion of *n-derivation*, which is map of algebras, which preserves multiplicative structure in some specific way. Particularly, a 0-derivation is just a multiplicative map, a 1-derivation is an ordinary derivation. We formulate the requirements for higher ∇_i-s in these terms: ∇_n should be a n-derivation. In a case when C is a free graded algebra, this allows proceed as follows: we can define ∇_n on generators and extend on whole C as a n-derivation. As an example we give construction of cooperations ∇_i for cubical chain complex of a space. Note, that the cooperation ∇_0, constructed by this procedure coincides with the cooperation, constructed in [9], and ∇_1 with cooperation, constructed in [3].

It is a pleasure to dedicate this article to 70'th birthday of Professor Nodar Berikashvili.

1. Derivations

1.1. High rank derivations. There are various types of maps of algebras $f : A \to B$, respecting multiplicative structure in some sense. First of all these are *multiplicative maps*, i. e. maps, satisfying $f(ab) = f(a)f(b)$. If two multiplicative maps $f, g : A \to B$ are given, then come in play (f, g)-*derivations*, i. e. maps $s : A \to B$ with property $s(ab) = s(a)f(b)+g(a)s(b)$. So multiplicative maps and derivations *decompose* on products differently. If A is a free algebra, i. e. if $A = T(V)$, here $T(V)$ is the tensor algebra generated by a graded module V, then each multiplicative map, as well as a derivation, is uniquely determined by the restriction on generator graded module V, but they have different *extension rules*: any linear map $\alpha : V \to B$ determines a multiplicative map $f_\alpha : A \to B$ extended from α by the rule

$$f_\alpha(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \alpha(a_1)\alpha(a_2)\cdots\alpha(a_n).$$

If, in addition $\beta, \beta' : V \to B$ are given (which determine a multiplicative maps $f_\beta, f_{\beta'} : A \to B$), then the same α can be extended as a $(f_\beta, f_{\beta'})$-derivation $s_\alpha : A \to B$ by the extension rule

$$s_\alpha(a_1 \otimes \cdots \otimes a_n) = \Sigma_{k=1}^{n} \beta'(a_1)\cdots\beta'(a_{k-1})\alpha(a_k)\beta(a_{k+1})\cdots\beta(a_n).$$

In this section we introduce *high rank derivations*, maps with special type of decomposability on products.

Bellow we assume, that A and B are graded algebras with products $\mu_A : A \otimes A \to A$ and $\mu_B : B \otimes B \to B$ respectively (we write μ for μ_A and μ_B).

Let us define a 0-*derivation* as a homomorphism of degree 0 $f : A \to B$ such, that

$$f\mu = \mu(f \otimes f),$$

in fact a 0-derivation is just a multiplicative morphism of graded algebras.

Suppose now that two 0-derivations $f_0^0, f_1^0 : A \to B$ are fixed. A 1-*derivation* with respect to data (f_0^0, f_1^0) is a homomorphism $f_0^1 : A \to B$ of degree 1, which decomposes by the rule

$$f_0^1\mu = \mu(f_0^1 \otimes f_0^0 + f_1^0 \otimes f_0^1),$$

This notion coincides with the usual notion of (f_0^0, f_1^0) derivation.

Now fix a following data

$$\begin{array}{ccccc} & f_0^1 & & f_1^1 & \\ f_0^0 & & f_1^0 & & f_2^0, \end{array}$$

where $f_0^0, f_1^0, f_2^0 : A \to B$ are 0-derivations, and $f_0^1, f_1^1 : A \to B$ are 1-derivations with respect to (f_0^0, f_1^0) and (f_1^0, f_2^0). A 2-*derivation* with respect to data above we define as a homomorphism of degree 2 $f_0^2 : A \to B$ which decomposes by the rule

$$f_0^2\mu = \mu(f_0^2 \otimes f_0^0 + f_1^1 \otimes f_0^1 + f_1^0 \otimes f_0^2).$$

Here is the general

Definition 1. Suppose a following data is given:

$$\begin{array}{ccccccccc} & & f_0^{n-1} & & f_1^{n-1} & & & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ & f_0^1 & & f_1^1 & \cdots\ f_{n-2}^1 & & & f_{n-1}^1 & \\ f_0^0 & & f_1^0 & \cdots & \cdots & \cdots & f_{n-1}^0 & & f_n^0, \end{array} \tag{1}$$

where each f_q^p is a p-derivation with respect to the collection $\{f_j^i\}$ with f_j^i from the triangle with f_q^p on the top. An n-derivation is a homomorphism of degree n, which decomposes by the rule

$$f_0^n\mu = \mu(\sum_{i=0}^{n} f_i^{n-i} \otimes f_0^i). \tag{2}$$

1.2. Universal property. Let A be a free graded algebra, i. e. the tensor algebra

$$T(V) = \Lambda + V + V \otimes V + V \otimes V \otimes V + \cdots = \sum_{i=0}^{\infty} \otimes^i V$$

of some free graded $\Lambda-$module V with usual grading and product.

Each morphism of graded algebras $f : T(V) \to B$ is uniquely determined by it's restriction on generators $f|_V$. More precisely tensor algebra has the following universal property: for an arbitrary homomorphism of graded modules $\alpha : V \to B$ there exists the unique multiplicative homomorphism (a 0-derivation) $f : T(V) \to B$, for which commutes the diagram

$$\begin{array}{ccccc} V & & \stackrel{i}{\longrightarrow} & & T(V) \\ \alpha & \searrow & & \swarrow & f \\ & & B & & \end{array},$$

here $i : V \to T(V)$ is the imbedding as the second direct summand of $T(V)$. The construction of f (i. e. the extension rule for α) is obvious:

$$f| \otimes^n V = \mu^n[(\alpha)^n],$$

here $\mu^n : \otimes^n B \to B$ is the iteration of the product $\mu : B \otimes B \to B$ and $(\alpha)^n = \alpha \otimes \cdots (n\text{-times}) \cdots \otimes \alpha$.

Tensor algebra has the similar universal property for 1-derivations too (see for example [7]): for an arbitrary data

$$\begin{array}{ccc} & \alpha_0^1 & \\ \alpha_0^0 & & \alpha_1^0 \end{array}$$

where $\alpha_0^0, \alpha_1^0 \in \mathrm{Hom}^0(V; B), \alpha_0^1 \in \mathrm{Hom}^1(V; B)$ there exist: 0-derivations $f_0^0, f_1^0 : T(V) \to B$ and 1-derivation $f_0^1 : T(V) \to B$ with respect to couple (f_0^0, f_1^0), such, that $f_q^p i = \alpha_q^p$. The construction of f_0^1 (i. e. the extension rule for α_0^1) is following:

$$f_0^1| \otimes^m V = \sum_{k=0}^{m-1} \mu^n[(\alpha_1^0)^k \otimes \alpha_0^1 \otimes (\alpha_0^0)^{m-k-1}],$$

here $(\alpha_i^0)^0$ means omitting of this factor.

Now we are going to show that $T(V)$ has the similar universal property for higher derivations too:

Proposition 1. *Tensor algebra $T(V)$ has the following universal property: for an arbitrary collection of homomorphisms*

$$\begin{array}{ccccccccc} & & & & \alpha_0^n & & & & \\ & & & \alpha_0^{n-1} & & \alpha_1^{n-1} & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & \alpha_0^1 & & \alpha_1^1 & \cdots & \alpha_{n-2}^1 & & \alpha_{n-1}^1 & \\ \alpha_0^0 & & \alpha_1^0 & \cdots & \cdots & \cdots & \alpha_{n-1}^0 & & \alpha_n^0 \end{array} \tag{3}$$

with $\alpha_q^p \in \mathrm{Hom}^p(V; B)$ there exists the unique collection

$$\begin{array}{ccccccccc} & & & & f_0^n & & & & \\ & & & f_0^{n-1} & & f_1^{n-1} & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & f_0^1 & & f_1^1 & \cdots & f_{n-2}^1 & & f_{n-1}^1 & \\ f_0^0 & & f_1^0 & \cdots & \cdots & \cdots & f_{n-1}^0 & & f_n^0 \end{array}$$

where each f_q^p is a p-derivation with respect to the collection $\{f_j^i\}$ with f_j^i from the triangle with f_q^p on the top, such, that $f_q^p i = \alpha_q^p$.

Proof. Here is the construction of f_0^n (the extension rule for α_0^n):

$$f_0^n \mid \otimes mV = \sum_{p=1}^{\min(n,m)} \sum_{k_1+k_2+\cdots+k_p=n} \sum_{m_1+m_2+\cdots+m_{p+1}=m-p}$$

$$\mu^m[(\alpha^0_{k_1+k_2+\cdots+k_p})^{m_1} \otimes \alpha^{k_p}_{k_1+k_2+\cdots+k_{p-1}} \otimes (\alpha^0_{k_1+k_2+\cdots+k_{p-1}})^{m_2} \otimes \cdots$$
$$\otimes \alpha^{k_3}_{k_1+k_2} \otimes (\alpha^0_{k_1+k_2})^{m_{p-1}} \otimes \alpha^{k_2}_{k_1} \otimes (\alpha^0_{k_1})^{m_p} \otimes \alpha^{k_1}_0 \otimes (\alpha^0_0)^{m_{p+1}}], \qquad (4)$$

here $k_i \geq 1, m_i \geq 0$, and $(\alpha_i^0)^0$ means omitting of this factor. The rest is routine verification. Note, that even $\alpha_0^n = 0$, it's extension f_0^n can be nonzero. □

1.3. DG-derivations. Now we suppose that (A, d) and (B, d) are *differential* graded algebras (DG-algebras) with differentials of degree-1.

Let us define a DG-0-derivation as a 0-derivation $f : A \to B$ with additional property $Df = 0$, (here D is the differential in $\mathrm{Hom}(A, B)$: $Df = d_B f - (-1)^{\deg f} f d_A$). In fact this means that f is just a multiplicative chain map.

Suppose now that two DG-derivations $f_0^0, f_1^0 : A \to B$ are fixed. A $DG-1-$derivation with respect to (f_0^0, f_1^0) we define as a 1-derivation $f_0^1 : A \to B$ with additional property $Df_0^1 = f_0^0 - f_1^0$. This notion coincides with the obvious notion of derivation homotopy.

Generally we introduce the following inductive

Definition 2. Suppose a following data is given:

$$\begin{array}{ccccccccccc} & & & f_0^{n-1} & & f_1^{n-1} & & & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & & & \\ & f_0^1 & & f_1^1 & \cdots & f_{n-2}^1 & & & f_{n-1}^1 & & \\ f_0^0 & & f_1^0 & \cdots & \cdots & \cdots & & f_{n-1}^0 & & & f_n^0, \end{array} \qquad (5)$$

where each f_q^p is a $DG-p-$derivation with respect to the collection $\{f_j^i\}$ with f_j^i from the triangle with f_q^p on the top. A $DG-n-$ derivation is an n-derivation with respect to this data $f_0^n : A \to B$ which, in addition, satisfies the condition

$$Df_0^n = f_0^{n-1} + (-1)^n f_1^{n-1}. \qquad (6)$$

Proposition 2. *Suppose $A = T(V)$ and $f_0^n : T(V) \to B$ be an n-derivation with respect to data* (5)*, then f_0^n is a DG-n-derivation if and only if the condition* (6) *is satisfied for generators, i. e. if*

$$Df_0^n|V = (f_0^{n-1} + (-1)^n f_1^{n-1})|V.$$

Proof. The proof follows from the fact, that Df_0^n and $f_0^{n-1}+(-1)^n f_1^{n-1}$ have same decomposition rules:

$$\begin{gathered}Df_0^n\mu=\mu\Sigma_{i=0}^n(Df_i^{n-i}\otimes f_0^i+(-1)^{n-i}f_i^{n-i}\otimes Df_0^i)=\\=\mu[\Sigma_{i=0}^{n-1}((f_i^{n-i-1}+(-1)^{n-i}f_{i+1}^{n-i-1})\otimes f_0^i)+\Sigma_{i=1}^n(-1)^{n-i}f_i^{n-i}\otimes\\(f_0^{i-1}+(-1)^i f_1^{i-1})]=\mu[\Sigma_{i=0}^{n-1}f_i^{n-i-1}\otimes f_0^i+\Sigma_{i=0}^{n-1}(-1)^{n-i}f_{i+1}^{n-i-1}\otimes f_0^i+\\+\Sigma_{i=1}^n(-1)^{n-i}f_i^{n-i}\otimes f_0^{i-1}+\Sigma_{i=1}^n(-1)^n f_i^{n-i}\otimes f_1^{i-1}]=\\=(f_0^{n-1}+(-1)^n f_1^{n-1})\mu.\quad\Box\end{gathered}$$

1.4. Twisting elements. Now we suppose that A is a *free* $DG-$algebra, i. e. $A=T(V)$ with some differential d. In this section we shall find the conditions which should satisfy a data (3) in order the induced n-derivation $f_0^n:T(V)\to B$ to be a DG-n-derivation.

Let us first analyze the structure of a differential $d:T(V)\to T(V)$ on $T(V)$ (see for example [7] or [6]).

Since this differential is required to be a derivation of degree -1, it is determined by its restriction

$$\beta=d\mid V:V\longrightarrow T(V),$$

which, in fact, consists of components

$$\beta=\{\beta_i:V\longrightarrow\otimes^i V)$$

and the extension rule is

$$d\mid\otimes^m V=\sum_{k=1}^m\sum_{i=1}^\infty(id)^{k-1}\otimes\beta_i\otimes(id)^{m-k}.$$

The composite dd is a derivation of degree -2, whence $dd=0$ if and only if the restriction $dd\mid V=0$. This condition for the collection $\{\beta_i\}$ means

$$\sum_{i+j=n}\sum_{k=1}^j[(id)^{k-1}\otimes\beta_i\otimes(id)^{j-k}]\beta_j=0$$

for all $n\geq 1$. Note that shifting dimensions $(V;\{\beta_i\})$ is an $A_\infty-$coalgebra, see [10].

Since of the universal property an n-derivation $f_0^n:T(V)\to B$ is uniquely determined by a by a data (3). We are interested what aditional condition should satisfy this data in order f_0^n to be a DG-n-derivation?

Definition 3. A data (3) we call n-twisting if the following condition is satisfied:

$$\begin{gathered}d\alpha_o^n+\sum_{p=1}^{\min(n,m)}\sum_{k_1+k_2+\cdots+k_p=n}\sum_{m_1+m_2+\cdots+m_{p+1}=m-p}\\\mu^m[(\alpha^0_{k_1+k_2+\cdots+k_p})^{m_1}\otimes\alpha^{k_p}_{k_1+k_2+\cdots+k_{p-1}}\otimes(\alpha^0_{k_1+k_2+\cdots+k_{p-1}})^{m_2}\otimes\cdots\\\otimes\alpha^{k_3}_{k_1+k_2}\otimes(\alpha^0_{k_1+k_2})^{m_{p-1}}\otimes\alpha^{k_2}_{k_1}\otimes(\alpha^0_{k_1})^{m_p}\otimes\alpha_0^{k_1}\otimes(\alpha_0^0)^{m_{p+1}}]\beta m=\\=\alpha_0^{n-1}+(-1)^n\alpha_1^{n-1}.\end{gathered}\tag{7}$$

Proposition 3. *The n-derivation f_0^n induced by a data* (3) *is a DG-n-derivation if and only if* (3) *is n-twisting.*

Proof. Since of Proposition 2 Df_0^n and $f_0^{n-1}+(-1)^n f_1^{n-1}$ coincide, iff they coincide on generators, i. e. iff $Df_0^n \mid V = (f_0^{n-1}+(-1)^n f_1^{n-1}) \mid V$. This condition for generating homomorphisms $\{\alpha_i\}$ means exactly (7). ☐

2. DG-Hopf Algebras with Steenrods ∇_i-coproducts

2.1. Steenrods ∇_i coproducts. We are going to study objects of type

$$(C, d, \nabla_0, \nabla_1, \nabla_2, \dots)$$

where (C, d, ∇_0) is a DG-coalgebra (with $\deg d = -1$) and $\nabla_i : C \to C \otimes C,\ i > 0$, satisfies

$$\deg \nabla_i = i,\ D\nabla_i = \nabla_{i-1} + (-1)^n T\nabla_{i-1}, \tag{8}$$

here D is the differential in $\mathrm{Hom}(C, C \otimes C)$ and $T : C \otimes C \to C \otimes C$ is the permutation map $T(a \otimes b) = (-1)^{\dim a \cdot \dim b} b \otimes a$. Such an object we shall call *DG colagebra with Steenrods ∇_i- coproducts.*

The main example is chain complex $C_*(X)$ with ∇_i dual to Steenrods $\smile_i$ −products.

Suppose that, in addition, on C an associative multiplication $\mu : C\otimes C \to C$ is given, which turns (C, d, μ) into a DG algebra, i. e. d is a derivation with respect to μ.

We are interested what kind of compatibility of ∇_i-s with μ is natural to require.

Usualy for $\nabla_0 : C \to C \otimes C$ it is required the multiplicativity, i. e.

$$\nabla_0\mu = \mu_{C\otimes C}(\nabla_0 \otimes \nabla_0),$$

here $\mu_{C\otimes C} = (\mu \otimes \mu)(id \otimes T \otimes id)$ is the induced multiplication on $C \otimes C$. This condition means, that (C, d, μ, ∇_0) is a *DG-Hopf algebra.* The example of such structure is $C_*(\Omega X)$.

As for the next cooperation, in [3] it is requred that ∇_1 should be a $(\nabla_0, T\nabla_1)$-derivation, i. e.

$$\nabla_1\mu = \mu C \otimes C(\nabla_1 \otimes \nabla_0 + T\nabla_0 \otimes \nabla_1).$$

Let us mention that an object $(C, d, \mu, \nabla_0, \nabla_1)$ is a HAH in the sense of [An], homotopy Hopf algebra in sense of [3] and a $B_\infty-$algebra in sense of [5].

The compatibility of higher ∇_i-s with the product μ is described in the following

Definition 4. A $DG-$Hopf algebra with Steenrods ∇_i coproducts

$$(C, d, \mu, \nabla_0, \nabla_1, \nabla_2, \dots)$$

we define as a $DG-$algebra (C, d, μ) with given sequence of cooperations

$$\{\nabla_i : C \longrightarrow C \otimes C, i = 0, 2, \dots\}$$

so that each ∇_n is a $DG-$ $n-$derivation with respect to data (5) with $f_q^p = T^q \nabla_p$, i. e. with respect to data

$$\begin{array}{ccccccccc}
 & & & \nabla_{n-1} & & T\nabla_{n-1} & & & \\
 & & \dots & \dots & \dots & \dots & \dots & & \\
 & \nabla_1 & & T\nabla_1 & \cdots & T^{n-2}\nabla_1 & & T^{n-1}\nabla_1 & \\
\nabla_0 & & T\nabla_0 & \cdots & \cdots & \cdots & T^{n-1}\nabla_0 & & T^n\nabla_0.
\end{array} \tag{9}$$

In other words cooperations $(\nabla_0, \nabla_1, \nabla_2, \dots)$ shall satisfy the conditions (see (6) and (2)):

$$cD\nabla_n = \nabla_{n-1} + (-1)^n T\nabla_{n-1} \tag{10}$$

and

$$\nabla_n \mu = \sum_{k=0}^{n} \mu_{C\otimes C}(\nabla_k \otimes T^k \nabla_{n-k}). \tag{11}$$

The advantage of this definition is that if $C = T(V)$ then it suffices to define the cooperations ∇_i on generators and extend them by the rule (4), then the needed conditions will be automatically satisfied.

Namely, from the Proposition 3 follows the

Proposition 4. *Suppose for a sequence of homomorphisms*

$$\{\alpha^0, \alpha^1, \dots;\ \alpha^i \in \mathrm{Hom}^i(V, T(V) \otimes T(V))\}$$

the sequence $\{\alpha_q^p = T^q \alpha^p\}$ *satisfies the condition* (7). *Then the sequence of cooperations* $\{\nabla_i\}$ *where each* ∇_n *is a* $DG - n$*-derivation, obtained by extension of* $\{\alpha^i\}$ *by the rule* (4) *forms on* $(T(V), d)$ *a structure of* $DG-$*Hopf algebra with Steenrods* ∇_i *coproducts.*

2.2. Steenrods ∇_i coproducts in cubical chain complex. First we construct cooperations ∇_i for the DG-algebra $I(\infty)$, which was used by Baues [3] for the construction of ∇_0 and ∇_1.

Let $I(\infty)$ be the free graded algebra generated by three elements e, e', e'' with degree $\deg e' = 0$, $\deg e'' = 0$, $\deg e = 1$. An element of $I(\infty)$ we shall write as $e_1 \cdots e_2 \cdots e_n$ with $e_i = e', e''$ or e. The differential $d : I(\infty) \to I(\infty)$ on generators is given by $d(e') = d(e'') = 0, d(e) = e'' - e'$ and is extended as a derivation.

We want to define a sequence of cooperations

$$\{\nabla_i : I(\infty) \to I(\infty) \otimes I(\infty),\ i = 0, 1, 2, \dots\}$$

satisfying the conditions (10) and (11).

Let's define a sequence of cooperations $\{\alpha^i\}$ on generators by

$$
\begin{gathered}
\alpha^0(e')=e'\otimes e',\ \alpha^0(e'')=e''\otimes e'',\ \alpha^0(e)=e'\otimes e+e\otimes e'';\\
\alpha^1(e')=0,\ \alpha^1(e'')=0,\ \alpha^1(e)=e\otimes e;\\
\alpha^i(e')=0,\ \alpha^i(e'')=0,\ \alpha^i(e)=0\quad for\quad i>2,
\end{gathered}
\tag{12}
$$

thus the suitable data (3) $\{\alpha^p_q=T^q\alpha^p\}$ now looks as

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 0 & & 0 & & & \\
 & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
 & & 0 & \cdots & \cdots & \cdots & 0 & & \\
 & \alpha^1 & & T\alpha^1 & \cdots & T^{n-2}\alpha^1 & & T^{n-1}\alpha^1 & \\
\alpha_0 & & T\alpha^0 & \cdots & \cdots & \cdots & T^{n-1}\alpha^0 & & T^n\alpha^0.
\end{array}
\tag{13}
$$

It is not hard to check that the this data satisfies the condition (7). Extending (13) by the rule (4), according to Proposition 4, we obtain the sequence $\{\nabla_i\}$ which forms on $I(\infty)$ a structure of DG-Hopf algebra with Steenrods ∇_i products.

Here are the formulae of ∇_n-s for the elements $e^m=e\cdot\cdots\cdot e\in I(\infty)$:

$$
\begin{gathered}
\nabla_n(e^m)=\sum_{p_1+\cdots+p_{n+1}+n=m}(T^n\alpha^0(e))^{p_{n+1}}\cdots T^{n-1}\alpha^1(e)\cdots(T^{n-1}\alpha^0(e))^{p_n}\times\\
\times T^{n-2}\alpha^1(e)\cdots(T^1\alpha^0(e))^{p_2}\cdot T^0\alpha^1(e)\cdot(T^0\alpha^0(e))^{p_1}=\\
=\sum_{p_1+\cdots+p_{n+1}+n=m}(T^n(e'\otimes e+e\otimes e''))^{p_{n+1}}\cdot T^{n-1}(e\otimes e)\times\\
\times(T^{n-1}(e'\otimes e+e\otimes e''))^{p_n}\cdot T^{n-2}(e\otimes e)\cdots(T^1(e'\otimes e+e\otimes e''))^{p_2}\times\\
\times T^0(e\otimes e)\cdot(T^0(e'\otimes e+e\otimes e''))^{p_1}
\end{gathered}
\tag{14}
$$

here $p_i\geq 0$ and $(e'\otimes e+e\otimes e'')^0$ means omitting of this factor.

Let us change the notation in order to make it closer to standard one. The cube I^n we represent as $(t,t,\ldots,t)$ where t is variable from $I=[0,1]$. For example I^2 is represented as (t,t) and it's 1-faces as $(0,t),(1,t),(t,0),(t,1)$.

Denote $e'=0, e''=1, e=t$. For a word $e_1\cdots e_n\in I(\infty)$ with $e_k=e',e''$ or e we shall write $(i_1,\ldots,i_n)$ with $i_k=0,1$ or t.

The above formula (14) for $\nabla_n(t^m)$ now looks as

$$
\begin{gathered}
\nabla_n(t,\ldots,t)=\sum_{p_1+\cdots+p_{n+1}+n=m}(T^n(0\otimes t+t\otimes 1))^{p_{n+1}}\cdot T^{n-1}(t\otimes t)\cdot\\
(T^{n-1}(0\otimes t+t\otimes 1))^{p_n}\cdot T^{n-2}(t\otimes t)\cdot\\
\cdots(T^1(0\otimes t+t\otimes 1))^{p_2}\cdot T^0(t\otimes t)\cdot(T^0(0\otimes t+t\otimes 1))^{p_1}.
\end{gathered}
\tag{15}
$$

Particularly for ∇_0 we have

$$
\nabla_0(t,\ldots,t)=(0\otimes t+t\otimes 1)^m,
$$

this formula can be rewritten in another form. Let us introduce two maps $\varphi_0, \varphi_1 : \{0,1\} \to \{0,1,t\}$ given by

$$\varphi_0(0) = 0, \varphi_0(1) = t; \varphi_1(0) = t, \varphi_1(1) = 1.$$

Then it is possible to show by induction, that

$$\nabla_0(t,\dots,t) = \sum_{(i_1,\dots,i_n)} [\varphi_0(i_1),\dots,\varphi_0(i_n)] \otimes [\varphi_1(i_1),\dots,\varphi_1(i_n)]$$

where the summation is taken over all vertexes $(i_1,\dots,i_n)$, $i_k = 0,1$, of I^n. This coproduct coincides with one, given in [9] and [3].

For ∇_1 we have

$$\nabla_1(t,\dots,t) = \sum_{p_1+p_2+1=m} (t\otimes 0 + 1 \otimes t))^{p_1} \cdot (t \otimes t) \cdot (0 \otimes t + t \otimes 1)^{p_2}.$$

Using above terminology, this formula lookes as:

$$\nabla_1(t,\dots,t) = \\ = \textstyle\sum_{(i_1,\dots,i_{k-1},t,i_{k+1},\dots,i_n)} (\varphi_1(i_1),\dots,\varphi_1(i_{k-1}),t,\varphi_0(i_{k+1}),\dots,\varphi_0(i_n)) \otimes \\ (\varphi_0(i_1),\dots,\varphi_0(i_{k-1}),t,\varphi_1(i_{k+1}),\dots,\varphi_1(i_n))$$

where the summation is taken over all 1-faces $(i_1,\dots,i_{k-1},t,i_{k+1},\dots,i_n)$, $i_k = 0,1$, of I^n. This coproduct coincides with one given in [3].

Generally, the formula (15) lookes as:

$$\begin{aligned}\nabla_s(t,\dots,t) = \textstyle\sum (\varphi_s(i_1),\dots,\varphi_s(i_{k_1-1}),t,\varphi_{s-1}(i_{k_1+1}),\dots,\varphi_{s-1}(i_{k_2-1}),t,\\ \varphi_{s-2}(i_{k_2+1}),\dots,\varphi_1(i_{k_s-1}),t,\varphi_0(i_{k_s+1}),\dots,\varphi_0(i_m)) \otimes \\ (\varphi_{s+1}(i_1),\dots,\varphi_{s+1}(i_{k_1-1}),t,\varphi_s(i_{k_1+1}),\dots,\varphi_s(i_{k_2-1}),t,\\ \varphi_{s-1}(i_{k_2+1}),\dots,\varphi_2(i_{k_s-1}),t,\varphi_1(i_{k_s+1}),\dots,\varphi_1(i_m))\end{aligned} \tag{16}$$

where the summation is taken over all $s-$faces

$$(i_1,\dots,i_{k_1-1},t,i_{k_1+1},\dots,i_{k_2-1},t,i_{k_2+1},\dots,i_{k_s-1},t,i_{k_s+1},\dots,i_m), \\ i_k = 0,1$$

of I^m and $\varphi_k = \varphi_{\mathrm{kmod}\,2}$.

Suppose now that X is a topological space and $CU_*(X)$ be it's singular cubical chain complex, i. e. $CU_m(X)$ is a free module, generated by singular cubes $\sigma^m : I^m \to X$. Then the coproducts $\nabla_s : I(\infty) \to I(\infty) \otimes I(\infty)$, given by (16) induce the coproducts $\bigtriangledown_s : CU_*(X) \to CU_*(X) \otimes CU_*(X)$: by formula (16) $\nabla_s(t,\dots,t) = \nabla(I^m)$ is the sum of tensor products of some faces of I^m, define $\bigtriangledown_s \sigma^n = (\sigma^n \otimes \sigma^n)\nabla_s$.

Remark. The formula (16) can be rewritten in terms of boundary operators $d_i^{\epsilon}, \epsilon = 0, 1$ of a cubical set Q:

$$\begin{aligned} \bigtriangledown_s(\sigma^n) = \sum d_1^{\varphi_s(i_1)} d_2^{\varphi_s(i_2)} \cdots d_{i_{k_1-1}}^{\varphi_s(i_{k_1-1})} d_{i_k}^t d_{i_{k_1+1}}^{\varphi_{s-1}(i_{k_1+1})} \cdots \\ d_{i_{k_2-1}}^{\varphi_{s-1}(i_{k_2-1})} d_{i_{k_2}}^t d_{i_{k_2+1}}^{\varphi_{s-2}(i_{k_2+1})} \cdots d_{i_{k_s-1}}^{\varphi_1(i_{k_s-1})} d_{i_{k_s}}^t d_{i_{k_s+1}}^{\varphi_0(i_{k_s+1})} \cdots \\ d_n^{\varphi_0(i_n)} \sigma^n \otimes d_1^{\varphi_{s+1}(i_1)} d_2^{\varphi_{s+1}(i_2)} \cdots d_{i_{k_1-1}}^{\varphi_{s+1}(i_{k_1-1})} d_{i_k}^t d_{i_{k_1+1}}^{\varphi_s(i_{k_1+1})} \cdots \\ d_{i_{k_2-1}}^{\varphi_s(i_{k_2-1})} d_{i_{k_2}}^t d_{i_{k_2+1}}^{\varphi_{s-1}(i_{k_2+1})} \cdots d_{i_{k_s-1}}^{\varphi_2(i_{k_s-1})} d_{i_{k_s}}^t d_{i_{k_s+1}}^{\varphi_1(i_{k_s+1})} \cdots d_n^{\varphi_1(i_n)} \sigma^n, \quad (17) \end{aligned}$$

here we assume $d_k^t = id$, and, as above the summation is taken over all $s-$faces

$$(i_1, \ldots, i_{k_1-1}, t, i_{k_1+1}, \ldots, i_{k_2-1}, t, i_{k_2+1}, \ldots, i_{k_s-1}, t, i_{k_s+1}, \ldots, i_n),\ i_k = 0, 1$$

of I^n and $\varphi_k = \varphi_{\mathrm{kmod}\,2}$. This formula determines the coproducts $\bigtriangledown_s$ not only in $CU_*(X)$ but in chain complex $C_*(Q)$ of an arbitrary cubical set Q.

In [8] it is shown, that the cobar construction $\Omega\bar{C}_*(X)$ of the singular chain complex of an 1-connected space X coincides with the chain complex of certain cubical set Q, thus there appear on the cobar construction the coproducts $\bigtriangledown_s$ given by (17). Moreover, there exists a map of cubical sets $Q \to Q(\Omega X)$, where $Q(\Omega X)$ is the cubical set of singular cubs of the loop space ΩX. It is clear that it preserves boundary operators d_i^{ϵ} and, consequently, coproducts $\bigtriangledown s$, thus these coproducts are geometric. Note, that $\bigtriangledown_0$ and $\bigtriangledown_1$ of $\Omega C_*(X)$ coincide with coproducts, constructed by Baues in [2] and [3].

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(Received 14.01.1999)

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