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ON THE DERIVATIVE OF A CONFORMALLY MAPPING FUNCTION

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Abstract. Using the methods of solution of the boundary value problem of linear conjugation, we study properties of a derivative function which maps conformally the unit circle on the finite simply connected domain whose boundary belong to certain subclasses of rectifiable lines.

Everywhere in the present paper D is a finite simply connected domain of a complex plane, bounded by a simple rectifiable curve Γ, $w=\omega(z)$ is a function mapping conformally the unit circle U ($|z|<1$) on D, and γ ($|z|=1$) is the boundary of the circle U.

Many works are devoted to the investigation of the derivative of a conformally mapping function ω (see, for e.g., [1], Ch. X, [2–6]). Following [7, 8] and using the method of solution of the boundary value problem of linear conjugation we investigate boundary properties of functions ω' in the case, where Γ belongs to some subclasses of rectifiable curves.

The main result of the present paper is Theorem 1. Its formulation and proof are given in §3. Remaining statements (cited in §4) including the well-known theorems of Lindelöf, Kellog, Smirnov ([1], Ch. X, §6) and Warschawski [5] are more or less obvious corollaries of Theorem 1, or they are proved by means of this theorem.

1. It is known ([10], Ch. III, §1) that if an analytic function ω maps conformally the unit circle U on a finite simply connected domain D bounded by a simple rectifiable curve Γ, then the derivative ω', which is different from zero everywhere in U, belongs to the Hardy class H_1 and hence for almost

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all $\theta \in [0, 2\pi]$ (in particular for those θ for which $\omega(e^{i\theta})$ is differentiable) there exists an angular boundary value of the function $\omega'(z)$, and

$$\lim_{z \overset{\wedge}{\to} e^{i\theta}} \omega'(z) = \omega(e^{i\theta}) = ie^{i\theta} \frac{d\omega(e^{i\theta})}{d\theta}. \tag{1}$$

Let $\omega'(0) > 0$ and $\lg \omega'(z) = \lg |\omega'(z)| + i \arg \omega'(z)$ be a branch of an analytic in U function for which $\arg \omega'(0) = 0$. Then for all $\theta \in [0, 2\pi]$ for which $\frac{d\omega(e^{i\theta})}{d\theta} \neq 0$, taking into account (1), we can write

$$\lim_{|z| \to 1} \arg \omega'(z) = \arg \omega'(e^{i\theta}) = \arg \frac{d\omega(e^{i\theta})}{d\theta} - \theta - \frac{\pi}{2}, \tag{2}$$

where $\arg \frac{d\omega(e^{i\theta})}{d\theta}$ is the quantity of the angle, made by the tangent to Γ at the point $\omega(e^{i\theta})$ and the Ox-axis, defined by equation (1) with regard for the fact that $\arg \omega'(0) = 0$.

2. Let the domain D belong to the Smirnov class ([10], Ch. III, §§12–15, or [1], Ch. X, §6). This means that

$$\omega'(z) = e^{\frac{1}{2\pi} \int_0^{2\pi} \lg |\omega'(e^{i\sigma})| \frac{e^{i\sigma}+z}{e^{i\sigma}-z} d\sigma} =$$

$$= [\omega'(0)]^{-1} e^{\frac{1}{\pi i} \int_{|\tau|=1} \frac{\lg |\omega'(\tau)|}{\tau - z} d\tau}, \quad \tau = e^{i\sigma}. \tag{3}$$

Consequently,

$$\lg \omega'(z) = \frac{1}{2\pi} \int_0^{2\pi} \lg |\omega'(e^{i\sigma})| \frac{e^{i\sigma} + z}{e^{i\sigma} - z} d\tau \tag{4}$$

(here we again mean a branch for which $\arg \omega'(0) = 0$).

Separating in (4) the real and imaginary parts, we obtain

$$\arg \omega'(z) = -\frac{1}{2\pi} \int_0^{2\pi} \lg |\omega'(e^{i\sigma})| \frac{2r \sin(\sigma - \theta) d\sigma}{1 - 2r \cos(\sigma - \theta) + r^2}, \quad z = re^{i\theta}, \tag{5}$$

whence passing to the limit and taking into account (2) we obtain

$$\lim_{|z| \to 1} \arg \omega'(z) = \arg \omega'(e^{i\theta}) = \widetilde{\lg} |\omega'(e^{i\theta})| = \arg \frac{d\omega(e^{i\theta})}{d\theta} - \theta - \frac{\pi}{2}, \tag{6}$$

where

$$\widetilde{\lg} |\omega'(e^{i\theta})| = -\frac{1}{2\pi} \int_0^{2\pi} \lg |\omega'(e^{i\sigma})| \operatorname{ctg} \frac{\sigma - \theta}{2} d\sigma$$

Let $\lg \omega'(z) \in H_1$ (i.e., $\widetilde{\lg}|\omega'(e^{i\theta})| \in L(0,2\pi)$). Then by the Cauchy formula we have

$$\lg \omega'(0) = \frac{1}{2\pi}\int_0^{2\pi}[\lg|\omega'(e^{i\sigma})| + i\arg\omega'(e^{i\sigma})]d\sigma.$$

Consequently,

$$\lg|\omega'(0)| = \frac{1}{2\pi}\int_0^{2\pi}\lg|\omega'(e^{i\sigma})|d\sigma$$

(i.e., if $\lg\omega'(z) \in H_1$, then the domain D belongs to the Smirnov class) ([1], Ch. X, §6). Moreover,

$$\frac{1}{2\pi}\int_0^{2\pi}\arg\omega'(e^{i\sigma})d\sigma = \arg\omega'(0). \tag{7}$$

For $\arg\omega'(0) = 0$ the equality (7) results in

$$\int_0^{2\pi}\widetilde{\lg}|\omega'(e^{i\sigma})|d\sigma = \int_0^{2\pi}\arg\omega'(e^{i\sigma})d\sigma = 0.$$

Using the equality

$$\int_\gamma \varphi(t)dt\int_\gamma \frac{f(\tau)d\tau}{\tau - t} = \int_\gamma f(\tau)d\tau\int_\gamma \frac{\varphi(t)dt}{\tau - t}, \tag{8}$$

where f and $\int_\Gamma \frac{f(\tau)d\tau}{\tau-t}$ are summable on γ, and φ satisfies the Hölder condition (see, for e.g., [11]), from (3) we obtain

$$\omega'(z) = \omega'(0)e^{\frac{1}{\pi}\int_{|\tau|=1}\frac{\arg\omega'(\tau)}{\tau-z}d\tau}. \tag{9}$$

Applying now (7), (9) and the following obvious equality

$$\frac{i}{2\pi}\int_0^{2\pi}\arg\omega'(\tau)\frac{\tau+z}{\tau-z}d\sigma =$$
$$= \frac{1}{\pi}\int_\gamma\frac{\arg\omega'(\tau)}{\tau - z}d\tau - \frac{i}{2\pi}\int_0^{2\pi}\arg\omega'(e^{i\sigma})d\sigma, \tag{10}$$

we can write

$$\lg\omega'(z) = \lg\omega'(0) + \frac{i}{2\pi}\int_0^{2\pi}\arg\omega'(e^{i\sigma})\frac{e^{i\sigma}+z}{e^{i\sigma}-z}d\sigma.$$

Separating in the last equality the real and imaginary parts, we get

$$\arg\omega'(z)=\frac{1}{2\pi}\int\limits_0^{2\pi}\arg\omega'(e^{i\sigma})\frac{(1-r^2)d\sigma}{1-2r\cos(\sigma-\theta)+r^2}; \tag{11}$$

(equality (11) follows from the Fichtenholz theorem as well ([10], p. 97).

Remark 1. Since $\widetilde{lg}|\omega'(e^{i\theta})|=\arg\omega'(e^{i\theta})$ is not always summable, for the validity of formulas (9) and (11) we have required the condition $\lg\omega'(z)\in H_1$ (which is equivalent to the condition $\arg\omega'(e^{i\theta})\in L(0,2\pi)$). If we replace (8) by the equality

$$(\widetilde{L})\int\limits_\gamma\varphi(t)dt\int\limits_\Gamma\frac{f(\tau)d\tau}{\tau-t}=\int\limits_\Gamma f(\tau)d\tau\int\limits_\gamma\frac{\varphi(t)dt}{\tau-t},$$

where f is summable, φ satisfies the Hölder condition on γ and $(\widetilde{L})\int$ is the integral being an extension of the Lebesgue integral (in which sense the function conjugate to the summable function is integrable and its integral is equal to zero) (see [11] or [12], pp. 38, 88) then formulas (9) and (11) can be obtained in a general case, i.e., when D is Smirnov's domain (without requirement $\lg\omega'(z)\in H_1$) and using in formulas (9) and (11) the $\widetilde{L}$-integral instead of the Lebesgue integral.

Using formulas (9) and (11) and also the well-known Lindelöf theorem and its generalizations [1–4], we can establish some properties of the function ω' for some classes of domains D (for example, for the domains with smooth or piecewise smooth boundaries, when (by Lindelöf theorem) the function $\arg\omega'(z)$ is respectively continuous or piecewise continuous).

But we shall take advantage of the other method which is again based on the representations (9) and (11). If in formulas (9) and (11) the density of the Cauchy type (and Poisson) integral $\arg\omega'(e^{i\theta})$ is obtained in terms of the boundary value of the harmonic in U function $\arg\omega'(z)$, then we can construct it on the basis of geometrical properties of the curve Γ.

3. Let Γ satisfy one of the following three conditions:

(1) Γ is a smooth curve, i.e., it has at every point a tangent which depends continuously on the point $t\in\Gamma$. Then one can define a continuous on $[0,2\pi]$ function $\alpha(e^{i\theta})$ representing a slope angle of the tangent to Γ at the point $\omega(e^{i\theta})$ to the real axis. Since $\alpha(2\pi)-\alpha(0)=2\pi$ ([13], Ch. III, §§18,19), the function $\alpha^*(e^{i\theta})=\alpha(e^{i\theta})-\theta-\frac{\pi}{2}$ is continuous, 2π periodic on $[0,2\pi]$ (that is, continuous on the circumference γ).

(2) Γ is a piecewise smooth curve with angular points t_k, $k=1,2,\dots,n$ and angle values $\pi\nu_k$, $0\le\nu_k<2$ respectively, where under the angle with the vertex $t_k\in\Gamma$ is meant the angle between the right and left tangents to Γ at the point t_k, interior with respect to D. In this case, similarly, as in the previous case, one can determine piecewise continuous functions

$\alpha(e^{i\theta})$ (whose increment in passing around γ is equal to 2π) and $\alpha_*(e^{i\theta})$ with discontinuities at the points θ_k, $\omega(e^{i\theta_k}) = t_k$ and jump values $\pi h_k = \pi - \pi\nu_k$.

(3) Local oscillation of the tangent to Γ is less than π. This means that every point $t \in \Gamma$ possesses on Γ a neighbourhood for an arbitrary pair of points (we mean those points at which there is a tangent), and the angle between positive directions of the corresponding to them tangents is less than π. Obviously, a set of above-mentioned intervals covers Γ; one can select from this set a finite subset which again covers Γ.

The function $\alpha(e^{i\theta})$ can be defined almost for all $\theta \in [0, 2\pi]$ as follows: Let $\omega_0 = \omega(e^{i\theta_0})$ be an arbitrary point of Γ at which there exists a tangent. Choosing at the point ω_0 any value $\alpha_0 = \alpha(e^{i\theta_0})$ of the angle made by the tangent and the Ox-axis, we define the values $\alpha(e^{i\theta})$ at the remaining points of the interval containing ω so that for every pair of points $\omega(e^{i\theta_1})$ and $\omega(e^{i\theta_2})$ (from the same interval) the equality

$$|\alpha(e^{i\theta_1}) - \alpha(e^{i\theta_2})| < \pi \tag{12}$$

is fulfilled.

Since the neighboring intervals of the finite system covering Γ intersect, we can extend subsequently the process of defining $\alpha(e^{i\theta})$ to all intervals, and hence to the entire Γ.

Having defined $\alpha(e^{i\theta})$ at the points with no tangent and retaining condition (12), we can obtain a bounded on $[0, 2\pi]$ function. Similarly to the aforementioned cases, since at the points $\omega(e^{i\theta})$ with a tangent we have $\alpha(e^{i(\theta+2\pi)}) - \alpha(e^{i\theta}) = 2\pi$ ([13], Ch. III, §§18,19), the 2π-periodic function $\alpha_*(e^{i\theta}) = \alpha(e^{i\theta}) - \theta - \frac{\pi}{2}$ has local oscillation which is less than π, that is, at every point $e^{i\theta_0}$

$$\lim_{\varepsilon\to 0}\Big[\sup_{\theta\in(\theta_0-\varepsilon,\theta_0+\varepsilon)} \alpha_*(e^{i\theta}) - \inf_{\theta\in(\theta_0-\varepsilon,\theta_0+\varepsilon)} \alpha_*(e^{i\theta})\Big] < \pi.$$

Theorem 1. *If Γ, satisfying one of the above three conditions, is smooth, piecewise smooth and contains interior angles $\pi\nu_k$, $0 \le \nu_k < 2$ or if the local oscillation of the tangent to Γ is less that π, then*

$$\omega'(z) = \omega'(0)\exp\left\{\frac{1}{\pi}\int\limits_{\gamma} \frac{\alpha_*(\tau)d\tau}{\tau - z}\right\} \tag{13}$$

and

$$\arg\omega'(z) = \frac{1}{2\pi}\int\limits_0^{2\pi} \alpha_*(e^{i\sigma})\frac{(1-r^2)d\sigma}{1 - 2r\cos(\sigma-\theta) + r^2} + 2n\pi, \quad z = re^{i\theta}, \tag{14}$$

where $\alpha_*(e^{i\theta}) = \alpha(e^{i\theta}) - \theta - \frac{\pi}{2}$ is the 2π-periodic function representing the angle of revolution of the tangent to γ at the point τ to the tangent Γ at the *point $\omega(\tau)$. For the conformal mapping, ω is respectively continuous, piecewise continuous or bounded on the interval $[0,2\pi]$, and its local oscillation is less than π.*

Proof. Consider the function which is analytic on the plane, cut along the unit circumference γ,

$$\Omega(z)=\begin{cases}\sqrt{\omega'(z)}, & |z|<1\\ \overline{\sqrt{\omega'(\frac{1}{\bar z})}}, & |z|>1\end{cases}$$

(for the sake of definiteness we shall mean a branch for which $\sqrt{\omega'(0)}>0$).

Because $\omega'\in H_1$, we have $\sqrt{\omega'(z)}\in H_2$, which is representable in U by the Cauchy type integral with density from the class $L_2(\Gamma)$. It can be easily verified that $\overline{\sqrt{\omega'(\frac{1}{\bar z})}}-\sqrt{\omega'(0)}$ is also representable outside $\overline{U}$ by the Cauchy type integral with density from the same class. Consequently, $\Omega-\Omega(0)$ is representable on the plane, cut along γ, by the Cauchy type integral with density from the class $L_2(\gamma)$.

Denote by $\Omega^+(\tau)$ and $\Omega^-(\tau)$ angular boundary values of the function Ω, at the point $\tau\in\gamma$, inside and outside γ. Taking into account equality (2), we can write

$$\Omega^+(\tau)=\lim_{z\to\tau}\sqrt{|\omega'(z)|e^{i\arg\omega'(z)}}=\sqrt{\omega'(\tau)}e^{i\frac{\arg\omega'(\tau)}{2}},$$

and analogously,

$$\Omega^-(\tau)=\sqrt{\omega'(\tau)}e^{-i\frac{\arg\omega'(\tau)}{2}}.$$

Since the both functions $\arg\omega'(\tau)$ and $\alpha_*(\tau)$ represent the angle of turn of the tangent to γ at the point τ with respect to tangent of Γ at the point $\omega(\tau)$, we write

$$G(\tau)\equiv\frac{\Omega^+(\tau)}{\Omega^-(\tau)}=e^{i\arg\omega'(\tau)}=e^{i\alpha_*(\tau)},\quad |\tau|=1. \tag{15}$$

Consider the function

$$\phi(z)=\exp\left\{\frac{1}{2\pi}\int_\gamma\frac{\alpha_*(\tau)d\tau}{\tau-z}\right\},\quad z\notin\gamma, \tag{16}$$

where $\alpha_*(\tau)$ satisfies one of the above three conditions.

Note that if ϕ is representable in U by the Cauchy type integral with density $L_p(\gamma)$, $p>1$, then $\phi(z)-1=\phi(0)\overline{\phi}\left(\frac{1}{\bar z}\right)-1$ is representable Outside $\overline{U}$ by the Cauchy type integral with the density from the same class. Consequently, if $p>1$, then $\phi(z)-1$ is representable on the plane, cut along γ, by the Cauchy type integral with density from $L_p(\gamma)$.

Using the Sokhotski-Plemelj formulas ([10], Ch. III, §§2,3), it is not difficult to show that the function ϕ (likewise the function Ω) satisfies the boundary condition

$$\phi^+(\tau) = G(\tau)\phi^-(\tau), \quad \tau \in \gamma, \tag{17}$$

where ϕ^+ and ϕ^- are the limit functions respectively inside and outside γ. From (15) and (17) we obtain

$$\Omega^+(\tau)(\phi^+(\tau))^{-1} = \Omega^-(\tau)(\phi^-(\tau))^{-1}. \tag{18}$$

Now, to satisfy ourselves that (13) is valid, it suffices to show that the function $\Omega(\phi)^{-1}$ is representable by the Cauchy type integral with a jump line. Indeed, if it is the case, then taking into consideration equality (18), we can conclude that Ω is analytic on the entire plane, and since it takes at infinity the value $\sqrt{\omega'(0)}$, we have $\Omega(z) = \sqrt{\omega'(0)}\phi(z)$ which for $|z| < 1$ leads to (13).

Consider now the questions dealing with the representation of the function $\Omega(\phi)^{-1} - \sqrt{\omega'(0)}$ by the Cauchy type integral.

If Γ is a smooth curve, then the function α_* is continuous on γ, and hence ϕ, $(\phi)^{-1} \in \bigcap_{p>0} H_p$ ([1], p. 402). This, by virtue of the above-said and the fact that Ω belongs to H_2, implies that the function $\Omega(\phi)^{-1} - \sqrt{\omega'(0)}$ is representable on the plane, cut along γ, by the Cauchy type integral with density from the class $L_{2-\varepsilon}(\gamma)$ (for arbitrarily small ε).

If Γ is a curve whose local oscillation of the tangent is less than π, then α_* is bounded on γ. For some $\delta > 0$ we have ϕ, $(\phi)^{-1} \in H_\delta$ ([14]). But since α_* is the function whose local oscillation is less than π, we obtain $(\phi^{-1})^+ \in L_2(\gamma)$, where $(\phi^{-1})^+$ is a limit function of $1/\phi$ ([15] or [12], p. 82). Consequently, by Smirnov's theorem ([10], p. 116), $(\phi)^{-1} \in H_2$. But then the function $\Omega(\phi)^{-1} - \sqrt{\omega'(0)}$ is representable on the plane, cut along γ, by the Cauchy type integral with density from $L_1(\gamma)$.

Let now Γ be a piecewise smooth curve (note that if a piecewise curve has no cusps, then local oscillations of its tangent is less than π). We show that under the conditions given above, for ($\nu_k < 2$) the function $\Omega(\phi)^{-1} - \sqrt{\omega'(0)}$ is representable by the Cauchy type integral.

Bearing this in mind, we consider on $[0, 2\pi]$ the 2π-periodic piecewise linear function χ whose graph is drawn up by rectilinear segments $[(\theta_k, \alpha_*(e^{i\theta_k^+})), (\theta_{k+1}, \alpha_*(e^{i\theta_{k+1}^-}))]$. $k = 1, 2, \dots, n$, $\theta_{n+1} = \theta_1 + 2\pi$ It is evident that χ has the same points of discontinuity θ_k ($t_k = \omega(e^{i\theta_k})$) with the same jumps $\pi h_k = \pi - \pi\nu_k$ ($\pi\nu_k$ is the size of the interior with respect to D angle with the vertex $t_k \in \Gamma$) as α_*. Therefore the difference $\alpha_* - \chi$ is the 2π-periodic continuous on $[0, 2\pi]$ function.

Taking the above arguments into consideration, we can write

$$\frac{1}{2\pi}\int\limits_{\gamma}\frac{\alpha_*(t)dt}{t-z}=\frac{1}{2\pi}\int\limits_{\gamma}\frac{(\alpha_*-\chi)(t)}{t-z}dt+\frac{1}{2\pi}\int\limits_{\gamma}\frac{\chi(t)dt}{t-z}=$$

$$=\frac{1}{2\pi}\int\limits_{\gamma}\frac{(\alpha_*-\chi)(t)}{t-z}dt+\sum_{k=1}^{n}\Big(-\frac{h_k}{2}\Big)\lg(c_k-z)+\Psi(z),\qquad(19)$$

where $t_k=\omega(c_k)=\omega(e^{i\theta_k})$, and Ψ is the function, analytic in U and continuous in $\overline{U}$ ([9], Ch. I, §26).

Using equation (19), we can represent the function ϕ as

$$\phi(z)=\prod_{k=1}^{n}(c_k-z)^{\frac{\nu_k-1}{2}}\phi_0(z),$$

where ϕ_0 and $(\phi_0)^{-1}$ belong in U to the classes H_p, for all $p>0$.

By the assumption, since $0\le\nu_k<2$, i.e., $1-\nu_k>-1$, it follows from the last representation of the function ϕ that $(\phi)^{-1}\in H_{2+\varepsilon}$ for some $\varepsilon>0$. Taking into account the fact that $\Omega\in H_2(U)$, we can conclude that the function $(\phi)^{-1}\Omega-1$ is representable on the plane, cut along γ, by the Cauchy type integral with density from the class $L_{1+\varepsilon_0}(\gamma)$ (again for some $\varepsilon_0>0$).

Thus the validity of formula (13) is proved in all three cases.

Now let us prove (14). Substituting in (10) instead of $\arg\omega'(e^{i\theta})$ the function α_* and applying (13), we can write

$$\lg\omega'(z)=\lg\omega'(0)+$$

$$+\frac{i}{2\pi}\int\limits_{0}^{2\pi}\alpha_*(\tau)\frac{\tau+z}{\tau-z}d\sigma+\frac{1}{2\pi}\int\limits_{0}^{2\pi}\alpha_*(e^{i\sigma})d\sigma+2k\pi,\qquad(20)$$

from which it follows that

$$\arg\omega'(z)=$$

$$=\frac{1}{2\pi}\int\limits_{0}^{2\pi}\alpha_*(e^{i\sigma})\frac{(1-r^2)d\sigma}{1-2r\cos(\sigma-\theta)+r^2}+\frac{1}{2\pi}\int\limits_{0}^{2\pi}\alpha_*(e^{i\sigma})d\sigma+2k\pi.\quad(21)$$

Passing to the limit, from (21) we obtain the equality

$$\alpha_*(e^{i\theta})=\arg\omega'(e^{i\theta})+2m\pi,\qquad(22)$$

and hence (14). ■

4. As it has been noted in Introduction, in this section we present assertions which are the consequences of Theorem 1 or are proved by means of this theorem.

Theorem 2. *If Γ satisfy the conditions of Theorem 1, i.e., is smooth, piecewise smooth with angles of size $\pi\nu_k$, $0 \le \nu_k < 2$, at the points in $t_k \in \Gamma$, or is local oscillation of the tangent to Γ is less than π, then $\arg\omega'(z)$ is bounded in U, and $\lg\omega' \in \bigcap_{p>0} H_p$.*

Theorem 3. *If Γ is a smooth curve, then $\omega' \in \bigcap_{p>0} H_p$ ([1], p. 110), and $(\omega')^{-1} \in \bigcap_{p>0} H_p$.*

If local oscillation of the tangent to Γ is less than π/λ, $\lambda \ge 1$ (in particular, if Γ a piecewise smooth curve with no cusps), then for some $\varepsilon > 0$ depending on λ, $(\omega')^{-1} \in H_{\lambda+\varepsilon}$,

Theorem 4 (Lindelöf, see, for e.g., [1], p. 409). *If Γ is a smooth curve, then $\arg\omega'(z)$ is continuous on $\overline{U}$, and*

$$\lim_{z \overset{\wedge}{\to} \to e^{i\theta_0}} \arg\omega'(z) = \arg\omega'(e^{i\theta_0}) = \gamma(e^{i\theta_0}) - \theta_0 - \frac{\pi}{2}, \quad \theta_0 \in [0, 2\pi].$$

Theorem 5. *If Γ is a piecewise curve with angular points t_k, $k = 1, 2, \dots, n$ and interior with respect to Γ angle sizes $\pi\nu_k$, $0 \le \nu_k < 2$, then*

(1) $\omega'(z) = \prod_{k=1}^{n}(z - c_k)^{\nu_k - 1}\phi_0(z)$, where $t_k = \omega(e^i\theta_k) = \omega(c_k)$, $\phi_0 \in \underset{p>0}{H}$;

(2) $\arg[\omega'(z)\prod_{k=1}^{n}(z - c_k)^{-\nu_k}]$ is continuous on U ([6]).

Theorem 6 (Kellog, see, for e.g., [1], p. 411). *If Γ is the Lyapunov curve, i.e., is the smooth curve, whose angle of the tangent slop to the real axis $t \in \Gamma$ at the point t, as the function of the point t(or of the length of the arc s), satisfies Hölder's condition with the exponent $\alpha \in (0, 1)$, then $\omega'(z)$ is continuous on U, and the functions $\omega'(\tau)$ and $\lg\omega'(\tau)$ satisfy on the circumference γ Hölder's condition with the same exponent α.*

Indeed, let $\tau = \psi(t)$ be an inverse to $t = \omega(\tau)$ ($\tau = e^{i\theta}$, $0 \le \theta \le 2\pi$) function. Then, taking into account the fact that Γ is the smooth curve and hence α_* is continuous, for any pairs of points τ_1, $\tau_2 \in \gamma$ we can write

$$\begin{gathered} |\alpha_*(\tau_1) - \alpha_*(\tau_2)| = |\alpha_*[\psi(t_1)] - \alpha_*[\psi(t_2)]| = \\ = |\varphi(t_1) - \varphi(t_2)| \le C|t_1 - t_2|^\alpha \le C_1|s_1 - s_2|^\alpha = \\ = C_1\left|\int_{\theta_1}^{\theta_2} |\omega'(e^{i\sigma})|d\sigma\right|^\alpha \le C_1\left(\int_{\theta_1}^{\theta_2} |\omega'(e^{i\sigma})|^2 d\sigma\right)^{1/2}\left(\int_{\theta_1}^{\theta_2} d\sigma\right)^{\frac{\alpha}{2}} \le \\ \le C_2|\theta_1 - \theta_2|^{\frac{\alpha}{2}} \le C_3|\tau_1 - \tau_2|^{\frac{\alpha}{2}}. \end{gathered}$$

Consequently, density of the Cauchy type integral representing the functions ω' and $\lg\omega'$, satisfies Hölder's condition. Therefore, by virtue of I. Privalov's theorem ([10], Ch. III, §4), ω' is continuous on $\overline{U}$

Taking into account the above-said, we can write

$$|\alpha_*(\tau_1)-\alpha_*(\tau_2)|\le C_2\left|\int_{\theta_1}^{\theta_2}|\omega'(e^{i\sigma})|d\sigma\right|^{\alpha}\le$$
$$\le MC_2|\theta_1-\theta_2|^{\alpha}<C_4|\tau_1-\tau_2|^{\alpha},\quad 0<\alpha<1,$$

where $M=\max_{|z|\le 1}|\omega'(z)|$.

From the above inequality and formula (13), by virtue again of I.Privalov's theorem, it follows that the theorem under consideration is valid.

Theorem 7 (Warschawski [5]). *If* Γ *is a piecewise Ljapunov curve with angular point* t_k, $k=1,2,\dots,n$ *and interior angle sizes* $\pi\nu_k$, $0<\nu_k<2$ *then*

$$\omega'(z)=\prod_{k=1}^{n}(z-c_k)^{\nu_k-1}\phi_0(z),\quad t_k=\omega(e^{i\theta_k})=\omega(c_k),\tag{23}$$

where ϕ_0 *is the function, different from zero, analytic in* U *and continuous on* $\overline{U}$.

Since Γ has no cusps, the function α_* has local less than π oscillation. Hence $\omega'\in H_p$, $p>1$. Taking into account the above-said, we shall show just as in proving Kellog's theorem that the function α_* on every arc $[\tau_k,\tau_{k+1}]$, $k=1,2,\dots,n$, $\tau_{k+1}=\tau_1$ of circumference γ satisfies Hölder's conditions with respect to the variable τ (or on the interval $[\theta,\theta_{k+1}]$ with respect to θ). That is, α_* is the piecewise Hölder's function with jumps at the points c_k, $\pi h_k=\pi(1-\nu_k)$. Then ([9], Ch. I, §26)

$$\frac{1}{\pi}\int_{\gamma}\frac{\alpha_*(\tau)d\tau}{\tau-z}=\sum_{k=1}^{n}(\nu_k-1)\lg(c_k-z)+\Psi(z),\tag{24}$$

where Ψ is the function, analytic in U and continuous on $\overline{U}$. From equations (24) we obtain (23).

Theorem 8 (V. Smirnov, see, for e.g., [1], p. 412). *If* Γ *is a smooth curve whose angle of tangent slop to the real axis* $\varphi(t)$, *as the function of the point* $t\in\Gamma$ (*or of the length of the arc* s) *is absolutely continuous function,* $\varphi'\in L_p$, $p>1$ *then* $\omega''(t)\in H_p$.

Since the function φ is absolutely continuous and its derivative belongs to L_p, $p>1$ it satisfies Hölder's condition. Hence, by Kellog's theorem, $\max_{|t|<1}|\omega'(t)|<\infty$. Therefore

$$|t_k'-t_k''|=|\omega(\tau_k')-\omega(\tau_k'')|=$$
$$=\left|\int_{\tau_k}^{\tau_k''}\omega'(\tau)d\tau\right|\le C\max_{|\tau|=1}|\omega'(\tau)||\tau_k'-\tau_k''|.\tag{25}$$

Let $\tau=\psi(t)$ be an inverse to $t=\omega(\tau)$ function. Then

$$|\alpha_*(\tau_k')-\alpha_*(\tau_k'')|= \\ =|\alpha_*[\psi(t_k')]-\alpha_*[\psi(t_k)'']|=|\varphi(t_k')-\varphi(t_k)''|. \tag{26}$$

From (25) and (26) it follows that α_* is also absolutely continuous. Moreover,

$$\frac{d\alpha_*}{d\tau}=\frac{d\alpha_*}{dt}\frac{dt}{d\tau}=\varphi'[\omega(\tau)]\omega'(\tau)\in L_p(\gamma).$$

From formula (13) we now find that

$$\omega''(z)=\omega'(z)\int_\gamma\frac{\alpha_*(\tau)d\tau}{(\tau-z)^2}=\frac{1}{\pi}\omega'(z)\int_\gamma\frac{\alpha_*'(\tau)d\tau}{\tau-z}\in H_p.$$

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