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SYMMETRICALLY CONTINUOUS FUNCTIONS OF TWO VARIABLES

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Abstract. The notions of symmetrically continuous and separately symmetrically continuous in the strong and angular sense are introduced for the functions of two variables. Between these two notions there exists the relation. Some of the well-known properties equivalent to the continuity with respect to either variable are not preserved for symmetric continuity.

Moreover, the notion of symmetrically $(*)$ continuous which is non- comparable with the notion of symmetrically continuous, is also introduced.

Everywhere below the function is assumed to take only finite values. It is known that the function $f(x)$ of one variable is called symmetrically continuous at the inner point x_0 of the interval (a,b) if the equality $\lim\limits_{h\to 0}[f(x_0+h)-f(x_0-h)]=0$ is fulfilled.

1. Symmetrically Continuous Functions of Two Variables

Definition 1.1. The function of two variables $f(x,y)$ is said to be symmetrically continuous at the point (x_0,y_0) if the equality

$$\lim_{(h,k)\to(0,0)}[f(x_0+h,y_0+k)-f(x_0-h,y_0-k)]=0 \tag{1.1}$$

is fulfilled.

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Proposition 1.1. If the function $f(x,y)$ is continuous at the point (x_0,y_0), then $f(x,y)$ is symmetrically continuous at the point (x_0,y_0). Moreover, at the given point there exists a symmetrically continuous function which is discontinuous at the same point.

The first part of this proposition is the consequence of the equality

$$f(x_0+h,y_0+k)-f(x_0-h,y_0-k)=$$
$$=[f(x_0+h,y_0+k)-f(x_0,y_0)]+[f(x_0,y_0)-f(x_0-h,y_0-k)].$$

To prove the second part of the proposition, we have to consider the function

$$\varphi(x,y)=\begin{cases}\sin\frac{1}{|x|}+\sin\frac{1}{|y|}, & \text{as} \quad x\cdot y\neq 0\\ 0, & \text{as} \quad x\cdot y=0.\end{cases} \tag{1.2}$$

This function is symmetrically continuous at the point $(0,0)$, since $|\varphi(h,k)-\varphi(-h,-k)|=0$. On the other hand, this function is discontinuous at the same point. Indeed, for the adding sequence to the point $(0,0)$, $(x_n,y_n)=\left(\frac{2}{(4n+1)\pi};\frac{2}{(4n+1)\pi}\right)$, we have $f(x_n,y_n)=\sin|\frac{\pi}{2}+2\pi n|+\sin|\frac{\pi}{2}+2\pi n|=2\cdot 1=2$.

Definition 1.2. The function $f(x,y)$ is said to be symmetrically continuous with respect to x(with respect to y) if the equality

$$\lim_{h\to 0}[f(x_0+h,y_0)-f(x_0-h,y_0)]=0 \tag{1.3}$$

$$\text{(respectively}\quad \lim_{k\to 0}[f(x_0,y_0+k)-f(x_0,y_0-k)]=0) \tag{1.3'}$$

is fulfilled.

The function $f(x,y)$ is said to be separately symmetrically continuous at the point (x_0,y_0) if $f(x,y)$ is symmetrically continuous at the point (x_0,y_0) with respect to each of the variables.

Proposition 1.2. If the function $f(x,y)$ is symmetrically continuous at the point (x_0,y_0), then $f(x,y)$ is separately symmetrically continuous at the point (x_0,y_0), and not vice versa.

Indeed, equations (1.3) and (1.3') for respectively $k=0$ and $h=0$ are the particular cases of equation (1.1). In order to prove the second part of the proposition, let us consider as an example the function which is separately symmetrically continuous at the same point as the function

$$U(x,y)=\begin{cases}\frac{x|y|}{x^2+y^2}, & \text{as} \quad (x,y)\neq(0,0)\\ 0, & \text{as} \quad (x,y)=(0,0).\end{cases} \tag{1.4}$$

The function $U(x,y)$ is symmetrically continuous at the point $(0,0)$ with respect to each of the variables, since $|U(h,0)-U(-h,0)|=|U(0,k)-U(0,-k)|=0$.

Let us now prove that the function $U(x,y)$ is not symmetrically continuous at the point $(0,0)$. Indeed, $|U(h,k)-U(-h,-k)| = 2|h||k|(h^2+k^2)^{-1}$.
This expression is equal to unity. If $h=k\neq 0$, then equation (1.1) is not fulfilled.

Definition 1.3. The function $f(x,y)$ is said to be symmetrically continuous in the strong sense with respect to x (with respect to y) at the point (x_0,y_0) if the equality

$$\lim_{(h,k)\to(0,0)}[f(x_0+h,y_0+k)-f(x_0-h,y_0+k)]=0 \tag{1.5}$$

$$\text{(respectively } \lim_{(h,k)\to(0,0)}[f(x_0+h,y_0+k)-f(x_0+h,y_0-k)]=0) \tag{1.5$'$}$$

is fulfilled.

The function $f(x,y)$ is said to be separately symmetrically continuous in the strong sense at the point (x_0,y_0) if $f(x,y)$ is symmetrically continuous in the strong sense at the point (x_0,y_0) with respect to each of the variables.

Proposition 1.3. If the function $f(x,y)$ is symmetrically continuous in the strong sense with respect to one of the variables at the point (x_0,y_0), then $f(x,y)$ is symmetrically continuous at the point (x_0,y_0) with respect to the same variable, and not vice versa.

Indeed, equation (1.3) (equation (1.3$'$)) is the particular case of equation (1.5) (respectively of equation (1.5$'$)) for $k=0$ (respectively for $h=0$).
To prove the second part of the proposition, we consider the function

$$g(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0).\end{cases} \tag{1.6}$$

This function is symmetrically continuous at the point $(0,0)$ with respect to x, since $|g(h,0)-g(-h,0)|=0$. Consider the difference $|g(h,k)-g(-h,k)|=\frac{2|h||k|}{h^2+k^2}$. This expression is equal to unity if $h=k\neq 0$. Hence $g(x,y)$ is not symmetrically continuous in the strong sense with respect to x or, analogously, with respect to y. Moreover, the following theorem is valid.

Theorem 1.1. *If the function $f(x,y)$ is separately symmetrically continuous in the strong sense at the point (x_0,y_0), then this function is symmetrically continuous at the same point, and not vice versa.*

Proof. We have the equality

$$f(x_0+h,y_0+k)-f(x_0-h,y_0-k)=$$
$$=[f(x_0+h,y_0+k)-f(x_0+h,y_0-k)]+[f(x_0+h,y_0-k)-f(x_0-h,y_0-k)].$$

Since the function $f(x,y)$ is separately continuous in the strong sense at the point (x_0,y_0), difference in square brackets is very small as $(h,k)\to(0,0)$. This means that the function $f(x,y)$ is symmetrically continuous at the point (x_0, y_0). Let us prove that the reverse proposition is wrong. To this end we consider the function $g(x, y)$ defined by equation (1.6). This function is symmetrically continuous at the point $(0, 0)$, since $|g(h, k) - g(-h, -k)| = 0$. But as it has been mentioned above, the function $g(x, y)$ is not symmetrically continuous in the strong sense at the point $(0, 0)$ with respect to none of the variables. ■

Remark 1.1. The continuity of the function of two variables is equivalent to the separately strong continuity of this function (see [1], Theorem 1). According to Theorem 1.1, the class of symmetrically continuous functions is essentically wide then the class of separatelly simmetrically continuius ones in strong sense.

1.3. In Theorem 1.1 there naturally arises the question as to whether the separate symmetric continuity in the strong sense at the point (x_0, y_0) causes by partitioning the continuity of the function $f(x, y)$ at the same point. The answer is negative. For example, the function

$$\psi(x, y) = \begin{cases} \frac{1}{x^2+y^2}, & \text{as} \quad (x, y) \neq (0, 0) \\ 0, & \text{as} \quad (x, y) = (0, 0) \end{cases}$$

is separately symmetrically continuous in the strong sense at the point $(0, 0)$. Indeed,

$$|\psi(h, k) - \psi(-h, k)| = 0, \quad |\psi(h, k) - \psi(h, -k)| = 0.$$

The function $\psi(x, y)$ is at the same time discontinuous at the point $(0, 0)$. Indeed, for the adding sequence to the point $(0, 0)$, $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right)$ we have

$$f(x_n, y_n) = f\Big(\frac{1}{n}, \frac{1}{n}\Big) = \frac{n^2}{2} \to \infty \neq f(0, 0).$$

Definition 1.4. The function $F(x, y)$ is said to be symmetrically continuous in the angular sense at the point (x_0, y_0) with respect to x(with respect to y) if the equality

$$\lim_{\substack{(h,k)\to(0,0) \\ |k|\le c|h|}} [f(x_0 + h, y_0 + k) - f(x_0 - h, y_0 + k)] = 0 \tag{1.7}$$

$$\Big(\text{respectively} \lim_{\substack{(h,k)\to(0,0) \\ |h|\le c|k|}} [f(x_0 + h, y_0 + k) - f(x_0 + h, y_0 - k)] = 0\Big) \tag{1.7$'$}$$

is fulfilled for any constant $c > 0$.

The function $f(x, y)$ is said to be separately symmetrically continuous in the angular sense at the point (x_0, y_0) if $f(x, y)$ is symmetrically continuous in the angular sense with respect to either variable.

Proposition 1.4. If the function $f(x,y)$ is symmetrically continuous in the strong sense at the point (x_0,y_0) with respect to one of the variables, then $f(x,y)$ is symmetrically continuous in the angular sense with respect to the same variable, and not vice versa.

Indeed, if equation (1.5) (equation (1.5′)) is fulfilled, then it will be fulfilled even in case $(h,k) \to (0,0)$, $|k| \le c|h|$, ($|h| \le c|k|$) for any constant $c > 0$, and not vice versa. Hence there takes place equation (1.7) (respectively, equation (1.7′)).

Consider the function

$$V(x,y) = \begin{cases} \frac{x^2}{y} + \frac{y^2}{x}, & \text{as} \quad x \cdot y \neq 0 \\ 0, & \text{as} \quad x \cdot y = 0. \end{cases} \tag{1.8}$$

This function is symmetrically continuous in the angular sense at the point $(0,0)$ with respect to x, since $|V(h,k) - V(-h,k)| = \frac{2k^2}{|h|}$ tends to zero for any constant $c > 0$, when $|k| \le c|h|$, $h \to 0$.

Let us now prove that $\lim\limits_{(h,k)\to(0,0)} \frac{2k^2}{|h|}$ does not exist. It we take $h = k^3$, then this expression is equal to $\frac{2}{|k|}$ which in its turn tends to $+\infty$ as $k \to 0$. Hence the function $V(x,y)$ is not symmetrically continuous in the strong sense at the point $(0,0)$.

Proposition 1.5. If the function $f(x,y)$ is symmetrically continuous in the angular sense at the point (x_0,y_0) with respect to x (with respect to y), then $f(x,y)$ is symmetrically continuous with respect to x(respectively, with respect to y), and not vice versa.

Indeed, if in equation (1.7) (in equation (1.7′)) we mean that $k = 0$ ($h = 0$), then we obtain equation (1.3) (respectively equation (1.3’)). There exists the function which is separately symmetrically continuous at the point (x_0,y_0) and does not the function which is separately symmetrically continuous in the angular sense at the same point. Such is the function

$$W(x,y) = \begin{cases} \frac{xy}{(|x|+|y|)^2}, & \text{as} \quad (x,y) \neq (0,0) \\ 0, & \text{as} \quad (x,y) = (0,0) \end{cases} \tag{1.9}$$

which is symmetrically continuous both with respect to x and to y at the point $(0,0)$. For example, with respect to x the above-said is a consequence of the equality $|W(h,0) - W(-h,0)| = 0$. But $W(x,y)$ is not separately symmetrically continuous in the angular sense at the same point. Indeed, let us consider this statement with respect to x. We have

$$|W(h,k) - W(-h,k)| = \Big| \frac{hk}{(|h|+|k|)^2} - \frac{(-h)k}{(|-h| + |-k|^2)} \Big| = \frac{2|h||k|}{(|h|+|k|)^2}.$$

If we take, for example, $k = h \neq 0$, then the above expression will be equal to $\frac{1}{2}$. Hence the function $W(x,y)$ is not symmetrically continuous in the angular sense with respect to x.

1.5. Interrelation between the separate symmetric angular continuity and symmetric continuity follows from

Theorem 1.2. *Symmetric continuity and separate symmetric continuity in the angular sense are non-comparable properties.*

Proof. First of all, let us prove that the symmetric continuity at the given point does not cause symmetric continuity in the angular sense with respect to any of the variables.

As is stated above, the function $g(x,y)$ defined by equation (1.6) is symmetrically continuous at the point $(0,0)$. Let us prove that $g(x,y)$ is not symmetrically continuous in the angular sense, for example, with respect to x at the same point. Indeed, $|g(h,k) - g(-h,k)| = 2|h||k|(h^2+k^2)^{-1}$. This expression is equal to unity when $k = h \neq 0$. Let us prove that the separate symmetric continuity in the angular sense at any point does not cause the symmetric continuity at the same point. As is stated above, the function $V(x,y)$ defined by equation (1.8) is separately symmetrically continuous in the angular sense at the point $(0,0)$. Prove that $V(x,y)$ is not symmetrically continuous at the same point.

Indeed, $|V(h,k) - V(-h,k)| = 2\left|\frac{k^2}{h} + \frac{h^2}{k}\right|$. If we take $k = h^2$, the $2|1+h^3| \to 2 \neq 0$ as $h \to 0$. ■

Remark 1.2. It is known that the continuity of the function of two variables is equivalent to the separate continuity in the angular sense (see [1], Theorem 1.1, Statement 2). According to Theorem 1.2, this property is not preserved for the symmetric continuity.

2. Symmetrically (∗)-continuity

2.1. The expression

$$\Delta^{\text{sym}} f(x_0,y_0;h,k) = f(x_0+h,y_0+k) + \\ + f(x_0-h,y_0-k) - f(x_0+h,y_0-k) - f(x_0-h,y_0+k) \qquad (2.1)$$

is called the second mixed symmetric difference (see, for example, [2]).

Definition 2.1. The function $f(x,y)$ is said to be $(*)$-continuous at the point (x_0,y_0) if we have

$$\lim_{(h,k)\to(0,0)} \Delta^{\text{sym}} f(x_0,y_0;h,k) = 0 \qquad (2.2)$$

Theorem 2.1. *The Simmetrically $(*)$-continuity and the symmetric continuity are non-comparable properties.*

Proof. The fact that the symmetric continuity is not a consequence of the $(*)$-continuity is proved by the function

$$U(x,y)=\begin{cases}\frac{1}{x}+\frac{1}{y}, & \text{as} \quad x\cdot y\neq 0\\ 0, & \text{as} \quad x\cdot y=0\end{cases} \tag{2.3}$$

at the point $(0,0)$. We have the equality $\Delta^{\mathrm{sym}}U(0,0;h,k)=0$. Thus $U(x,y)$ is symmetrically $(*)$-continuous at the point $(0,0)$. This function is not symmetrically continuous at the same point.

Indeed,

$$|U(h,k)-U(-h,-k)|=$$
$$=2\Big|\frac{1}{h}+\frac{1}{k}\Big|\to\begin{cases}+\infty, & \text{as} \quad h=k\\ 0, & \text{as} \quad h=-k,\end{cases} \quad \text{when} \ (h,k)\to(0,0).$$

Hence the symmetric continuity is not a consequence of the $(*)$-continuity.

Moreover, there exists a function which is symmetrically $(*)$-continuous at the point $(0,0)$ but not symmetrically continuous at the same point. Such is the function

$$F(x,y)=\begin{cases}\frac{1}{xy}, & \text{as} \quad x\cdot y\neq 0\\ 0, & \text{as} \quad x\cdot y=0,\end{cases}$$

whose symmetric continuity at the point $(0,0)$ is a consequence of the equation $|F(h,k)-F(-h,-k)|=0$, but $\Delta^{\mathrm{sym}}F(0,0,h,k)=\frac{4}{hk}\nrightarrow 0$, when $(h,k)\to(0,0)$. ■

Remark 2.1. If $\alpha(x)$ and $\beta(y)$ are the functions of one variable which are finite at neighbourhood of the points $x=x_0$ and $y=y_0$ respectively, then the function of two variables $f(x,y)=\alpha(x)+\beta(y)$ is symmetrically $(*)$-continuous at the point (x_0,y_0).

2.2. Between the symmetric continuity in the strong sense with one variable and the symmetrically $(*)$-continuity there is the relation.

Theorem 2.2. *If the function $f(x,y)$ is symmetrically continuous in the strong sense at the point (x_0,y_0) with respect to any variable, then $f(x,y)$ is symmetrically $(*)$-continuous at the point (x_0,y_0), and not vice versa.*

Proof. We have the equality

$$\Delta^{\mathrm{sym}}f(x_0,y_0;h,k)=f(x_0+h,y_0+k)+f(x_0-h,y_0-k)-$$
$$-f(x_0-h,y_0+k)-f(x_0+h,y_0-k)=[f(x_0+h,y_0+k)-f(x_0-h,y_0+k)]-$$
$$-[f(x_0+h,y_0-k)-f(x_0-h,y_0-k)].$$

If the function $f(x,y)$ is symmetrically continuous in the strong sense at the point (x_0,y_0) with respect to x, then the expressions in the square brackets tend to zero as $(h,k)\to(0,0)$. Thus the function $f(x,y)$ is symmetrically $(*)$-continuous at the point (x_0,y_0). The reverse proposition is wrong. In order to prove it, we consider the function $U(x,y)$ defined by equation (2.3). This function is symmetrically $(*)$-continuous at the point $(0,0)$ and at the same time symmetrically continuous in the strong sense at the point $(0,0)$ with respect to x (with respect to y).

Indeed, $|U(h,k) - U(-h,k)| = \frac{2}{|h|} \nrightarrow 0, \quad \text{as} \quad h \to 0.$ ■

REFERENCES

1. O. P. Dzagnidze, On the differentiability of functions of two variables and indefinite double integrals. *Proc. A. Razmadze Math. Inst.* **106**(1993), 7–48.

2. D. Rienne, A restricted symmetric derivative for continuous functions of two variables. *Real Anal. Exchange* **16**(1990-91), 559–564.

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