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ON DIVERGENT ORTHOGONAL SERIES BY THE METHODS OF SUMMABILITY WITH A VARIABLE ORDER

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Introduction

In the theory of orthogonal series one of the fundamental results is the result obtained by D. Menshov and H. Rademacher (see [1] and [2]).

Theorem A (Menshov-Rademacher). *Let* $\{\varphi_n(x)\}_{n=1}^{\infty}$ *be any orthonormal system on* $[0,1]$. *If the orthogonal series*

$$\sum_{n=1}^{\infty} c_n\, \varphi_n(x) \tag{1}$$

is such that

$$\sum_{n=1}^{\infty} c_n^2 \ln^2 n < \infty,$$

then the series (1) *converges almost everywhere on* $[0,1]$.

D. Menshov proved (see [3]) that the Theorem A can not be strengthened for the class of orthonormal systems. In particular, the following theorem holds.

Theorem B (Menshov). *There exists an orthonormal on* $[0,1]$ *system of polynomials* $\{P_n(x)\}$ *such that the following proposition holds.*

Let $\{W(n)\}$ *be such sequence of nonnegative numbers that*

$$\lim_{n\to\infty} \frac{W(n)}{\ln^2 n} = 0,$$

then there exists such sequence $\{c_n\}$ *of numbers, that the series*

$$\sum_{n=1}^{\infty} c_n\, P_n(x)$$

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diverges everywhere on $[0,1]$ *and*

$$\sum_{n=1}^{\infty} c_n^2 \, W(n) < \infty.$$

Theorem B has been strengthened by K. Tandori (see [4]). In particular, the following theorem holds.

Theorem C (Tandori). *For any sequence* $c_n \downarrow 0$ *for which*

$$\sum_{n=1}^{\infty} c_n^2 \, \ln^2 n = \infty,$$

there exists the orthogonal series

$$\sum_{n=1}^{\infty} c_n \, \psi_n(x)$$

such that this series diverges everywhere on $[0,1]$.

For the Cesáro method of (c,α) summability the following theorem holds (see [5]).

Theorem D (Menshov). *If coefficients of the series* (1) *are such that*

$$\sum_{n=2}^{\infty} c_n^2 \, \ln^2 \ln n < \infty,$$

then the series (1) *is summable almost everywhere on* $[0,1]$ *by the* (c,α) *method for an arbitrary* $\alpha > 0$.

D. Menshov proved that the Theorem D can not be strengthened (see [6]). Namely, the following theorem holds.

Theorem E (Menshov). *For any sequence of nonnegative numbers* $\{W(n)\}$ *for which*

$$\lim_{n\to\infty} \frac{W(n)}{\ln^2 \ln n} = 0,$$

there exists the series (1) *such that*

$$\sum_{n=2}^{\infty} c_n^2 \, W(n) < \infty$$

and this series diverges almost everywhere on $[0,1]$ *by the* (c,α) *method for any* $\alpha > 0$.

In the present work we consider the methods of summability with a variable order and represent our theorems connected with these methods.

Notations, Definitions and Statement of the Results

Let $M(\alpha)$ be a triangular matrix $M(\alpha) = \|\lambda_n(\alpha,k)\|$, whose every element depends on the parameter α, where $\alpha \in [0,+\infty)$ and $\lambda_n(\alpha,k) = 0$ for any $n \geq 0$ and $k > n$.

We consider the matrices whose elements for arbitrary $n \geq 0$ and $k = 0,1,\dots,n$ satisfy the following conditions

(i) $\lambda_n(0,k) = 1$ and $0 \leq \lambda_k(\alpha,k) \leq \lambda_n(\alpha,k) \leq 1$, where $\alpha \in [0,+\infty)$;

(ii) for any k, the function $\lambda_k(\alpha,k)$ is differentiable with respect to α.

Consider the numerical series

$$\sum_{k=0}^{\infty} u_k. \tag{2}$$

By S_n we denote partial sums of the series (2) and by $\tau_n(\alpha)$ the means of the series (2) with respect to the matrix $M(\alpha)$, i.e.,

$$S_n = \sum_{k=0}^{n} u_k$$

and

$$\tau_n(\alpha) = \sum_{k=0}^{n} \lambda_n(\alpha,k) \cdot u_k.$$

If there exists a finite limit

$$\lim_{n\to+\infty} \tau_n(\alpha) = S,$$

then we say that the series (2) is summable by the $M(\alpha)$ method to the number S.

If for the sequence $\{\alpha_n\}$ of nonnegative numbers there exists a finite limit

$$\lim_{n\to\infty} \tau_n(\alpha_n) = S,$$

then we say that the series (2) is summable to the number S by the method $M(\alpha_n)$ with a variable order.

If $\alpha_n = \alpha$ for any $n \geq 0$, then, obviously, the $M(\alpha_n)$ summability coincides with the $M(\alpha)$ summability, but if $\alpha_n = 0$ for any $n \geq 0$, then since $\lambda_n(0,k) = 1$, the $M(0)$ summability coincides with the convergence of the series (2).

It should be noted that the Cesáro method of (c,α) summability is a particular case of $M(\alpha)$ summability.

The Cesáro method of summability with a variable order, i.e., the (c,α_n) method has been introduced by Menshov (see [7]).

Below, the writing (A) $\leq$ (B), where (A) and (B) are two methods of summability denotes that if the series (2) is summable by the method (A) to the number S, then this series is also summable by the method (B) to the same number S, while the writing

$$\text{(A)} < \text{(B)}$$

denotes that (A)$\leq$(B) and there exists such series (2) which is not summable by the method (A), but is summable by the method (B).

It is known (see [7]) that if $\alpha_n \geq 0$ and $\alpha_n = \overline{\overline{o}}(n)$, then

$$(c, 0) \leq (c, \alpha_n).$$

We have constructed an example of the divergent series (2) with $u_k \to 0$ which is (c, α_n) summable, where $\sum\limits_{n=0}^{\infty} \alpha_n^2 < \infty$, i.e, it is shown that

$$(c, 0) < (c, \alpha_n)$$

and, in addition, $\sum\limits_{n=0}^{\infty} \alpha_n^2 < \infty$.

We denote partial sums of the series (1) by $S_n(x)$ and all means of the series (1) with respect to the $M(\alpha_n)$ method by $\tau_n(\alpha_n, x)$, i.e.,

$$S_n(x) = \sum_{k=0}^{n} c_k \, \varphi_k(x)$$

and

$$\tau_n(\alpha_n, x) = \sum_{k=0}^{n} \lambda_n(\alpha_n, k) \, c_k \, \varphi_k(x).$$

For every k, for the derivative of the function $\lambda_k(\alpha, k)$ we introduce the notation

$$\gamma_k^2 = \sup_{\alpha \geq 0} \big(\lambda_k'(\alpha, k)\big)^2.$$

For $M(\alpha_n)$ summability method, the following theorem holds.

Theorem 1. *If the sequence of nonnegative numbers* $\{\alpha_n\}$ *is such that*

$$\sum_{n=0}^{\infty} \alpha_n^2 \cdot \sum_{k=0}^{n} c_k^2 \, \gamma_k^2 < \infty,$$

then

$$\lim_{n \to \infty} \big(S_n(x) - \tau_n(\alpha_n, x)\big) = 0, \quad \textit{for almost all} \quad x \in [0, 1].$$

Using this theorem and the known results presented in Introduction, we have proved the theorems on the divergence of orthogonal series by the $M(\alpha_n)$ methods with a variable order. Namely, the following theorems hold.

Theorem 2. *Let the matrix $M(\alpha)$ be such that $\gamma_k^2 = \underset{=}{O}(\ln^2 k)$. Then for any $\{\alpha_n\}$ and $\{W(n)\}$ sequences of nonnegative numbers for which $\sum\limits_{n=0}^{\infty} \alpha_n^2 < \infty$ and $\lim\limits_{n\to\infty} \frac{W(n)}{\ln^2 n} = 0$, there exists the orthogonal series*

$$\sum_{n=0}^{\infty} c_n \, \psi_n(x),$$

which diverges by the $M(\alpha_n)$ method almost everywhere on $[0,1]$ and

$$\sum_{n=1}^{\infty} c_n^2 \, W(n) < \infty.$$

Theorem 3. *For any $\{\alpha_n\}$ sequence of nonnegative numbers for which $\sum\limits_{n=0}^{\infty} \alpha_n^2 < \infty$, there exits the sequence of numbers $c_n \downarrow$ such that the following propositions are valid:*

1) *there exists the orthogonal series*

$$\sum_{n=0}^{\infty} c_n \, \psi_n(x),$$

which diverges almost everywhere on $[0,1]$ by the (c, α_n) method;

2) *every orthogonal series*

$$\sum_{n=0}^{\infty} c_n \, \varphi_n(x),$$

with the same c_n coefficients is (c, α) summable almost everywhere on $[0,1]$, for any $\alpha > 0$.

Let us introduce the notion of the Riesz M-method of summability with order α and the notion of the Riesz M-method of summability with a variable order.

Let $M = \|\lambda_n(k)\|$ be a triangular matrix such that for any n,

$$0 < \lambda_k(k) \le \lambda_n(k) \le 1, \quad k = 0, 1, \ldots, n.$$

For every number $\alpha \ge 0$, the matrix $\|\lambda_n^\alpha(k)\|$ we denote by M^α.

$$\text{i.e.,} \quad M^\alpha = \big\|\lambda_n^\alpha(k)\big\|.$$

The means of the series (1) corresponding to the matrix M^α we denote by $R_n^\alpha(x)$, i.e.,

$$R_n^\alpha(x) = \sum_{k=0}^{n} \lambda_n^\alpha(k) \cdot c_k \, \varphi_k(x). \tag{3}$$

Definition 1. The method of summability corresponding to the matrix M^α we call the Riesz M-method of order α, and the means $R_n^\alpha(x)$ defined by equalities (3) we call the Riesz M-means of order α of the series (1) at the point x.

Definition 2. If for the $\{\alpha_n\}$ sequence of nonnegative numbers there exists a finite limit

$$\lim_{n\to\infty} R_n^{\alpha_n}(x) = S,$$

then we say that the series (1) is summable by the Riesz M-method with a variable order to the number S at the point x.

Here we present some corollaries of Theorem 1 and Theorem 2 for the Riesz M-method of summability with a variable order.

Corollary 1. *If the sequence of nonnegative numbers $\{\alpha_n\}$ is such that*

$$\sum_{n=1}^{\infty} \alpha_n^2 \cdot \sum_{k=1}^{n} c_k^2 \cdot \ln^2 \lambda_k(k) < \infty,$$

then

$$\lim_{n\to\infty} \big(S_n(x) - R_n^{\alpha_n}(x)\big) = 0 \quad \text{for almost all} \quad x \in [0,1].$$

Corollary 2. *Let the matrix $M = \|\lambda_n(k)\|$ be such that for some number $\delta > 0$,*

$$\frac{1}{(k+1)^\delta} \le \lambda_k(k) \le \lambda_n(k) \le 1, \quad k = 0,1,\dots,n.$$

Then for arbitrary sequences of nonnegative numbers $\{\alpha_n\}$ and $\{W(n)\}$ for which $\sum\limits_{n=0}^{\infty} \alpha_n^2 < \infty$ and $\lim\limits_{n\to\infty} \frac{W(n)}{\ln^2 n} = 0$, there exists the orthogonal series

$$\sum_{n=0}^{\infty} c_n\, \psi_n(x)$$

for which the $R_n^{\alpha_n}(x)$ means diverge almost everywhere on $[0,1]$ and

$$\sum_{n=1}^{\infty} c_n^2\, W(n) < \infty.$$

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