Proc. A. Razmadze Math. Inst. 159(2012), 147–151

A. Kharazishvili

SOME PROPERTIES OF *at*-SETS AND *ot*-SETS IN A HILBERT SPACE

In this note we will be dealing with those sets in a Hilbert space H (over the field of reals), all three-point subsets of which have a certain geometric property. Several combinatorial and set-theoretical features of such sets will be indicated and discussed.

Below, the symbol $\mathbf{N}$ denotes the set of all natural numbers. The cardinality of $\mathbf{N}$ is denoted by ω (which is usually identified with $\mathbf{N}$).

$\mathbf{R}$ stands for the real line. More generally, for any natural number m, the symbol $\mathbf{R}^m$ denotes the m-dimensional Euclidean space (consequently, $\mathbf{R} = \mathbf{R}^1$). In our further consideration we always assume that $m \geq 2$.

$\mathbf{c}$ is the cardinality of the continuum, i.e., $\mathbf{c} = \mathrm{card}(\mathbf{R}) = 2^{\omega}$.

If $\mathbf{a}$ is a cardinal number, then the symbol $\mathbf{a}^{+}$ stands for the least cardinal number strictly greater than $\mathbf{a}$.

Let X be a subset of a Hilbert (or, more generally, pre-Hilbert) space H over $\mathbf{R}$. We shall say that X is an *at*-set (respectively, *rt*-set, *ot*-set) if every three-element subset of X forms an acute-angled (respectively, right-angled, obtuse-angled) triangle.

Example 1. Let X be a subset of $\mathbf{R}^m$ such that any three points from X form either acute-angled or right-angled triangle. Then $\mathrm{card}(X) \leq 2^m$ (see [1], [2], [3]). Moreover, if X is an *at*-set in $\mathbf{R}^m$, then $\mathrm{card}(X) < 2^m$.

Example 2. If X is an *at*-set in $\mathbf{R}^3$, then $\mathrm{card}(X) \leq 5$ and there exists an *at*-set $Y \subset \mathbf{R}^3$ such that $\mathrm{card}(Y) = 5$. For sufficiently large natural numbers m, it is possible to indicate an *at*-set $Z \subset \mathbf{R}^m$ whose cardinality is of exponential growth with respect to m, i.e. $\mathrm{card}(Z) \geq \alpha^m$ where $\alpha > 1$ is some real number not depending on m. In [1] and [4] this fact is proved by using probabilistic methods. However, a purely combinatorial proof of the same fact can also be given. Namely, denote by V_m the set of all vertices of the m-dimensional unit cube in $\mathbf{R}^m$ and let $r(m)$ be the number of all right angles in the triangles whose vertices belong to V_m. Then we have the equality

$$r(m) = 2^m((3^m + 1)/2 - 2^m),$$

2010 *Mathematics Subject Classification.* 52A15, 52B05, 52B10, 52B35, 52B45.
Key words and phrases. Hilbert space, *at*-set, *ot*-set, *rt*-set, Ramsey's theorem.

with the aid of which the required result can be obtained by using Stirling's classical formula for the asymptotic behavior of $m!$.

Example 3. Let $X = \{e_i : i \in I\}$ be an orthogonal system of unit vectors in H. Then any three distinct points of X form an equilateral triangle, so X automatically turns out to be an *at*-set. If we take in H the set $Y = X \cup \{0\}$, then it is clear that any three distinct points of Y form either acute-angled triangle or right-angled triangle.

The above example is almost trivial. The next example seems to be much more interesting.

Example 4. Let H denote an infinite-dimensional separable Hilbert space. It can be shown that there exists a set $X \subset H$ such that:
(a) the cardinality of X is equal to $\mathbf{c}$;
(b) any three distinct points of X form an acute-angled triangle.

The existence of X follows directly from the well-known result of infinite combinatorics stating that there is an almost disjoint family of infinite subsets of $\mathbf{N}$, whose cardinality is equal to $\mathbf{c}$. Indeed, without loss of generality, we may identify H with the standard Hilbert space

$$\mathbf{l}_2 = \{t \in \mathbf{R}^{\mathbf{N}} : \sum\{(t(n))^2 : n \in \mathbf{N}\} < +\infty\}.$$

Let $\{N_j : j \in J\}$ be a family of infinite subsets of $\mathbf{N}$ such that:
(1) $\mathrm{card}(J) = \mathbf{c}$;
(2) $\mathrm{card}(N_j \cap N_{j'})$ is finite for any two distinct indices $j \in J$ and $j' \in J$.

Now, for each $j \in J$, define the element $x_j \in \mathbf{l}_2$ by the formula

$$x_j(n) = (1/2^n)\chi_j(n) \quad (n \in \mathbf{N}),$$

where χ_j denotes the characteristic function of the set $N_j \subset \mathbf{N}$.

Putting $X = \{x_j : j \in J\}$, it is easy to check that any three distinct points of X form an acute-angled triangle (cf. [8] where a more complicated argument for establishing the existence of X with the above-mentioned properties (a) and (b) is presented).

Example 5. Let H be again an infinite-dimensional separable Hilbert space over $\mathbf{R}$. We may identify H with the canonical Hilbert space $\mathbf{L}_2[0,1]$. Consider the mapping $g : [0,1] \to \mathbf{L}_2[0,1]$ defined by the formula

$$g(t) = \chi_{[0,t]} \quad (t \in [0,1]),$$

where $\chi_{[0,t]}$ denotes the characteristic function of $[0,t]$. This g is injective and continuous. It can readily be seen that $X = g([0,1])$ is an *rt*-set in $\mathbf{L}_2[0,1]$. This X is usually called the Wiener curve (it is homeomorphic to $[0,1]$; see also Remark 1 below).

Example 6. Take the segment $[0,1/2]$ and consider the mapping $f : [0,1/2] \to \mathbf{l}_2$ defined by the formula

$$f(t) = (t, t^2, t^3, \dots, t^n, \dots) \quad (t \in [0,1/2]).$$

This f is injective and continuous. It can easily be verified that $X = f([0, 1/2])$ is an *ot*-subset of $\mathbf{l}_2$.

We say that an *at*-set (respectively, *rt*-set, *ot*-set) $X \subset H$ is maximal if there is no *at*-set (respectively, *rt*-set, *ot*-set) in H properly containing X.

As a straightforward consequence of the Kuratowski-Zorn lemma, we get that any *at*-set (respectively, *rt*-set, *ot*-set) in a Hilbert space H over $\mathbf{R}$ is contained in some maximal *at*-set (respectively, maximal *rt*-set, maximal *ot*-set).

As far as we know, the following problem remain unsolved.

Problem 1. Give a characterization of all maximal *at*-subsets (respectively, maximal *ot*-subsets) of H.

Remark 1. In [6] a certain characterization of all *rt*-sets in H is given in terms of linear orderings.

It is not difficult to show that no finite *ot*-subset of H can be maximal. On the other hand, the following statement was proved in [7].

Theorem 1. *If $m \geq 2$, then there exists a countable locally finite maximal ot-set in the space $\mathbf{R}^m$.*

Example 7. In the Euclidean plane $\mathbf{R}^2$ consider the half-circumference of the unit circle, from which one of its endpoints is removed, i.e., consider the set

$$X = \{(\cos(\phi), \sin(\phi)) : 0 \leq \phi < 2\pi\}.$$

It can be demonstrated that X is a maximal *ot*-subset of $\mathbf{R}^2$.

Theorem 1 and Example 7 show us that there exist maximal *ot*-sets whose cardinalities are ω and $\mathbf{c}$ respectively. Keeping in mind this fact, it is natural to formulate the next unsolved problem concerning *ot*-sets in Euclidean space.

Problem 2. Let $m \geq 2$ be a natural number and let κ be a cardinal number from the open interval $]\omega, \mathbf{c}[$. Does there exist a maximal *ot*-set in $\mathbf{R}^m$ whose cardinality is equal to κ?

Obviously, under the Continuum Hypothesis the above problem becomes trivial.

Let $\varepsilon > 0$ be a real number and let $\triangle$ be a triangle in H whose side-lengths are a_1, a_2, and a_3. We shall say that $\triangle$ is an equilateral triangle with exactness to ε if the inequalities

$$1 - \varepsilon < a_i/a_j < 1 + \varepsilon$$

hold true whenever $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3\}$.

Theorem 2. *Let X be a subset of H. Then the disjunction of the following two statements is satisfied:*

(1) *X is separable;*

(2) *for any* $\varepsilon > 0$*, there exists an infinite set* $Y \subset X$ *such that all three-point subsets of* Y *form equilateral triangles with exactness to* ε*.*

The proof of Theorem 2 is based on the infinite (countable) version of the well-known combinatorial theorem of Ramsey (see [11]).

As a consequence of Theorem 2, we obtain the next statement.

Theorem 3. *Let* X *be a set in* H *such that every three-point subset of* X *forms either right-angled or obtuse-angled triangle. Then* X *is separable. Therefore,* $\mathrm{card}(X) \leq \mathbf{c}$.

It directly follows from Theorem 3 that any *ot*-subset (*rt*-subset) X of H is separable, so satisfies the inequality $\mathrm{card}(X) \leq \mathbf{c}$.

The following question naturally arises: is it true, for a finite set $Z \subset \mathbf{R}^m$ containing sufficiently many points no three of which are collinear, that there exists an *ot*-set $Y \subset Z$ containing the prescribed number of points?

It turns out that the answer to this question is positive. Indeed, taking into account Example 1 and the finite version of Ramsey's theorem [11], it is not difficult to prove that, for each $k \in \mathbf{N}$, there exists a natural number $p = p(k, m)$ having the following property:

(*) any set $Z \subset \mathbf{R}^m$ with $\mathrm{card}(Z) \geq p$, no three points of which are collinear, contains some *ot*-set Y with $\mathrm{card}(Y) = k$.

The infinite (countable) version of Ramsey's theorem yields a natural analogue of the above result, which can be formulated as follows:

(**) if X is an arbitrary infinite subset of $\mathbf{R}^m$ no three points of which are collinear, then there exists an infinite *ot*-set $Y \subset X$.

Remark 2. (**) does not admit a generalization to the case of uncountable sets in $\mathbf{R}^m$. More precisely, if $X \subset \mathbf{R}^m$ is an uncountable set no three points of which are collinear, then we cannot assert (in general) that X contains an uncountable *ot*-subset. The corresponding counterexample can be constructed by assuming the Continuum Hypothesis, with the aid of an appropriate Luzin or Sierpiński subset of $\mathbf{R}^m$ (extensive information on Luzin and Sierpiński sets may be found in [9] and [10]).

Theorem 4. *Under the Continuum Hypothesis, there exists an uncountable set* X *in a separable Hilbert space* H*, such that:*

(a) *all points of* X *are in general position;*

(b) *every at-subset* (*rt-subset, ot-subset*) *of* X *is at most countable.*

Notice that in Theorem 4 the role of X is played by a certain Luzin set in H. Actually, X has a much stronger property than property (b). Namely, every uncountable subset of X contains three-point sets which form a triangle almost similar to any given triangle.

It is useful to compare Theorem 4 with Examples 4, 5 and 6.

Remark 3. It follows from the well-known Erdös-Rado combinatorial theorem [5] that if $\mathbf{a} \geq \mathbf{c}$ is a cardinal number and X is a subset of a pre-Hilbert space H satisfying the inequality $\text{card}(X) \geq (2^{\mathbf{a}})^{+}$, then there exists a set $Y \subset X$ such that $\text{card}(Y) \geq \mathbf{a}^{+}$ and all three-element subsets of Y form equilateral triangles. Obviously, this Y is an *at*-set in H.

References

1. M. Aigner and G. M. Ziegler, Proofs from The Book. Including illustrations by Karl H. Hofmann. Third edition. *Springer-Verlag, Berlin*, 2004.
2. V. G. Boltyanskii and I. Ts. Gokhberg, Theorems and Problems from Combinatorial Geometry. (Russian) *Izd. Nauka, Moscow*, 1965.
3. L. Danzer and B. Grünbaum, Über zwei Probleme bezüglich konvexer Körper von P. Erdös und von V. L. Klee. (German) *Math. Z.* **79** (1962), 95–99.
4. P. Erdös and Z. Füredi, The greatest angle among n points in the d-dimensional Euclidean space. *Combinatorial mathematics (Marseille-Luminy, 1981)*, 275–283, North-Holland Math. Stud., 75, *North-Holland, Amsterdam*, 1983.
5. P. Erdös and R. Rado, A partition calculus in set theory. *Bull. Amer. Math. Soc.* **62** (1956), 427–489.
6. A. B. Kharazishvili, Selected Topics in Geometry of Euclidean spaces. (Russian) *Izd. Tbil. Gos. Univ., Tbilisi*, 1978 .
7. A. B. Kharazishvili, On maximal *ot*-subsets of the Euclidean plane. *Georgian Math. J.* **10** (2003), No. 1, 127–131.
8. P. Komjáth and V. Totik, Problems and theorems in classical set theory. Problem Books in Mathematics. *Springer, New York*, 2006.
9. K. Kuratowski, Topology. Vol. I. New edition, revised and augmented. Translated from the French by J. Jaworowski *Academic Press, New York-London; Panstwowe Wydawnictwo Naukowe, Warsaw*, 1966.
10. J. C. Oxtoby, Measure and category. A survey of the analogies between topological and measure spaces. Graduate Texts in Mathematics, Vol. 2. *Springer-Verlag, New York-Berlin*, 1971.
11. F. P. Ramsey, On a problem of formal logic. *Proc. London Math. Soc.*, Vol. 30, 1930, 264–286.

Author's address:

A. Razmadze Mathemetical Institute
I. Javakhishvili Tbilisi State University
2, University Str., Tbilisi 0186
Georgia
E-mail: kharaz2@yahoo.com