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ON THE INTEGRABILITY OF A MAXIMAL FUNCTION CORRESPONDING TO A TRANSLATION INVARIANT BASIS COMPOSED OF CUBIC INTERVALS

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Let us call a family B of measurable subsets of $\mathbb{R}^n$ a basis if for each $x \in \mathbb{R}^n$ there exists a sequence of sets from B containing x and having diameters which tend to zero. The maximal operator M_B corresponding to B is defined as follows: for $f \in L(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$

$$M_B(f)(x) = \sup_{E \in B,\, x \in E} \frac{1}{|E|} \int_E |f|.$$

The basis B is said to be translation invariant if for every $E \in B$ any translation of the set E belongs to the basis B. The basis composed of all cubic intervals of $\mathbb{R}^n$ is denoted by $\mathbf{Q}_n$. $M_{\mathbf{Q}_n}$ is called the Hardy–Littlewood maximal operator.

Assume that an increasing and continuous function Φ defined on $[0, \infty)$ is such that $\Phi(0) = 0$, $\Phi(t) > 0$ $(t > 0)$ and $\varliminf\limits_{t\to\infty} \frac{\Phi(t)}{t} > 0$. Then the class

$$\Phi(L)(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to T : \ f \ \text{ is measurable and } \int_{\mathbb{R}^n} \Phi(|f|) < \infty \right\}$$

is called an integral class.

Denote by $\Phi(L)(I^n)$, where $I^n = (0,1)^n$, the class of all functions $f \in \Phi(L)(I^n)$ whose support is in I^n.

The following theorem is valid.

Theorem A. *For a function $f \in L(I^n)$ the following conditions are equivalent:*

$$f \in L \ln^+(I^n), \tag{1}$$

$$M_{\mathbf{Q}_n}(f) \in L(I^n). \tag{2}$$

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The implication (1)⇒(2) was proved by G. Hardy and J. Littlewood [1] for $n = 1$, and by Wiener for any $n \geq 2$. The inverse implication (2)⇒(1) was proved by E. Stein [3] and also by O. Tsereteli [4] for $n = 1$.

K. Hare and A. Stokolos [5] showed that the above-presented theorem does not extend to the maximal operator corresponding to a translation invariant basis composed of cubic intervals. In particular they proved the following

Theorem B. *There exists a translation invariant basis B composed of cubic intervals, for which there is a function $f \in L(I^1)$ such that $M_B(f) \in L(I^1)$ but $f \notin L \ln^+ L(I^1)$.*

Therefore for translation invariant basis composed of cubic intervals, the integrability of a maximal function, as different from the case of the basis (or a Hardy–Littlewood maximal function) composed of all cubic intervals, does not lead to the improvement of integral properties so that it could belong to the class $L \ln^+ L(I^n)$.

There arises the question: does there exist nontrivial (i.e. narrower than $L(I^n)$) class $\Phi(L)(I^n)$ wider than the class $L \ln^+ L(I^n)$ such that for a translation invariant basis B composed of cubic intervals the integrability of $M_B(f)$ makes f belong to the class $\Phi(L)(I^n)$?

The following theorem gives a negative answer to this question.

Theorem. *For any integral class $\Phi(L)(I^n)$ narrower than $L(I^n)$ there exists a translation invariant basis B composed of cubic intervals for which there exists a function f such that $M_B(f) \in L(I^n)$, but $f \notin \Phi(L)(I^n)$.*

Therefore the influence of the integrability condition of the maximal function corresponding to a translation invariant basis B composed of cubic intervals on integral properties of the function itself can be arbitrarily weak.

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