Proc. A. Razmadze Math. Inst. 149(2009), 126–129

R. Koplatadze and G. Giorgadze

ON OSCILLATORY PROPERTIES OF SOLUTIONS OF THIRD-DIMENSIONAL LINEAR DIFFERENTIAL SYSTEMS WITH DEVIATING ARGUMENTS

(Reported on 22.01.2009)

1. Introduction

The problem of oscillation of solutions is well studied for differential equations of high order. In particular, many interesting results of optimal types are obtained (see, for example [1], [2]). Analogous results for two-dimensional linear systems were obtained in [3]. The purpuse of the present paper is to establish some optimal sufficient conditions for the oscillation of solutions of three-dimensional linear systems.

Let us consider the following linear system

$$\begin{aligned} x_1'(t) &= p_i(t)\, x_{i+1}(\tau_i(t)) \quad (i = 1, 2), \\ x_3'(t) &= -p_3(t)\, x_1(\tau_3(t)), \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} &p_i \in L_{\rm loc}(T_+; R_+), \quad \tau_i \in C(R_+; R_+), \\ &\lim_{t\to+\infty} \tau_i(t) = +\infty \quad (i = 1, 2, 3), \quad \tau_1'(t) \ge 0. \end{aligned} \tag{1.2}$$

Below, we will assume that the following conditions

$$\int\limits_0^{+\infty} p_i(t)\, dt = +\infty \quad (i = 1, 2) \tag{1.3}$$

hold.

Definition 1.1. Let $t_0 \in R_+$.A continuous vector function $x = (x_i)_{i=1}^3 : [\tau_0; +\infty) \to R^3$ is said to be a proper solution of system (1.1) if it is locally absolutely continuous on $[t_0, +\infty)$, almost everywhere on this interval the equality (1.1) if fulfilled, and $\sup\{\|x(s)\|;\ s \in [t_0; +\infty)\} > 0$, for $t \ge t_0$, where $\tau_0 = \inf\{\tau_*(t);\ t \ge t_0\}$, $\tau_*(t) = \min\{\tau_i(t);\ i = 1, 2, 3\}$.

2000 *Mathematics Subject Classification:* 34K06, 34K11.
Key words and phrases. Proper solutions, differential systems, oscillatory solutions.

Definition 1.2. A proper solution of the system (1.1) is said to be oscillatory if every component of this solution has a sequence of zeroes tending to $+\infty$. Otherwise the solution is said to be nonoscillatory.

2. Sufficient Conditions for Oscillation of Solutions

Theorem 2.1. *Assume that the conditions* (1.2), (1.3) *and*

$$\int_0^{+\infty} p_3(t)\, h(t)\, h(\tau_3(t))\, dt = +\infty, \qquad \int_0^{+\infty} p_3(t)\, h^2(\tau_3(t))\, dt = +\infty \tag{2.1}$$

are fulfilled. If, moreover for any $\lambda \in [1,2]$ *the inequality*

$$\limsup_{\varepsilon\to 0+}\bigg(\liminf_{t\to+\infty} h^{-\lambda-\delta_{2\varepsilon}(\lambda)}(t)\int_0^t \tau_1'(s)\bigg(\int_s^t p_1(\xi)d\xi\bigg)\times \\ \times \big(h(\sigma(s))\big)^{\delta_{2\varepsilon}(\lambda)+\delta_{1\varepsilon}(\lambda)}\int_{\tau_2(\tau_1(s))}^{+\infty} p_3(\xi)\big(h(\tau_3(\xi))\big)^{\lambda-\delta_{1\varepsilon}(\lambda)}d\xi\, ds\bigg) > 1$$

holds, then each proper solution of system (1.1) *either is oscillatory or satisfies the conditions*

$$\big\|x_i(t)\big\| \downarrow 0 \quad \textit{for} \quad t\uparrow +\infty \quad (i=1,2,3), \tag{2.2}$$

where

$$h(t) = \int_0^t p_1(s)\, ds, \quad \sigma(t) = \inf\big\{\min(s,\tau_3(s)) : s\geq t\big\}, \tag{2.3}$$

$$\delta_{1\varepsilon}(\lambda) = \begin{cases} 0 & \textit{for} \quad \lambda = 1, \\ \varepsilon & \textit{for} \quad \lambda\in(1,2], \end{cases} \qquad \delta_{2\varepsilon}(\lambda) = \begin{cases} 0 & \textit{for} \quad \lambda = 2, \\ \varepsilon & \textit{for} \quad \lambda\in[1,2). \end{cases} \tag{2.4}$$

Theorem 2.2. *Assume that the conditions* (1.2), (1.3), (2.1) *and*

$$\liminf_{t\to+\infty}\frac{h(\sigma(t))}{h(t)} > 0 \tag{2.5}$$

are fulfilled. If, moreover for any $\lambda\in[1,2]$

$$\limsup_{\varepsilon\to 0+}\bigg(\liminf_{t\to+\infty} h^{-\lambda-\delta_{2\varepsilon}(\lambda)}(t)\int_0^t \tau_1'(s)\bigg(\int_s^t p_1(\xi)d\xi\bigg)\times \\ \times \big(h(s)\big)^{\delta_{2\varepsilon}(\lambda)+\delta_{1\varepsilon}(\lambda)}\int_{\tau_2(\tau_1(s))}^{+\infty} p_3(\xi)\big(h(\tau_3(\xi))\big)^{\lambda-\delta_{1\varepsilon}(\lambda)}d\xi\, ds\bigg) > 1,$$

then each proper solution of system (1.1) *either is oscillatory or satisfies condition* (2.2)*, where the functions* $h, \sigma, \delta_{1\varepsilon}, \delta_{2\varepsilon}$ *are defined by* (2.3) *and* (2.4)*.*

Theorem 2.3. *Assume that the conditions* (1.2) *and* (2.1) *hold, if*

$$p_1(t) = c_1 > 0, \quad p_2(t) \geq c_2 > 0 \quad \textit{for} \quad t \in R_+. \tag{2.6}$$

If, moreover for any $\lambda \in [1, 2]$

$$\limsup_{\varepsilon \to 0+} \left(\liminf_{t \to +\infty} \tau_1'(t)\, t^{2-\lambda+\delta_{1\varepsilon}(\lambda)} \int\limits_{\tau_2(\tau_1(t))}^{+\infty} p_3(\xi)\big(\tau_3(\xi)\big)^{\lambda-\delta_{1\varepsilon}(\lambda)} d\xi \right) > \frac{\lambda(\lambda-1)}{c_1\, c_2},$$

then each proper solution of the system (1.1) *either is oscillatory or satisfies the condition* (2.2)*, where* $\delta_{1\varepsilon}$ *is defined by the first equality of* (2.4)*.*

Corollary 2.1. *Assume that the conditions* (1.2)*,* (2.1)*,* (2.6) *hold and* $\tau_1(t) = \alpha_1 t$ *for* $t \in R^+$*, where* $\alpha_1 > 0$*. If, moreover for any* $\lambda \in [1, 2]$ *the following inequality*

$$\limsup_{\varepsilon \to 0+} \left(\liminf_{t \to +\infty} t^{2-\lambda+\delta_{1\varepsilon}(\lambda)} \int\limits_{\tau_2(\alpha_1(t))}^{+\infty} p_3(\xi)\big(\tau_3(\xi)\big)^{\lambda-\delta_{1\varepsilon}(\lambda)} d\xi \right) > \\ > \frac{\lambda(\lambda-1)}{c_1\, c_2\, \alpha_1} \tag{2.7}$$

holds, then each proper solution of system (1.1) *either is oscillatory or satisfies condition* (2.2)*.*

Remark 2.1. If should be mentioned that we can not change the inequality (2.7) by the nonstrong one because in this case Corollary 2.1 will not be true.

The next theorem demonstrates that condition (2.7) is optimal.

Theorem 2.4. *Let us assume that* $\alpha_i, c_i \in (0, +\infty)$ $(i = 1, 2, 3)$*. Then every proper solution of the system*

$$x_i'(t) = c_i\, x_{i+1}(\alpha_i t) \quad (i = 1, 2),$$
$$x_3'(t) = -\frac{c_3}{t^3}\, x_1(\alpha_3 t),$$

either is oscillatory or satisfies the condition (2.2)*, if and only if the inequality*

$$c_1 c_2 c_3 \alpha_1^{-1} \alpha_2^{-1} > \max\big\{\lambda(\lambda-1)(2-\lambda)(\alpha_1\alpha_2\alpha_3)^{-\lambda};\ \lambda \in [1, 2]\big\}$$

holds.

References

1. R. Koplatadze, On oscillatory properties of solutions of functional-differential equations. *Mem. Differential Equations Math, Phys.* **3** (1994).
2. R. Koplatadze, Quasi-linear functional differential equations with Property A. *J. Math. Anal. Appl.* **330** (2007), No. 1, 483–510.
3. R. Koplatadze, N. Partsvania and I. P. Stavroulakis, Asymptotic behaviour of solutions of two-dimensional linear differential systems with deviating arguments. *Arch. Mathh.* (*Brno*) **39** (2003), No. 3, 213–232.

Authors' Addresses:

R. Koplatadze
Department of Mathematics of
I. Javakhishvili Tbilisi State
University,
2, University str., 0143 Tbilisi
Georgia

G. Giorgadze
Department of Mathematics
Georgian Technical University
77, M. Kostava St., Tbilisi 0175
Georgia