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ON OSCILLATORY PROPERTIES OF SOLUTIONS OF GENERALIZED EMDEN-FOWLER TYPE DIFFERENTIAL EQUATIONS

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1. Introduction

Consider the Emden-Fowler type ordinary differential equation

$$u^{(n)}(t) + p(t)\,|u(t)|^{\lambda}\,\operatorname{sign} u(t) = 0, \tag{1.1}$$

where $p \in L_{\mathrm{loc}}(R_+; R_+)$, $\lambda \in (0, +\infty)$.

When $\lambda = 1$, the equation (1.1) is the linear differential equation

$$u^{(n)}(t) + p(t)\,u(t) = 0. \tag{1.2}$$

In 1893, A. Kneser posed the problem of finding conditions for the equation (1.2) to have properties similar to those of

$$u^{(n)}(t) + u(t) = 0. \tag{1.2$'$}$$

Later, in 1961, the property that the equation (1.2) may have only such solutions as (1.2$'$) was called Property **A** by Kondrat'ev:

Definition. We say that the equation (1.2) ((1.1)) has Property **A** if any of its proper solutions is oscillatory when n is even and either is oscillatory or satisfies

$$|u^{(i)}(t)| \downarrow 0, \quad t \uparrow +\infty \quad (i = 0, \dots, n-1)$$

when n is odd.

The essential result in case of the equation (1.2) was given by Kondrat'ev [1].

Theorem (Kondrat'ev [1]). *Let*

$$p(t) \geq \frac{M_n + \varepsilon}{t^n}\,, \quad t \geq 1,$$

where $\varepsilon > 0$ *and*

$$M_n = \max\big\{ -\lambda(\lambda-1)\cdots(\lambda-n+1) : \lambda \in [0, n-1]\big\}.$$

Then the equation (1.2) has Property **A**.

An integral generalization of this result was obtained by Chanturia [2], [3].

Analogous results for the equation (1.1), where $\lambda \neq 1$, were obtained in 60ies of the last century.

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2. “Almost Linear” Generalized Emden-Fowler Type Ordinary Differential Equations

Consider the following ordinary differential equation

$$u^{(n)}(t)+p(t)\,|u(t)|^{\mu(t)}\,\operatorname{sign}u(t)=0, \tag{2.1}$$

where $p\in L_{loc}(R_+;R_+)$, $\mu\in C(R_+;(0,+\infty))$, $\lim\limits_{t\to+\infty}\mu(t)=1$. We say that the equation (2.1) is “almost linear”, if the condition $\lim\limits_{t\to+\infty}\mu(t)=1$ is fulfilled, while if there exists $\lambda\in(0,1)$ ($\lambda\in(1,+\infty)$) such that $\mu(t)\le\lambda$ ($\mu(t)\ge\lambda$) for $t\in R_+$, then we say that the equation (2.1) is an essentially nonlinear differential equation. To simplify things, consider the “almost linear” differential equation of the following type:

$$u^{(n)}(t)+p(t)\,|u(t)|^{1+\frac{d}{\ln t}}\,\operatorname{sign}u(t)=0,\quad t\ge a>1, \tag{1.1$'$}$$

where $d\in R$.

Theorem 2.1. *Let*

$$\liminf_{t\to+\infty}\, t\int\limits_t^{+\infty}s^{n-2}p(s)ds$$
$$>\max\big\{-e^{-\lambda d}\lambda(\lambda-1)\cdots(\lambda-n+1):\lambda\in[0,n-1]\big\}. \tag{2.2}$$

Then the equation (2.1′) *has Property* **A***.*

Remark 2.1. (2.2) is the optimal condition for any $d\in R$. The inequality (2.2) cannot be replaced by the nonstrict one. In general the theorem will not remain true.

Remark 2.2. For $d=0$ this result gives a theorem of Chanturia [3], which is the integral generalization of Kondrat’ev’s result.

Theorem 2.1 and its generalization are published in the paper [4].

Theorem 2.2. *Let*

$$\liminf_{t\to+\infty}\,\frac{1}{t}\int\limits_0^{t}s^{n}\,p(s)\,ds$$
$$>\max\big\{-e^{-\lambda d}\lambda(\lambda-1)\cdots(\lambda-n+1):\lambda\in[0,n-1]\big\}. \tag{2.3}$$

Then (2.1′) *has Property* **A***.*

Remark 2.3. The inequality (2.3) cannot be replaced by the nonstrict one.

Remark 2.4. For $d=0$ from this result follows a result of R. Koplatadze [5].

Now consider the “almost linear” differential equation

$$u^{(n)}(t)+\frac{c}{t^n}\,|u(t)|^{1+\frac{d}{\ln t}}\,\operatorname{sign}u(t)=0. \tag{2.4}$$

Theorem 2.3. *Let $c\in(0,+\infty)$, $d\in R$. Then for the equation* (2.4) *to have Property* **A***, it is necessary and sufficient that*

$$c>\max\big\{-e^{-\lambda d}\lambda(\lambda-1)\cdots(\lambda-n+1):\lambda\in[0,n-1]\big\}.$$

For showing difference and similarity between linear and “almost linear” differential equations we will consider a simple example. Consider the equation

$$u^{(n)}(t)+\frac{M_n}{t^n}\,u(t)=0,$$

where M_n is Kondrat'ev's constant $\big(M_n = \max\big\{-\lambda(\lambda-1)\cdots(\lambda-n+1) : \lambda \in [0,n-1]\big\}\big)$. It is obvious that this equation has not Property **A**, but for any $d>0$ the equation

$$u^{(n)}(t) + \frac{M_n}{t^n}\,|u(t)|^{1+\frac{d}{\ln t}}\ \operatorname{sign} u(t) = 0$$

has Property **A**.

On the other hand, for any $d>0$ exist $\varepsilon = \varepsilon(d) > 0$ such that, the equation

$$u^{(n)}(t) + \frac{M_n+\varepsilon}{t^n}\,u(t) = 0$$

have Property **A** and the equation

$$u^{(n)}(t) + \frac{M_n+\varepsilon}{t^n}\,|u(t)|^{1-\frac{d}{\ln t}}\ \operatorname{sign} u(t) = 0$$

does not have Property **A**.

Furthermore, note that the author has obtained analogous results for differential equations with deviating arguments as well as for integro-differential equations. For each class the obtained results are optimal, that is, a strict inequality cannot be replaced by a non-strict one.

3. Essentially Nonlinear Generalized Differential Equation of Emden-Fowler Type with Advanced Argument

In this section we will consider the following equation

$$u^{(n)}(t) + p(t)\,|u(\sigma(t))|^{\mu(t)}\ \operatorname{sign} u(\sigma(t)) = 0. \tag{3.1}$$

Everywhere below it will be assumed that the conditions

$$p \in L_{loc}(R_+;R_+), \quad \sigma(t) \ge t, \quad 0 < \mu(t) \le \mu_0 < 1, \quad t \in R_+\,,$$

are fulfilled.

Theorem 3.1. *Let*

$$\limsup_{t\to+\infty} \frac{\sigma(t)}{t} < +\infty. \tag{3.2}$$

Then the condition

$$\int^{+\infty} t^{(n-1)\,\mu(t)} p(t)\,dt = +\infty$$

is necessary and sufficient for the equation (3.1) *to have Property* **A**.

Remark 3.1. From Theorem 3.1, when $\sigma(t) \equiv t$ and $\mu(t) \equiv \mu_0 < 1$, follows a theorem of I. Licko and M. Svec [6].

Corollary 3.1. *Let the condition* (3.2) *be fulfilled and*

$$\limsup_{t\to+\infty} t^{\mu(t)} < +\infty.$$

Then the condition

$$\int^{+\infty} p(t)\,dt = +\infty$$

is necessary and sufficient for the equation (3.1) *to have Property* **A**.

Remark 3.2. Note that a necessary and sufficient condition of this kind which does not depend on the order of the equation, is given for the first time.

Remark 3.3. One can give examples of the equation

$$u^{(n)}(t) + p(t)\,|u(t)|^{\mu(t)} \operatorname{sign} u(t) = 0, \tag{3.3}$$

with $\mu(t) \downarrow \mu_0$, $(\mu(t) \uparrow \mu_0)$, $0 < \mu_0 < 1$, such that the equation (3.3) has Property **A** (does not have Property **A**) but the corresponding "limiting" equation

$$u^{(n)}(t) + p(t)\,|u(t)|^{\mu_0} \operatorname{sign} u(t) = 0$$

does not have Property **A** (has Property **A**).

Theorem 3.2. *Let there exist* $\beta \ge 1$ *such that*

$$\liminf_{t\to+\infty} \frac{\sigma(t)}{t^\beta} > 0, \quad \beta\,\mu(t) \le 1.$$

Then the condition

$$\int^{+\infty} t^{(n-2)\beta\mu(t)+\mu(t)}\, p(t)\, dt = +\infty$$

is sufficient for the equation (3.1) *to have Property* **A**.

Remark 3.4. Theorem 3.2 is an essential generalization of Theorem 1.3 [7] (even in case $\mu(t) \equiv \mu_0 < 1$).

Theorem 3.3. *Let there exist* $\beta > 1$ *such that*

$$\liminf_{t\to+\infty} \frac{\sigma(t)}{t^\beta} > 0, \quad \mu(t)\,\beta \ge 1.$$

Then the condition

$$\int^{+\infty} t^{n-2+\mu(t)}\, p(t)\, dt = +\infty$$

is sufficient for the equation (3.1) *to have Property* **A**.

Theorem 3.4. *Let* n *be odd and*

$$\liminf_{t\to+\infty} \frac{t\,\sigma(t)}{t^{2/\mu(t)}} > 0.$$

Then the condition

$$\int^{+\infty} t^{n-1}\, p(t)\, dt = +\infty \tag{3.4}$$

is necessary and sufficient for the equation (3.1) *to have Property* **A**.

Remark 3.5. Theorem 3.4 is an essential generalization of Theorem 1.2 [7] (even in case $\mu(t) \equiv \mu_0 < 1$).

It is obvious that the number $2/\mu_0 - 1$ $(\mu(t) \equiv \mu_0)$ defines the set of the functions σ for which the condition (3.4) is necessary and sufficient. It turns out that the number $2/\mu_0 - 1$ is optimal.

Namely, for any $\varepsilon > 0$ and $\mu_0 \in \left[\frac{1}{1+\varepsilon}, 1\right)$ there exists the set of the functions p and σ such that

$$\liminf_{t\to+\infty} \frac{t\,\sigma(t)}{t^{2/\mu_0}} = 0, \qquad \lim_{t\to+\infty} \frac{t\,\sigma(t)}{t^{2/\mu_0-\varepsilon}} = +\infty,$$

the condition (3.4) is fulfilled, but the equation

$$u^{(n)}(t) + p(t)\,\big|u(\sigma(t))\big|^{\mu_0} \operatorname{sign} u(\sigma(t)) = 0$$

has not Property **A**.

Analogous results are obtained when

$$\sigma(t) \le t, \quad \mu(t) \ge \mu_0 > 1 \quad \text{for} \quad t \ge 0.$$

They are generalizations of some results by R. Koplatadze [8].

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